INTRODUCTION

In the classification of simple finite-dimensional complex Lie superalgebras — due to Kac (cf. [12]) — a special one-parameter family occurs, whose elements $g_a$ depend on a parameter $a \in \mathbb{C} \setminus \{0, -1\}$. These are “generically non-isomorphic”, and all isomorphisms between them are encoded in a free action of the symmetric group $S_3$ on the family $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$. It was pointed out in [12] that the Cartan matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}$ used to define this Lie superalgebra had already appeared in [18], as a Cartan matrix of a one-parameter family of 16-dimensional simple Lie algebras over a field $k$ of characteristic 2 with $a \in k \setminus \{0, 1\}$.

For any $a \in \{1, -2, -1/2\}$ one has $g_a \cong \mathfrak{osp}(4, 2)$, which is of type $D(2, 1)$: thus Kac called each $g_a$ to be “of type $D(2, 1; a)$” — while $D(m, n)$ is the type of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m, 2n)$. For the same reason, some authors, for example [3] — cf. also [2] — use instead notation $\mathfrak{osp}(4, 2; a)$. By general theory, one can complete each of the (simple) Lie superalgebras $g_a$ and form a so-called super Harish-Chandra pair: and then one associates to the latter a corresponding complex Lie supergroup, say $G_a$, whose tangent Lie superalgebra is $g_a$ — as prescribed in Kac’ classification of simple algebraic supergroups, cf. [11]. All these $G_a$’s form a family $\{G_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$, which bears a free $S_3$-action that induces the $S_3$-action on $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$. The starting point of the present paper is the following question: can we “take the limit” (in some sense) of $g_a$ for $a$ approaching to the “singular values” $a = 0$ and $a = -1$? And if yes, what is the structure of the resulting “limit” Lie superalgebra? Similarly, we raise the same questions for the family of the supergroups $G_a$.

In this article, we show that there are several ways to answer, in the positive, these questions. In fact, we present five possible ways to complete the family of simple Lie superalgebras $D(2, 1; a)$ with additional Lie superalgebras for the “singular values” $a \in \{0, -1\}$. Each one of these new, extra objects can be thought of as a “limit” of the older ones; however, the existence of different options show that such “limits” have no intrinsic meaning, but strongly depend on some choice — roughly, on “how you approach the singular point”. For each of these choices, the corresponding new objects that are
“limits” of the (original) simple Lie superalgebras $D(2, 1; a)$ happen to be non-simple, and we describe explicitly their structure, which is different for the different choices. Therefore, we extend the old family $\{g_a = \mathfrak{osp}(4, 2; a)\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras to five larger families, indexed by the points of $\mathbb{P}^1(\mathbb{C}) \cup \{\ast\}$, whose elements at “non-singular values” $a \in \{0, -1, \infty, \ast\}$ are non-simple — which is why we call them “degenerations” — and (when comparing one family with a different one) non-isomorphic.

By the way, our analysis is by no means exhaustive: one can still provide further ways to complete the family of the simple Lie superalgebras right in the same spirit but with different outcomes. Our goal here is only to explain the existence and non-uniqueness of such constructions. A few words about our construction. First, instead of working with Lie superalgebras $g_a$ indexed by a single parameter $a \in \mathbb{C} \setminus \{0, -1\}$ — later extended to $a \in \mathbb{C}$ — we rather deal with a multiparameter $\boldsymbol{\sigma} \in V := \{(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3 \mid \sum_i \sigma_i = 0\}$. The starting point is a construction — due to Kaplansky, cf. [13]; see also [15] — that for each $\boldsymbol{\sigma} \in V$ provides a Lie superalgebra $g_{\boldsymbol{\sigma}}$: this yields a full family of Lie superalgebras $\{g_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in V}$, forming a bundle over $V$, naturally endowed with an action of the group $G := \mathbb{C}^\times \times \mathfrak{S}_3$ via Lie superalgebra isomorphisms. For each $\boldsymbol{\sigma}$ in the “general locus” $V^\times := V \setminus (\bigcup_{i=1}^3 \{\sigma_i = 0\})$ we have $g_{\boldsymbol{\sigma}} \cong g_a$ for some $a \in \mathbb{C} \setminus \{0, -1\}$ so the original family $\{g_a = \mathfrak{osp}(4, 2; a)\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras is taken into account; in addition, the $g_{\boldsymbol{\sigma}}$’s are well-defined also at singular values $\boldsymbol{\sigma} \in V \cap (\bigcup_{i=1}^3 \{\sigma_i = 0\})$, but there they are non-simple instead.

