"Quantum Groups and Noncommutative Geometry. Perspectives on Quantum Geometry",
pp. 80–95, M. Marcolli, D. Parashar (eds.), Aspects of Mathematics **E41**,
Vieweg+Teubner, Wiesbaden, 2011 — **DOI:** 10.1007/978-3-8348-9831-9_4
The original publication is available at www.springerlink.com/content/t64184631908j166/

Quantum Duality Principle for Quantum Grassmannians

R. Fioresi[♭,1], F. Gavarini[♯]

[♭] *Dipartimento di Matematica, Università di Bologna*
piazza di Porta S. Donato, 5 — I-40127 Bologna, ITALY
e-mail: fioresi@dm.unibo.it

[♯] *Dipartimento di Matematica, Università di Roma "Tor Vergata"*
via della ricerca scientifica 1 — I-00133 Roma, ITALY
e-mail: gavarini@mat.uniroma2.it

Keywords: *Quantum Grassmann Varieties.*
2000 *MSC:* Primary 20G42, 14M15; Secondary 17B37, 17B62.

1. Introduction

In the theory of quantum groups, the geometrical objects that one takes into consideration are affine algebraic Poisson groups and their infinitesimal counterparts, namely Lie bialgebras. By "quantization" of either of these, one means a suitable one-parameter deformation of one of the Hopf algebras associated with them. They are respectively the algebra of regular function $\mathcal{O}(G)$, for a Poisson group G, and the universal enveloping algebra $U(\mathfrak{g})$, for a Lie bialgebra $\mathfrak{g}$. Deformations of $\mathcal{O}(G)$ are called *quantum function algebras* (QFA), and are often denoted with $\mathcal{O}_q(G)$, while deformations of $U(\mathfrak{g})$ are called *quantum universal enveloping algebras* (QUEA), denoted with $U_q(\mathfrak{g})$.

The quantum duality principle (QDP), after its formulation in [**9, 10, 11**], provides a recipe to get a QFA out of a QUEA, and vice-versa. This involves a change of the underlying geometric object, according to Poisson duality, in the following sense. Starting from a QUEA over a Lie bialgebra $\mathfrak{g} = Lie(G)$, one gets a QFA for a dual Poisson group G^*. Starting instead from a QFA over a Poisson group G, one gets a QUEA over the dual Lie bialgebra $\mathfrak{g}^*$.

[1]Partially supported by the University of Bologna, funds for selected research topics.

In [**3**], this principle is extended to the wider context of homogeneous Poisson G–spaces. One describes these spaces, in global or in infinitesimal terms, using suitable subsets of $\mathcal{O}(G)$ or of $U(\mathfrak{g})$. Indeed, each homogeneous G–space M can be realized as G/K for some closed subgroup K of G (this amounts to fixing a point in M: it is shown in [**3**], §1.2, how to select such a point). Thus we can deal with either the space or the subgroup. Now, K can be coded in infinitesimal terms by $U(\mathfrak{k})$, where $\mathfrak{k} := \mathit{Lie}(K)$, and in global terms by $\mathcal{I}(K) := \big\{ \varphi \in \mathcal{O}(G) \,\big|\, \varphi(K) = 0 \big\}$, the defining ideal of K. Instead, G/K can be encoded infinitesimally by $U(\mathfrak{g})\,\mathfrak{k}$ and globally by $\mathcal{O}\big(G/K\big) \equiv \mathcal{O}(G)^K$, the algebra of K–invariants in $\mathcal{O}(G)$. Note that $U(\mathfrak{g})\big/U(\mathfrak{g})\,\mathfrak{k}$ identifies with the set of left-invariant differential operators on G/K, or the set of K–invariant, left-invariant differential operators on G.

These constructions *all* make sense in formal geometry, i.e. when dealing simply with formal groups and formal homogeneous spaces, as in [**3**]. Instead, if one looks for *global* geometry, then one construction might fail, namely the description of G/K via its function algebra $\mathcal{O}\big(G/K\big) = \mathcal{O}(G)^K$. In fact, this makes sense — i.e., $\mathcal{O}(G)^K$ is enough to describe G/K — if and only if the variety G/K is *quasi-affine*. In particular, this is not the case if G/K is projective, like, for instance, when G/K is a Grassmann variety.

By "quantization" of the homogeneous space G/K one means any quantum deformation (in suitable sense) of any one of the four algebraic objects mentioned before which describe either G/K or K. Moreover one requires that given an infinitesimal or a global quantization for the group G, denoted by $U_q(\mathfrak{g})$ or $\mathcal{O}_q(G)$ respectively, the quantization of the homogeneous space admits a $U_q(\mathfrak{g})$–action or a $\mathcal{O}_q(G)$–coaction respectively, which yields a quantum deformation of the algebraic counterpart of the G–action on G/K.

The QDP for homogeneous G–spaces (cf. [**3**]) starts from an infinitesimal (global) quantization of a G–space, say G/K, and provides a global (infinitesimal) quantization for the Poisson dual G^*–space. The latter is $G^*/K^\perp$ (with $\mathit{Lie}\big(K^\perp\big) = \mathfrak{k}^\perp$, the orthogonal subspace — with respect to the natural pairing between $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$ — to $\mathfrak{k}$ inside $\mathfrak{g}^*$). In particular, the principle gives a concrete recipe

$$\mathcal{O}_q\big(G/K\big) \;\;\circ\!\!-\!-\!-\!-\!\rightsquigarrow\;\; \mathcal{O}_q\big(G/K\big)^\vee =: U_q\big(\mathfrak{k}^\perp\big)$$

in which the right-hand side is a quantization of $U\big(\mathfrak{k}^\perp\big)$.

However, this recipe makes no sense when $\mathcal{O}_q\big(G/K\big)$ is not available. In the non-formal setting this is the case whenever G/K is not quasi-affine, e.g. when it is projective.

In this paper we show how to solve this problem in the special case of the Grassmann varieties, taking G as the general linear group and $K = P$ a maximal parabolic subgroup. We adapt the basic ideas of the original QDP recipe to these new ingredients, and we obtain a new recipe

$$\mathcal{O}_q\big(G/P\big) \;\circ\!\!-\!-\!-\!-\!\rightsquigarrow\; \widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$$

which perfectly makes sense, and yields the same kind of result as predicted by the QDP for the quasi-affine case. In particular, $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$ is a quantization of $U\big(\mathfrak{p}^\perp\big)$, obtained through a $(q-1)$–adic completion process.

Our construction goes as follows.

