INTRODUCTION

The study of supergroups is a chapter of supergeometry, i.e. geometry in a $\mathbb{Z}_2$-graded sense. In particular, the relevant structure sheaves of (commutative) algebras sitting on top of the topological spaces one works with are replaced with sheaves of (commutative) superalgebras.

When dealing with differential supergeometry, our “superspaces” are supermanifolds, that are real smooth, real analytic or complex holomorphic (depending on the context): any such supermanifold can be considered as a classical (i.e. non-super) manifold — of the appropriate type — endowed with a suitable sheaf of commutative superalgebras. The supergroups in this context are then Lie supergroups, of smooth, analytic or holomorphic type according to the chosen setup.

For every Lie supergroup $G$ there exists a special pair of objects, say $(G_0, \mathfrak{g})$, that is naturally associated with it: $G_0$ is the classical Lie group underlying $G$ — roughly given by “killing the odd part” of the structure sheaf on $G$ — while $\mathfrak{g} = \text{Lie}(G)$ is the tangent Lie superalgebra of $G$, and these two objects are “compatible” in a natural sense. More in general, any similar pair $(K_+, \mathfrak{k})$ made by a Lie group $K$ and a Lie superalgebra $\mathfrak{k}$ obeying the same compatibility constraints is called “super Harish-Chandra pair” (a terminology first found in [8]), or just sHCP for short: in fact, this notion is tailored in such a way that mapping $G \mapsto (G_0, \mathfrak{g})$ yields a functor, call it $\Phi$, from the category of Lie supergroups — either smooth, analytic or holomorphic — to the category of super Harish-Chandra pairs — of smooth, analytic or holomorphic type respectively.

The key question is: can one recover a Lie supergroup out of its associated sHCP? More precisely: is there any functor $\Psi$ from sHCPs to Lie supergroups which be a quasi-inverse for $\Phi$, so that the two categories be equivalent? And how much explicit such a functor (if any) is?

A first answer to this question was given by Kostant and by Koszul in the real smooth case (see [17] and [18]), with equivalent methods, providing an explicit quasi-inverse for $\Phi$. Later on, Vishnyakova (see [24]) fixed the complex holomorphic case, and her proof works for the real analytic case as well. More recently, this result was increasingly extended to the setup of algebraic supergeometry, i.e. for algebraic supergroups (and corresponding sHCP’s), over fields and then over rings (see [7], [21], [22]). It is worth remarking, though, that all these subsequent results were, in the end, further improvements of the original idea by Koszul (while Kostant’s method was a slight variation of that): indeed, Koszul defines a Lie supergroup out of a sHCP $(K_+, \mathfrak{t})$ as a super-ringed space, just defining the “proper” sheaf of (commutative) superalgebras onto $K_+$ by means of $\mathfrak{t}$; the authors of the successive, above mentioned papers just re-worked this same recipe.
In this paper I present a new method to solve that problem, i.e. I provide a different, more concrete functor $\Psi$ from sHCP’s to Lie supergroups that is quasi-inverse to $\Phi$. The starting idea is to follow a different approach to supergeometry, à la Grothendieck: namely instead of thinking of supermanifolds as being super-ringed manifolds (i.e. classical manifolds endowed with a sheaf of commutative superalgebras), one studies (or directly defines) them through their “functor of points”. Thus, if $M$ is a supermanifold, then for each commutative superalgebra $A$ one has the manifold $M(A)$ of $A$–points of $M$; in fact, in order to recover the full supermanifold $M$ one can restrict this functor to a smaller category, namely that of Weil superalgebras — roughly, those which are direct sum of a copy of our ground field plus a finite-dimensional nilpotent ideal. Conversely, functors from Weil superalgebras to manifolds enjoying some additional properties do correspond to Lie supergroups (i.e., they are the functor of points of some Lie supergroup): so one can directly call “Lie supergroup” any such special functor. This functorial point of view allows to unify several different approaches to supergeometry (see [2]) and also to treat infinite-dimensional supermanifolds (see [1]). For a broader discussion of the interplay between different approaches to supergeometry we refer to classical sources as [3], [8], [19], [25] or more recent ones like [2], [4], [6], [23].