Thus Kaplansky’s family of Lie superalgebras provides a first solution to our problem. In addition, we re-visit this construction and devise five recipes to construct similar families, as follows. For $\boldsymbol{\sigma} \in V^\times$, we fix in $g_{\boldsymbol{\sigma}}$ a particular $\mathbb{C}$-basis, call it $B$, in such a way that the structure constants are polynomials in $\boldsymbol{\sigma}$. When we replace $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with a formal parameter $x = (x_1, x_2, x_3)$, the previous multiplication table defines a Lie superalgebra structure on the free $\mathbb{C}[x]$-module with basis $B$, denoted by $g_B(x)$. Then for each $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in V$ the quotient $g_B(\boldsymbol{\sigma}) := g_B(x)/\langle x_i - \sigma_i \rangle_{i=1,2,3}g_B(x)$ is a well-defined complex Lie superalgebra, such that $g_B(\boldsymbol{\sigma}) \cong g_{\boldsymbol{\sigma}}$ for $\boldsymbol{\sigma} \in V^\times$; thus we get a whole family $\{g_B(\boldsymbol{\sigma})\}_{\boldsymbol{\sigma} \in V}$ as requested, that actually depends on the choice of the basis $B$. We present five explicit examples that give rise to five different outcomes — one being Kaplansky’s family. Indeed, at each point of the “singular locus” $V \cap (\bigcup_{i=1}^3 \{\sigma_i = 0\})$ these families present different (non-isomorphic) non-simple Lie superalgebras, that we describe in detail. As a second contribution, we perform a parallel construction at the level of Lie supergroups: namely, for each $\boldsymbol{\sigma} \in V$ we “complete” each Lie superalgebra $g_B(\boldsymbol{\sigma})$ to form a super Harish-Chandra pair, and then take the corresponding (complex holomorphic) Lie supergroup. This yields a family $\{G_B(\boldsymbol{\sigma})\}_{\boldsymbol{\sigma} \in V}$ of Lie supergroups, with $G_{\boldsymbol{\sigma}}$ isomorphic to $G_a$ for a suitable $a \in \mathbb{C} \setminus \{0, -1\}$ for non-singular values of $\boldsymbol{\sigma}$, while $G_B(\boldsymbol{\sigma})$ is not simple for singular values instead; moreover, the group $G := \mathbb{C}^\times \times \mathfrak{S}_3$ freely acts on this family via Lie supergroup isomorphisms. In other words, we complete the “old” family of the simple Lie supergroups $G_a$’s (isomorphic to suitable $G_{\boldsymbol{\sigma}}$’s) by suitably adding new, non-simple Lie supergroups at singular values of $\boldsymbol{\sigma}$. The construction depends on $B$, and with our five, previously fixed choices we find five different families: for each
of them, we describe explicitly the non-simple supergroups $G_\sigma$ at singular values of $\sigma$ — which are referred to as “degenerations” of the (previously known, simple) $G_a$’s.

This analysis might be reformulated in the language of deformation theory of supermanifolds — e.g., as treated in [17]. However, this goes beyond the scope of the present article. This article is organized as follows. In Section 2, we briefly recall the basic algebraic background necessary for this work, in particular, some language about supermathematics. In Section 3, we introduce our Lie superalgebras $g_\sigma = \mathfrak{osp}(4, 2; \sigma)$. Several integral forms of the Lie superalgebra $g_\sigma$ are introduced in Section 4. In particular, as an application, the structure of their singular degenerations is studied in detail (Theorems 4.1, 4.2, 4.3, 4.4 and 4.5). Section 5 is the last highlight of this paper: we introduce and analyze the Lie supergroups whose Lie superalgebras are studied in Section 4, and we describe the (non-simple) structure of their degenerations — i.e., the member of the families at singular values of $\sigma$ (Theorems 5.1, 5.2, 5.3, 5.4 and 5.5).

As the main objects treated in this article have many special features, most of the above descriptions are given in a down-to-earth manner, so that even the readers who are not familiar with the subject could follow easily our exposition.

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**References**