First, we consider the embedding of the Grassmannian G/P (where $G := GL_n$ or $G := SL_n$, and P is a parabolic subgroup of G) inside a projective space, given by Plücker coordinates. This will give us the first new ingredient:

$$\mathcal{O}\big(G/P\big) \;:=\; \text{ring of homogeneous coordinates on } G/P \quad .$$

Many quantizations $\mathcal{O}_q\big(G/P\big)$ of $\mathcal{O}\big(G/P\big)$ already exist in the literature (see, e.g., [**6, 12, 13**]). All these quantizations, which are equivalent, come together with a quantization of the natural G–action on G/P.

In the original recipe (see [**3**]) $\mathcal{O}_q\big(G/K\big) \circ\!\!-\!-\!-\!-\!\rightsquigarrow \mathcal{O}_q\big(G/K\big)^{\vee}$ of the QDP (when G/K is quasi affine) we need to look at a neighborhood of the special point eK (where $e \in G$ is the identity), and at a quantization of it. Therefore, we shall replace the projective variety G/P with such an affine neighborhood, namely the big cell of G/P. This amounts to realize the algebra of regular functions on the big cell as a "homogeneous localization" of $\mathcal{O}\big(G/P\big)$, say $\mathcal{O}^{\,loc}\big(G/P\big)$, by inverting a suitable element. We then do the same at the quantum level, via the inversion of a suitable almost central element in $\mathcal{O}_q\big(G/P\big)$ — which lifts the previous one in $\mathcal{O}\big(G/P\big)$. The result is a quantization $\mathcal{O}_q^{\,loc}\big(G/P\big)$ of the coordinate ring of the big cell.

Hence we are able to *define* $\mathcal{O}_q\big(G/P\big)^{\vee} := \mathcal{O}_q^{\,loc}\big(G/P\big)^{\vee}$, where the right-hand side is given by the original QDP recipe applied to the big cell as an affine variety (we can forget any group action at this step). By the very construction, this $\mathcal{O}_q\big(G/P\big)^{\vee}$ should be a quantization of $U\big(\mathfrak{p}^\perp\big)$ (as an algebra). Indeed, we prove that this is the case, so we might think at $\mathcal{O}_q\big(G/P\big)^{\vee}$ as a quantization (of infinitesimal type) of the variety $G^*/P^\perp$. On the other hand, the construction does not ensure that $\mathcal{O}_q\big(G/P\big)^{\vee}$ also admits a quantization of the G^*–action on $G^*/P^\perp$ (just like the big cell is not a G–space). As a last step, we look at $\mathcal{O}_q\widehat{\big(G\big/P\big)}^\vee$, the $(q-1)$–adic completion of $\mathcal{O}_q\big(G\big/P\big)^\vee$. Of course, it is again a quantization of $U\big(\mathfrak{p}^\perp\big)$ (as an algebra). But in addition, it admits a coaction of the $(q-1)$–adic completion of $\mathcal{O}_q(G)^\vee$ — which is a quantization of $U(\mathfrak{g}^*)$. This coaction yields a quantization of the infinitesimal G^*–action on $G^*\big/P^\perp$. Therefore, in a nutshell, $\mathcal{O}_q\widehat{\big(G\big/P\big)}^\vee$ is a quantization of $G^*\big/P^\perp$ *as a homogeneous G^*–space*, in the sense explained above.

Notice that our arguments could be applied to any *projective* homogeneous G–space X, *up to having the initial data to start with*. Namely, one needs an embedding of X inside a projective space, a quantization (compatible with the G–action) of the ring of homogeneous coordinates of X (w.r.t. such an embedding), and a quantization of a suitable open dense affine subset of X. This program is carried out in detail in a separate work (see [**2**]).

Finally, this paper is organized as follows.

In section 2 we fix the notation, and we describe the Manin deformations of the general linear group (as a Poisson group), and of its Lie bialgebra, together with its dual. In section 3 we briefly recall results concerning the constructions of the quantum Grassmannian $\mathcal{O}_q\big(G\big/P\big)$ and its quantum big cell $\mathcal{O}_q^{loc}\big(G\big/P\big)$. These are known results, treated in detail in [**6, 7**]. Finally, in section 4 we extend the original QDP to build $\mathcal{O}_q\big(G\big/P\big)^\vee$, and we show that its $(q-1)$–adic completion is a quantization of the homogeneous G^*–space $G^*\big/P^\perp$ dual to the Grassmannian $G\big/P$.

Acknowledgements

The first author wishes to thank the Dipartimento di Matematica “Tor Vergata”, and in particular Prof. V. Baldoni and Prof. E. Strickland, for the warm hospitality during the period in which this paper was written.

Both authors also thank D. Parashar and M. Marcolli for their kind invitation to the workshop “Quantum Groups and Noncommutative Geometry” held at MPIM in Bonn during August 6–8, 2007.

2. The Poisson Lie group $GL_n(\Bbbk)$ and its quantum deformation

Let $\Bbbk$ be any field of characteristic zero.

In this section we want to recall the construction of a quantum deformation of the Poisson Lie group $GL_n := GL_n(\Bbbk)$. We will also describe explicitly the bialgebra structure of its Lie algebra $\mathfrak{gl}_n :=$

$\mathfrak{gl}_n(\Bbbk)$ in a way that fits our purposes, that is to obtain a quantum duality principle for the Grassmann varieties for GL_n (see §4).

Let $\Bbbk_q = \Bbbk\big[q, q^{-1}\big]$ (where q is an indeterminate), the ring of Laurent polynomials over q, and let $\Bbbk(q)$ be the field of rational functions in q.

DEFINITION 2.1. The *quantum matrix algebra* is defined as

$$\mathcal{O}_q(M_{m\times n}) \;=\; \Bbbk_q\big\langle \{\, x_{ij} \,\}_{1\leq i\leq m}^{1\leq j\leq n} \big\rangle \Big/ I_M$$

where the x_{ij}'s are non commutative indeterminates, and I_M is the two-sided ideal generated by the *Manin relations*

$$\begin{aligned} x_{ij}\, x_{ik} \;&=\; q\, x_{ik}\, x_{ij} \;, \qquad x_{ji}\, x_{ki} \;=\; q\, x_{ki}\, x_{ji} \qquad \forall\;\, j<k \\ x_{ij}\, x_{kl} \;&=\; x_{kl}\, x_{ij} \qquad\qquad\qquad \forall\;\, i<k\,,\; j>l \;\text{ or }\; i>k\,,\; j<l \\ x_{ij}\, x_{kl} \,-\, x_{kl}\, x_{ij} \;&=\; \big(q-q^{-1}\big)\, x_{kj}\, x_{il} \qquad \forall\;\, i<k\,,\; j<l \end{aligned}$$

Warning: sometimes these relations appear with q exchanged with q^{-1}.