Now, if we want a functor $\Psi$ from sHCP’s to Lie supergroups, we need a Lie supergroup $G_P$ for each sHCP $\mathcal{P}$; using the functorial point of view, in order to have $G_P$ as a functor we need a Lie group $G_P(A)$ for each Weil superalgebra $A$, and their definition must be natural in $A$: moreover, one still has to show that the resulting functor have those additional properties that make it into a Lie supergroup. Finally, all this should aim to find a $\Psi$ that is quasi-inverse to $\Phi$ — and this fixes ultimate bounds to the construction we aim to.

Bearing all this in mind, the construction that I present goes as follows. Given a super Harish-Chandra pair $\mathcal{P} = (G_+, g)$, for each Weil superalgebra, say $A$, I define a group $G_P(A)$ abstractly, by generators and relations: this definition is uniform and natural with respect to $A$, hence it yields a functor from Weil algebras to (abstract) groups, call it $G_P$. As key step in the work, one proves that $G_P$ admits a “global splitting”, i.e. it is the direct product of $G_+$ times a totally odd affine superspace (isomorphic to $g_1$, the odd part of $g$): as both these are supermanifolds, it turns out that $G_P$ itself is a supermanifold as well, and in fact it is a Lie supergroup because (as a functor) it is group-valued too. One more step proves that the construction of $G_P$ is natural in $\mathcal{P}$, so it yields a functor $\Psi$ from sHCP’s to Lie supergroups: this is our candidate to be a quasi-inverse to $\Phi$.

It is immediate to check that $\Phi \circ \Psi$ is isomorphic to the identity functor onto sHCP’s; on the other hand, proving that $\Psi \circ \Phi$ is isomorphic to the identity functor onto Lie supergroups is much more demanding. In fact, to get the latter we need to know that every Lie supergroup $G$ has a “global splitting” on its own: this implies that $G$ and $\Psi(\Phi(G))$ share the same structure, in that both are the direct product of $G_0$ and $g_1 = (\text{Lie}(G))_1$. Now, the fact that a “Global Splitting Theorem” for Lie supergroups does hold true is (more or less) known among specialists; however, we need it stated in a genuine geometrical form, while it is usually given in sheaf-theoretic terms, so in the end we work it out explicitly. In fact, we find two different formulations of such a result: this is why, building upon them, we can provide two versions, $\Psi^o$ and $\Psi^e$, of a functor $\Psi$ as required.
Two last words are still in order:

(a) The recipe given here — for $\Psi^\circ$ — was originally presented in [15] to solve the same problem in the context of algebraic supergeometry; adapting this idea to the differential setup (i.e. to Lie supergroups and their sHCP’s), however, is definitely not straightforward. The second recipe instead — introducing the functor $\Psi^e$ — is entirely original; with some work, it can be adapted to the setup of algebraic supergroups and sHCP’s too.

(b) We deal here with Lie supergroups (and sHCP’s) of finite dimension; nevertheless, our construction of the functor $\Psi$ is perfectly fit for the infinite dimensional case too — still following the functorial approach, as in [1]. This requires extra technicalities which go beyond our scopes, so we do not fulfill that task; however, the core strategy to follow is already displayed hereafter.

Finally, the paper is organized as follows. In section 2, I fix language and notations. Section 3 introduces the notion of super Harish-Chandra pair and the natural functor $\Phi$ from Lie supergroups to super Harish-Chandra pairs. Section 4 presents structure results for Lie supergroups, in particular about “global splittings”: more or less, these results are known (or should be known), but I could not find them in literature (in the form I need them), so I wrote them down myself.

The core of the paper is in sections 5 and 6. In section 5, I introduce two definitions of functor $\Psi$, namely $\Psi^\circ$ and $\Psi^e$, and I prove key structure results for the Lie supergroups $G_\rho := \Psi(\mathcal{P})$ — with $\Psi \in \{ \Psi^\circ, \Psi^e \}$; in fact, in both cases the very definition of $G_\rho$ and the results about its structure are “prescribed” by the structure results of section 4 for Lie supergroups in general. In section 6 then I prove, using the structure results of sections 4 and 5 (mainly the “Global Splitting Theorems”), that both versions of functor $\Psi$ are indeed quasi-inverse to $\Phi$, as expected.

Finally, section 7 treats special cases and applications.

Acknowledgements
The author thanks Alexander Alldridge, Claudio Carmeli and Rita Fioresi for many valuable conversations.

References