For simplicity we will denote $\mathcal{O}_q(M_{n\times n})$ with $\mathcal{O}_q(M_n)$.

There is a coalgebra structure on $\mathcal{O}_q(M_n)$, given by

$$\Delta(x_{ij}) \;=\; \sum_{k=1}^{n} x_{ik}\otimes x_{kj} \quad , \qquad \epsilon(x_{ij}) \;=\; \delta_{ij} \qquad\qquad (\,1\leq i\,, j\leq n\,)$$

The *quantum general linear group* and the *quantum special linear group* are defined in the following way:

$$\mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)[T]\Big/\big(TD_q-1\,, 1-TD_q\big)$$

$$\mathcal{O}_q(SL_n) := \mathcal{O}_q(M_n)\Big/\big(D_q-1\big)$$

where $D_q \;:=\; \sum_{\sigma\in\mathcal{S}_n} (-q)^{\ell(\sigma)}\, x_{1\,\sigma(1)}\cdots x_{n\,\sigma(n)}$ is a central element, called the *quantum determinant*.

Note: We use the same letter to denote the generators x_{ij} of $\mathcal{O}_q(M_{m\times n})$, of $\mathcal{O}_q(GL_n)$ and of $\mathcal{O}_q(SL_n)$: the context will make clear where they sit.

The algebra $\mathcal{O}_q(GL_n)$ is a quantization of the algebra $\mathcal{O}(GL_n)$ of regular functions on the affine algebraic group GL_n, in the following sense: $\mathcal{O}_q(GL_n)\big/(q-1)\,\mathcal{O}_q(GL_n)$ is isomorphic to $\mathcal{O}(GL_n)$ as a Hopf algebra (over the field $\Bbbk$). Similarly, $\mathcal{O}_q(SL_n)$ is a quantization of the algebra $\mathcal{O}(SL_n)$ of regular functions on SL_n. Both $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are Hopf algebras, that is, they also have the antipode. For more details on these constructions see for example [**1**], pg. 215.

By general theory, $\mathcal{O}(GL_n)$ inherits from $\mathcal{O}_q(GL_n)$ a Poisson bracket, which makes it into a Poisson Hopf algebra, so that GL_n becomes a

Poisson group. We want to describe now its Poisson bracket. Recall that

$$\mathcal{O}(GL_n) \;=\; \Bbbk\big[\{\,\bar{x}_{ij}\,\}_{i,j=1,\dots,n}\big][t]\Big/\big(t\,d-1\big)$$

where $d := det\big(\bar{x}_{i,j}\big)_{i,j=1,\dots,n}$ is the usual determinant. Setting $\bar{x} = \pi(x)$ for $\pi : \mathcal{O}_q(GL_n) \longrightarrow \mathcal{O}(GL_n)$, the Poisson structure is given (as usual) by

$$\big\{\bar{a}\,,\bar{b}\big\} \;:=\; (q-1)^{-1}\,(a\,b-b\,a)\Big|_{q=1} \qquad \forall\;\bar{a}\,,\bar{b}\in\mathcal{O}(GL_n)\;\;.$$

In terms of generators, we have

$$\big\{\bar{x}_{ij}\,,\bar{x}_{ik}\big\} \;=\; \bar{x}_{ij}\,\bar{x}_{ik} \quad \forall\;\; j<k\;, \qquad \big\{\bar{x}_{ij}\,,\bar{x}_{\ell k}\big\} \;=\; 0 \qquad \forall\;\; i<\ell\,,k<j$$

$$\big\{\bar{x}_{ij}\,,\bar{x}_{\ell j}\big\} \;=\; \bar{x}_{ij}\,\bar{x}_{\ell j} \quad \forall\;\; i<\ell\,, \qquad \big\{\bar{x}_{ij}\,,\bar{x}_{\ell k}\big\} \;=\; 2\,\bar{x}_{ij}\,\bar{x}_{\ell k} \quad \forall\;\; i<\ell\,,j<k$$

$$\big\{d^{-1},\bar{x}_{ij}\big\} \;=\; 0\;, \qquad \big\{d\,,\bar{x}_{ij}\big\} \;=\; 0 \qquad \forall\;\; i,j=1,\dots,n\,.$$

As GL_n is a Poisson Lie group, its Lie algebra $\mathfrak{gl}_n$ has a Lie bialgebra structure (see [**1**], pg. 24). To describe it, let us denote with E_{ij} the elementary matrices, which form a basis of $\mathfrak{gl}_n$. Define ($\forall\; i=1,\dots,n{-}1$, $j=1,\dots,n$)

$$e_i := E_{i,i+1}\;, \quad g_j := E_{j,j}\;, \quad f_i := E_{i+1,i}\;, \quad h_i := g_i - g_{i+1}$$

Then $\big\{\,e_i\,,\,f_i\,,\,g_j\;\big|\;i=1,\dots,n-1,\;j=1,\dots,n\,\big\}$ is a set of Lie algebra generators of $\mathfrak{gl}_n$, and a Lie cobracket is defined on $\mathfrak{gl}_n$ by

$$\delta(e_i) \;=\; h_i\wedge e_i \;\;, \quad \delta(g_j) \;=\; 0 \;\;, \quad \delta(f_i) \;=\; h_i\wedge f_i \qquad\qquad \forall\;\; i,j.$$

This cobracket makes $\mathfrak{gl}_n$ itself into a *Lie bialgebra*: this is the so-called *standard* Lie bialgebra structure on $\mathfrak{gl}_n$. It follows immediately that $U(\mathfrak{gl}_n)$ is a co-Poisson Hopf algebra, whose co-Poisson bracket is the (unique) extension of the Lie cobracket of $\mathfrak{gl}_n$ while the Hopf structure is the standard one.

Similar constructions hold for the group SL_n. One simply drops the generator d^{-1}, imposes the relation $d=1$, in the description of $\mathcal{O}(SL_n)$, and replaces the g_s's with the h_i's ($i=1,\dots,n$) when describing $\mathfrak{sl}_n$.

Since $\mathfrak{gl}_n$ is a Lie bialgebra, its dual space $\mathfrak{gl}_n^*$ admits a Lie bialgebra structure, dual to the one of $\mathfrak{gl}_n$. Let $\big\{\,\mathrm{E}_{ij} := E_{ij}^*\;\big|\;i,j=1,\dots,n\,\big\}$ be the basis of $\mathfrak{gl}_n^*$ dual to the basis of elementary matrices for $\mathfrak{gl}_n$. As a Lie algebra, $\mathfrak{gl}_n^*$ can be realized as the subset of $\mathfrak{gl}_n\oplus\mathfrak{gl}_n$ of all pairs

$$\left(\begin{pmatrix} -m_{11} & 0 & \cdots & 0 \\ m_{21} & -m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & 0 \\ m_{n,1} & m_{n,2} & \cdots & -m_{n,n} \end{pmatrix}, \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} & m_{1,n} \\ 0 & m_{22} & \cdots & m_{2,n-1} & m_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ 0 & 0 & \cdots & 0 & m_{n,n} \end{pmatrix}\right)$$

with its natural structure of Lie subalgebra of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$. In fact, the elements E_{ij} correspond to elements in $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ in the following way:

$\mathrm{E}_{ij} \cong \big(E_{ij}\,,0\big)\;\forall\, i>j\,,\;\; \mathrm{E}_{ij} \cong \big(-E_{ij}\,,+E_{ij}\big)\;\forall\, i=j\,,\;\; \mathrm{E}_{ij} \cong \big(0\,,E_{ij}\big)\;\forall\, i<j$

Then the Lie bracket of $\mathfrak{gl}_n^*$ is given by

$$\big[\mathrm{E}_{i,j}\,,\,\mathrm{E}_{h,k}\big] = \delta_{j,h}\,\mathrm{E}_{i,k} - \delta_{k,i}\,\mathrm{E}_{h,j}\;\;,\quad \forall\;\, i\leq j\,,\, h\leq k \;\;\text{ and }\;\; \forall\;\, i>j\,,\, h>k$$

$$\big[\mathrm{E}_{i,j}\,,\,\mathrm{E}_{h,k}\big] = \delta_{k,i}\,\mathrm{E}_{h,j} - \delta_{j,h}\,\mathrm{E}_{i,k}\;\;,\quad \forall\;\, i=j\,,\, h>k \;\;\text{ and }\;\; \forall\;\, i>j\,,\, h=k$$

$$\big[\mathrm{E}_{i,j}\,,\,\mathrm{E}_{h,k}\big] = 0\;, \qquad\qquad \forall\;\, i<j\,,\, h>k \;\;\text{ and }\;\; \forall\;\, i>j\,,\, h<k$$

Note that the elements $(\,1 \leq i \leq n{-}1\,,\; 1 \leq j \leq n\,)$

$$\mathrm{e}_i \,=\, e_i^* \,=\, \mathrm{E}_{i,i+1}\;\;,\qquad \mathrm{f}_i \,=\, f_i^* \,=\, \mathrm{E}_{i+1,i}\;\;,\qquad \mathrm{g}_j \,=\, g_j^* \,=\, \mathrm{E}_{jj}$$

are Lie algebra generators of $\mathfrak{gl}_n^*$. In terms of them, the Lie bracket reads

$$\big[\mathrm{e}_i\,,\mathrm{f}_j\big] \,=\, 0\;, \qquad \big[\mathrm{g}_i\,,\mathrm{e}_j\big] \,=\, \delta_{ij}\,\mathrm{e}_i\;, \qquad \big[\mathrm{g}_i\,,\mathrm{f}_j\big] \,=\, \delta_{ij}\,\mathrm{f}_j \qquad \forall\;\, i,j\;.$$

On the other hand, the Lie cobracket structure of $\mathfrak{gl}_n^*$ is given by

$$\delta\big(\mathrm{E}_{i,j}\big) \;=\; \sum_{k=1}^{n} \mathrm{E}_{i,k} \wedge \mathrm{E}_{k,j} \qquad\qquad \forall\;\, i,j=1,\dots,n$$

where $x \wedge y := x \otimes y - y \otimes x$.

Finally, all these formulæ also provide a presentation of $U\big(\mathfrak{gl}_n^*\big)$ as a co-Poisson Hopf algebra.

A similar description holds for $\mathfrak{sl}_n^* = \mathfrak{gl}_n^* \big/ Z\big(\mathfrak{gl}_n^*\big)$, where $Z\big(\mathfrak{gl}_n^*\big)$ is the centre of $\mathfrak{gl}_n^*$, generated by $\mathfrak{l}_n := \mathrm{g}_1 + \cdots + \mathrm{g}_n$. The construction is immediate by looking at the embedding $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n$.

3.The quantum Grassmannian and its big cell

In this section we want to briefly recall the construction of a quantum deformation of the Grassmannian of r–spaces inside an n–dimensional vector space and its big cell, as they appear in [**6, 7**]. The quantum Grassmannian ring will be obtained as a quantum homogeneous space, namely its deformation will come together with a deformation of the natural coaction of the function algebra of the general linear group on it. The deformation will also depend on a specific embedding (the Plücker one) of the Grassmann variety into a projective space. This deformation is very natural, in fact it embeds into the deformation of its big cell ring. Let's see explicitly these constructions.

Let $G := GL_n$, and let P and P_1 be the standard parabolic subgroups

$$P \,:=\, \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right\} \subset GL_n \quad , \qquad P_1 \,:=\, P \bigcap SL_n$$

where A is a square matrix of size r, with $0 < r < n$.

DEFINITION 3.1. The *quantum Grassmannian coordinate ring* $\mathcal{O}_q\big(G/P\big)$ with respect to the Plücker embedding is the subalgebra of $\mathcal{O}_q(GL_n)$ generated by the quantum minors (called *quantum Plücker coordinates*)

$$D^I \;=\; D^{i_1 \dots i_r} \;:=\; \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)}\, x_{i_1\,\sigma(1)}\, x_{i_2\,\sigma(2)} \cdots x_{i_r\,\sigma(r)}$$

for every ordered r–tuple of indices $I = \{i_1 < \cdots < i_r\}$.

<u>*Remark:*</u> Equivalently, $\mathcal{O}_q\big(G/P\big)$ may be defined in the same way but with $\mathcal{O}_q(SL_n)$ instead of $\mathcal{O}_q(GL_n)$.

The algebra $\mathcal{O}_q\big(G/P\big)$ is a quantization of the Grassmannian G/P in the usual sense: the $\Bbbk$–algebra $\mathcal{O}_q\big(G/P\big)\Big/(q-1)\,\mathcal{O}_q\big(G/P\big)$ is isomorphic to $\mathcal{O}\big(G/P\big)$, the algebra of homogeneous coordinates of G/P with respect to the Plücker embedding. In addition, $\mathcal{O}_q\big(G/P\big)$ has an important property w.r.t. $\mathcal{O}_q(G)$, given by the following result:

PROPOSITION 3.2.

$$\mathcal{O}_q\big(G/P\big) \bigcap \, (q-1)\,\mathcal{O}_q(G) \;=\; (q-1)\,\mathcal{O}_q\big(G/P\big)$$

Proof. By Theorem 3.5 in [**13**], we have that certain products of minors $\{p_i\}_{i \in I}$ form a basis of $\mathcal{O}_q\big(G/P\big)$ over $\Bbbk_q$. Thus, a generic element in $\mathcal{O}_q\big(G/P\big) \bigcap (q-1)\,\mathcal{O}_q(G)$ can be written as

$$\textstyle\sum_{i \in I} \alpha_i\, p_i \;=\; (q-1)\,\phi \tag{3.1}$$

for some $\phi \in \mathcal{O}_q(G)$. Moreover, the specialization map

$$\pi_G \,:\, \mathcal{O}_q(G) \relbar\joinrel\twoheadrightarrow \mathcal{O}_q(G)\Big/(q-1)\,\mathcal{O}_q(G) \;=\; \mathcal{O}(G)$$

maps $\{p_i\}_{i \in I}$ onto a basis $\big\{\pi_G(p_i)\big\}_{i \in I}$ of $\mathcal{O}\big(G/P\big)$, the latter being a subalgebra of $\mathcal{O}(G)$. Therefore, applying π_G to (3.1) we get $\sum_{i \in I} \overline{\alpha_i}\, \pi_G(p_i) \;=\; 0$, where $\overline{\alpha_i} := \alpha_i \mod (q-1)\,\Bbbk_q$, for all $i \in I$. This forces $\alpha_i \in (q-1)\,\Bbbk_q$ for all i, by the linear independence of the $\pi_G(p_i)$'s, whence the claim. $\square$

An immediate consequence of Proposition 3.2 is that the canonical map

$$\mathcal{O}_q\big(G/P\big)\Big/(q-1)\,\mathcal{O}_q\big(G/P\big) \longrightarrow \mathcal{O}_q(G)\Big/(q-1)\,\mathcal{O}_q(G)$$

is *injective*. Therefore, the specialization map

$$\pi_{G/P} \,:\, \mathcal{O}_q\big(G/P\big) \relbar\joinrel\twoheadrightarrow \mathcal{O}_q\big(G/P\big)\Big/(q-1)\,\mathcal{O}_q\big(G/P\big)$$

coincides with the restriction to $\mathcal{O}_q\big(G/P\big)$ of the specialization map

$$\pi_G : \mathcal{O}_q(G) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_q(G)\Big/(q-1)\,\mathcal{O}_q(G) \quad .$$

Moreover — from a geometrical point of view — the key consequence of this property is that P *is a coisotropic subgroup* of the Poisson group G. This implies the existence of a well defined Poisson structure on the algebra $\mathcal{O}\big(G/P\big)$, inherited from the one in $\mathcal{O}(G)$.

Observation 3.3. The quantum deformation $\mathcal{O}_q\big(G/P\big)$ comes naturally equipped with a coaction of $\mathcal{O}_q(GL_n)$ — or, similarly, of $\mathcal{O}_q(SL_n)$ — on it, obtained by restricting the comultiplication Δ. This reads

$$\begin{array}{rccc} \Delta\big|_{\mathcal{O}_q(G/P)} : & \mathcal{O}_q\big(G/P\big) & \longrightarrow & \mathcal{O}_q(G) \otimes \mathcal{O}_q\big(G/P\big) \\ & D^I & \mapsto & \sum_K D^I_K \otimes D^K \end{array}$$

where, for any $I = (i_1 \dots i_r)$, $K = (k_1 \dots k_r)$, with $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq k_1 < \cdots < k_r \leq n$, we denote by D^I_K the *quantum minor*

$$D^I_K \equiv D^{i_1 \dots i_r}_{k_1 \dots k_r} := \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)}\, x_{i_1 k_{\sigma(1)}}\, x_{i_2 k_{\sigma(2)}} \cdots x_{i_r k_{\sigma(r)}} \quad .$$

This provides a quantization of the natural coaction of $\mathcal{O}(G)$ onto $\mathcal{O}\big(G/P\big)$.

The ring $\mathcal{O}_q\big(G/P\big)$ is fully described in [**6**] in terms of generators and relations. We refer the reader to this work for further details.

We now turn to the construction of the quantum big cell ring.

Definition 3.4. Let $I_0 = (1 \dots r)$, $D_0 := D^{I_0}$. Define

$$\mathcal{O}_q(G)\big[D_0^{-1}\big] := \mathcal{O}_q(G)[T]\Big/\big(T\,D_0 - 1\,, D_0\,T - 1\big)$$

Moreover, we define the *big cell ring* $\mathcal{O}_q^{\,loc}\big(G/P\big)$ to be the $\Bbbk_q$–subalgebra of $\mathcal{O}_q(G)\big[D_0^{-1}\big]$ generated by the elements

$$t_{ij} := (-q)^{r-j}\, D^{1 \dots \widehat{j} \dots r\, i}\, D_0^{-1} \qquad \forall \;\; i\,, j \;:\; 1 \leq j \leq r < i \leq n$$

(see [**7**] for more details).

As in the commutative setting, we have the following result:

Proposition 3.5. $\mathcal{O}_q^{\,loc}\big(G/P\big) \cong \mathcal{O}_q\big(G/P\big)\big[D_0^{-1}\big]_{proj}$, *where the right-hand side denotes the degree-zero component of the quotient ring* $\mathcal{O}_q\big(G/P\big)[T]\Big/\big(TD_0 - 1\,, D_0\,T - 1\big)$.

Proof. In the classical setting, the analogous result is proved by this argument: one uses the so-called "straightening relations" to get rid of the extra minors (see, for example, [**4**], §2). Here the argument works essentially the same, using the *quantum straightening* (or *Plücker*) *relations* (see [**6**], §4, [**13**], formula (3.2)(c) and Note I, Note II). $\square$

Remark 3.6. As before, we have that

$$\mathcal{O}_q^{loc}\big(G/P\big) \bigcap (q-1)\,\mathcal{O}_q^{loc}(G) \;=\; (q-1)\,\mathcal{O}_q^{loc}\big(G/P\big)$$

This can be easily deduced from Proposition 3.2, taking into account Proposition 3.5. As a consequence, the map

$$\mathcal{O}_q^{loc}\big(G/P\big)\Big/(q-1)\,\mathcal{O}_q^{loc}\big(G/P\big) \longrightarrow \mathcal{O}_q^{loc}(G)\Big/(q-1)\,\mathcal{O}_q^{loc}(G)$$

is *injective*, so that the specialization map

$$\pi_{G/P}^{loc}\colon\; \mathcal{O}_q^{loc}\big(G/P\big) \twoheadrightarrow \mathcal{O}_q^{loc}\big(G/P\big)\Big/(q-1)\,\mathcal{O}_q^{loc}\big(G/P\big)$$

coincides with the restriction of the specialization map

$$\pi_G^{loc} \,:\; \mathcal{O}_q^{loc}(G) \twoheadrightarrow \mathcal{O}_q^{loc}(G)\Big/(q-1)\,\mathcal{O}_q^{loc}(G) \quad .$$

The following proposition gives a description of the algebra $\mathcal{O}_q^{loc}\big(G/P\big)$:

Proposition 3.7. *The big cell ring is isomorphic to a matrix algebra, via the map*

$$\begin{array}{ccc} \mathcal{O}_q^{loc}\big(G/P\big) & \longrightarrow & \mathcal{O}_q\big(M_{(n-r)\times r}\big) \\ t_{ij} & \mapsto & x_{ij} \end{array} \qquad \forall\ 1 \leq j \leq r < i \leq n$$

In particular, the generators t_{ij}'s satisfy the Manin relations.

Proof. See [**7**], Proposition 1.9. $\square$

4. The Quantum Duality Principle for quantum Grassmannians

The quantum duality principle (QDP), originally due to Drinfeld [**5**] and later formalized in [**9**] and extended in [**10, 11**] by Gavarini, is a functorial recipe to obtain a quantum group starting from a given one. The main ingredients are the "Drinfeld functors", which are equivalences between the category of QFA's and the category of QUEA's. Ciccoli and Gavarini extended this principle to the setting of homogeneous spaces. More precisely, in [**3**] they developed the QDP for homogeneous spaces in the *local setting*, i.e. for quantum groups of formal type (where topological Hopf algebras are taken into account). If one tries to find a global version of the QDP for non quasi-affine homogeneous spaces, then problems arise from the very beginning, as explained in §1. The case of *projective* homogeneous spaces has been solved in [**2**], where the original version of the Drinfeld-like functor for which the (global) QDP recipe should fail is suitably modified.

In this section, we apply the general recipe for projective homogeneous spaces to the Grassmannian G/P. The result is a quantization of the homogeneous space *dual* (in the sense of Poisson duality, see [**3**]) to G/P, just as the QDP recipe predicts in the setting of [**3**].

We begin recalling the Drinfeld functor ${}^\vee : QFA \longrightarrow QUEA$.

Definition 4.1. Let G be an affine algebraic group over $\Bbbk$, and $\mathcal{O}_q(G)$ a quantization of its function algebra. Let J be the augmentation ideal of $\mathcal{O}_q(G)$, i.e. the kernel of the counit $\epsilon : \mathcal{O}_q(G) \longrightarrow \Bbbk$. We define

$$\mathcal{O}_q(G)^\vee := \big\langle (q-1)^{-1} J \big\rangle = \sum_{n=0}^{\infty} (q-1)^{-n} J^n \quad \big(\subset \mathcal{O}_q(G) \otimes_{\Bbbk_q} \Bbbk(q) \big) .$$

It turns out that $\mathcal{O}_q(G)^\vee$ is a quantization of $U(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual Lie bialgebra to the Lie bialgebra $\mathfrak{g} = Lie(G)$. So $\mathcal{O}_q(G)^\vee$ is a QUEA, and an infinitesimal quantization for any Poisson group G^* dual to G, i.e. such that $Lie\big(G^*\big) \cong \mathfrak{g}^*$ as Lie bialgebras. Moreover, the association $\mathcal{O}_q(G) \mapsto \mathcal{O}_q(G)^\vee$ yields a functor from QFA's to QUEA's (see [**10, 11**] for more details).

Remark 4.2. Let $G = GL_n$. Then $\mathcal{O}_q(G)^\vee$ is generated, as a unital subalgebra of $\mathcal{O}_q(G) \otimes_{\Bbbk_q} \Bbbk(q)$, by the elements

$$\mathcal{D}_- := (q-1)^{-1} \big(D_q^{-1} - 1 \big) , \qquad \chi_{ij} := (q-1)^{-1} \big(x_{ij} - \delta_{ij} \big) \qquad \forall\, i,j = 1,\dots,n$$

where the x_{ij}'s are the generators of $\mathcal{O}_q(G)$. As $x_{ij} = \delta_{ij} + (q-1)\,\chi_{ij} \in \mathcal{O}_q(G)^\vee$, we have an obvious embedding of $\mathcal{O}_q(G)$ into $\mathcal{O}_q(G)^\vee$.

In the same spirit — mimicking the construction in [**3**] — we now want to define $\mathcal{O}_q\big(G/P\big)^\vee$ when G/P is the Grassmannian.

Let $G = GL_n$, and let P be the maximal parabolic subgroup of §3.

Definition 4.3. Let ϵ' be the natural extension to $\mathcal{O}_q^{\,loc}(G/P)$ of the restriction to $\mathcal{O}_q(G/P)$ of the counit of $\mathcal{O}_q(G)$, and let $J_{G/P}^{\,loc} := Ker(\epsilon')$. We define (as a subset of $\mathcal{O}_q^{\,loc}\big(G/P\big) \otimes_{\Bbbk_q} \Bbbk(q)$)

$$\mathcal{O}_q\big(G/P\big)^\vee := \big\langle\, (q-1)^{-1} J_{G/P}^{\,loc} \,\big\rangle = \sum_{n=0}^{\infty} (q-1)^{-n} \big(J_{G/P}^{\,loc}\big)^n \quad .$$

It is worth pointing out that $\mathcal{O}_q\big(G/P\big)^\vee$ is *not* a "quantum homogeneous space" for $\mathcal{O}_q(G)^\vee$ in any natural way, i.e. it does not admit a coaction of $\mathcal{O}_q(G)^\vee$. This is a consequence of the fact that there is no natural coaction of $\mathcal{O}_q(G)$ on $\mathcal{O}_q^{\,loc}\big(G/P\big)$. Now we examine this more closely.

Since $\mathcal{O}_q\big(G/P\big)^\vee$ is not contained in $\mathcal{O}_q(G)^\vee$, we cannot have a $\mathcal{O}_q(G)^\vee$ coaction induced by the coproduct. This would be the case if $\mathcal{O}_q\big(G/P\big)^\vee$ were a (one-sided) *coideal* of $\mathcal{O}_q(G)^\vee$; but this is not true because $\mathcal{O}_q^{loc}\big(G/P\big)$ is not a (right) coideal of $\mathcal{O}_q(G)$. This reflects the geometrical fact that the big cell of G/P is not a G–space itself. Nevertheless, we shall find a way around this problem simply by *enlarging* $\mathcal{O}_q\big(G/P\big)^\vee$ and $\mathcal{O}_q(G)^\vee$, i.e. by taking their $(q-1)$–adic completion (which will not affect their behavior at $q=1$).

To begin, we provide a concrete description of $\mathcal{O}_q\big(G/P\big)^\vee$:

Proposition 4.4.

$$\mathcal{O}_q\big(G/P\big)^\vee \;=\; \Bbbk_q\big\langle \{\, \mu_{ij} \,\}_{i=r+1,\dots,n}^{j=1,\dots,r} \big\rangle \Big/ I_M$$

where $\mu_{ij} := (q-1)^{-1}\, t_{ij}$ *(for all i and j), I_M is the ideal of the Manin relations among the μ_{ij}'s, and* $t_{ij} = (-q)^{r-j}\, D^{1 \dots \widehat{j} \dots r\, i}\, D_0^{-1}$ *(for all i and j).*

Proof. Trivial from definitions and Proposition 3.7. $\square$

We now explain the relation between $\mathcal{O}_q\big(G/P\big)^\vee$ and $\mathcal{O}_q(G)^\vee$. The starting point is the following special property:

Proposition 4.5.

$$\mathcal{O}_q\big(G/P\big)^\vee \bigcap\, (q-1)\, \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \;=\; (q-1)\, \mathcal{O}_q\big(G/P\big)^\vee$$

Proof. It is the same as for Proposition 3.2. $\square$

Remark 4.6. As a direct consequence of Proposition 4.5, the canonical map

$$\mathcal{O}_q\big(G/P\big)^\vee \Big/ (q-1)\, \mathcal{O}_q\big(G/P\big)^\vee \longrightarrow \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \Big/ (q-1)\, \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big]$$

is in fact *injective*: therefore, the specialization map

$$\pi^\vee_{G/P} \,:\, \mathcal{O}_q\big(G/P\big)^\vee \relbar\joinrel\twoheadrightarrow \mathcal{O}_q\big(G/P\big)^\vee \Big/ (q-1)\, \mathcal{O}_q\big(G/P\big)^\vee$$

coincides with the restriction to $\mathcal{O}_q\big(G/P\big)^\vee$ of the specialization map

$$\pi^\vee_G \,:\, \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \relbar\joinrel\twoheadrightarrow \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \Big/ (q-1)\, \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \quad .$$

From now on, let $\widehat{A}$ denote the $(q-1)$–adic completion of any $\Bbbk_q$–algebra A. Note that $\widehat{A}$ and A have the same specialization at $q=1$, i.e. $A/(q-1)\,A$ and $\widehat{A}/(q-1)\,\widehat{A}$ are canonically isomorphic. When $A = \mathcal{O}_q(G)$, note also that $\widehat{\mathcal{O}_q(G)}$ is naturally a complete topological Hopf $\Bbbk_q$–algebra.

The next result shows why it is relevant to introduce such completions.

Lemma 4.7. *$\mathcal{O}_q(G)^\vee\big[D_0^{-1}\big]$ naturally embeds into $\widehat{\mathcal{O}_q(G)}^\vee$.*

Proof. By remark 4.2 we have that $\mathcal{O}_q(G)^\vee$ is generated by the elements (for all $i,j=1,\dots,n$)

$$\mathcal{D}_- := (q-1)^{-1}\big(D_q^{-1}-1\big)\ , \qquad \chi_{ij} := (q-1)^{-1}\big(x_{ij}-\delta_{ij}\big)$$

inside $\mathcal{O}_q(G)\otimes_{\Bbbk_q}\Bbbk(q)$. On the other hand, observe that

$$x_{ij} = (q-1)\,\chi_{i,j} \in (q-1)\,\mathcal{O}_q(G)^\vee \qquad \forall\ i\neq j$$

and

$$x_{\ell\ell} = 1+(q-1)\,\chi_{\ell\ell} \in \big(1+(q-1)\,\mathcal{O}_q(G)^\vee\big) \qquad \forall\ \ell\ .$$

Then, if we expand explicitly the q–determinant $D_0 := D^{I_0}$, we immediately see that $D_0 \in \big(1+(q-1)\,\mathcal{O}_q(G)^\vee\big)$ as well. Thus D_0 is invertible in $\widehat{\mathcal{O}_q(G)}^\vee$, and so the natural immersion $\mathcal{O}_q(G)^\vee \hookrightarrow \widehat{\mathcal{O}_q(G)}^\vee$ canonically extends to an immersion $\mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \hookrightarrow \widehat{\mathcal{O}_q(G)}^\vee$. $\square$

Corollary 4.8.

(a) The specializations at $q=1$ of $\mathcal{O}_q(G)^\vee$, $\mathcal{O}_q(G)^\vee\big[D_0^{-1}\big]$ and $\widehat{\mathcal{O}_q(G)}^\vee$ are canonically isomorphic. More precisely, the chain

$$\mathcal{O}_q(G)^\vee \hookrightarrow \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \hookrightarrow \widehat{\mathcal{O}_q(G)}^\vee$$

of canonical embeddings induces at $\underline{q=1}$ a chain of isomorphisms.

(b) $\mathcal{O}_q\big(G/P\big)^\vee$ embeds into $\widehat{\mathcal{O}_q(G)}^\vee$ via the chain of embeddings

$$\mathcal{O}_q\big(G/P\big)^\vee \hookrightarrow \mathcal{O}_q(G)^\vee\big[D_0^{-1}\big] \hookrightarrow \widehat{\mathcal{O}_q(G)}^\vee$$

(c) $$\mathcal{O}_q\big(G/P\big)^\vee \bigcap\ (q-1)\,\widehat{\mathcal{O}_q(G)}^\vee = (q-1)\,\mathcal{O}_q\big(G/P\big)^\vee\ .$$

Proof. Part *(a)* and *(b)* are trivial, and *(c)* follows from them. $\square$

Notice that part *(c)* of Corollary 4.8 also implies that

$$\mathcal{O}_q\big(G/P\big)^\vee\Big|_{q=1} := \mathcal{O}_q\big(G/P\big)^\vee\Big/(q-1)\,\mathcal{O}_q\big(G/P\big)^\vee$$

is a subalgebra of

$$\widehat{\mathcal{O}_q(G)}^\vee\Big|_{q=1} = \mathcal{O}_q(G)^\vee\Big|_{q=1} := \mathcal{O}_q(G)^\vee\Big/(q-1)\,\mathcal{O}_q(G)^\vee \cong U(\mathfrak{g}^*)$$

just because the specialization map

$$\pi^\vee_{G/P} :\ \mathcal{O}_q\big(G/P\big)^\vee \twoheadrightarrow \mathcal{O}_q\big(G/P\big)^\vee\Big/(q-1)\,\mathcal{O}_q\big(G/P\big)^\vee$$

coincides with the restriction to $\mathcal{O}_q\big(G/P\big)^\vee$ of the specialization map

$$\widehat{\pi^\vee_G} :\ \widehat{\mathcal{O}_q(G)}^\vee \twoheadrightarrow \widehat{\mathcal{O}_q(G)}^\vee\Big/(q-1)\,\widehat{\mathcal{O}_q(G)}^\vee\ .$$

Now we want to see what is $\mathcal{O}_q\big(G/P\big)^\vee\Big|_{q=1}$ inside $U\big(\mathfrak{gl}_n^{\,*}\big)$. In other words, we want to understand what is the space that $\mathcal{O}_q\big(G/P\big)^\vee$ is quantizing.

Proposition 4.9.

$$\mathcal{O}_q\big(G/P\big)^\vee\Big|_{q=1} \;=\; U\big(\mathfrak{p}^\perp\big)$$

as a subalgebra of $\mathcal{O}_q(G)^\vee\Big|_{q=1} = U\big(\mathfrak{gl}_n^{\,*}\big)$, *where* $\mathfrak{p}^\perp$ *is the orthogonal subspace to* $\mathfrak{p} := \mathit{Lie}\,(P)$ *inside* $\mathfrak{gl}_n^{\,*}$.

Proof. Thanks to the previous discussion, it is enough to show that

$$\pi_G^\vee\Big(\mathcal{O}_q\big(G/P\big)^\vee\Big) \;=\; U\big(\mathfrak{p}^\perp\big) \;\subseteq\; U\big(\mathfrak{gl}_n^{\,*}\big) \;=\; \mathcal{O}_q(G)^\vee\Big|_{q=1} \quad .$$

To do this, we describe the isomorphism $\mathcal{O}_q(G)^\vee\Big|_{q=1} \cong U\big(\mathfrak{gl}_n^{\,*}\big)$ (cf. [**8**]). First, recall that $\mathcal{O}_q(G)^\vee$ is generated by the elements (see Remark 4.2)

$$\mathcal{D}_- \;:=\; (q-1)^{-1}\,\big(D_q^{-1}-1\big)\;, \qquad \chi_{ij} \;:=\; (q-1)^{-1}\,\big(x_{ij}-\delta_{ij}\big)$$

(for all $i,j\,=\,1,\dots,n$) inside $\mathcal{O}_q(G)\otimes_{\Bbbk_q}\Bbbk(q)$. In terms of these generators, the isomorphism reads

$$\begin{gathered} \mathcal{O}_q(G)^\vee\Big|_{q=1} \xrightarrow{\qquad} U\big(\mathfrak{gl}_n^{\,*}\big) \\ \overline{\mathcal{D}_-} \mapsto -(\mathrm{E}_{1,1}+\cdots+\mathrm{E}_{n,n})\;, \qquad \overline{\chi_{i,j}} \mapsto \mathrm{E}_{i,j} \qquad \forall\; i\,,j\;. \end{gathered}$$

where we used notation $\overline{X} := X \mod (q-1)\,\mathcal{O}_q(G)^\vee$. Indeed, from $\overline{\chi_{i,j}} \mapsto \mathrm{E}_{i,j}$ and $(q-1)^{-1}\big(D_q-1\big) \in \mathcal{O}_q(G)^\vee$, one gets $\overline{D_q} \mapsto 1$ and $\overline{(q-1)^{-1}\big(D_q-1\big)} \mapsto \mathrm{E}_{1,1}+\cdots+\mathrm{E}_{n,n}$. Moreover, the relation $D_q\,D_q^{-1} = 1$ in $\mathcal{O}_q(G)$ implies $D_q\,\mathcal{D}_- = -(q-1)^{-1}\big(D_q-1\big)$ in $\mathcal{O}_q(G)^\vee$, so $\overline{\mathcal{D}_-} \mapsto -(\mathrm{E}_{1,1}+\cdots+\mathrm{E}_{n,n})$ as claimed (cf. [**8**], §3, or [**10**], §7). In other words, the specialization $\pi_G^\vee : \mathcal{O}_q(G)^\vee \twoheadrightarrow U\big(\mathfrak{gl}_n^{\,*}\big)$ is given by

$$\pi_G^\vee\big(\mathcal{D}_-\big) \;=\; -(\mathrm{E}_{1,1}+\cdots+\mathrm{E}_{n,n})\;, \qquad \pi_G^\vee\big(\chi_{i,j}\big) \;=\; \mathrm{E}_{i,j} \qquad \forall\; i\,,j\;.$$

If we look at $\widehat{\mathcal{O}_q(G)}^\vee$, things are even simpler. Since

$$D_q \in \Big(1+(q-1)\,\mathcal{O}_q(G)^\vee\Big) \subset \Big(1+(q-1)\,\widehat{\mathcal{O}_q(G)}^\vee\Big)\;,$$

then $D_q^{-1} \in \Big(1+(q-1)\,\widehat{\mathcal{O}_q(G)}^\vee\Big)$, and the generator $\mathcal{D}_-$ can be dropped. The specialization map $\widehat{\pi_{G/P}^\vee}$ of course is still described by formulæ as above.

Now let's compute $\pi^\vee_{G/P}\Big(\mathcal{O}_q\big(G/P\big)^\vee\Big) = \widehat{\pi^\vee_G}\Big(\mathcal{O}_q\big(G/P\big)^\vee\Big)$. Recall that $\mathcal{O}_q\big(G/P\big)^\vee$ is generated by the μ_{ij}'s, with

$$\mu_{ij} \;:=\; (q-1)^{-1}\, t_{ij} \;=\; (q-1)^{-1}\, (-q)^{r-j}\, D^{1\,\dots\,\widehat{j}\,\dots\,r\,i}\, D_0^{-1}$$

for $i = r+1, \dots, n$, and $j = 1, \dots, r$; thus we must compute $\widehat{\pi^\vee_G}\big(\mu_{ij}\big)$.

By definition, for every $i \neq j$ the element $x_{ij} = (q-1)\,\chi_{ij}$ is mapped to 0 by $\widehat{\pi^\vee_G}$. Instead, for each ℓ the element $x_{\ell\,\ell} = 1 + (q-1)\,\chi_{\ell\,\ell}$ is mapped to 1 (by $\widehat{\pi^\vee_G}$ again). But then, expanding the q–determinants one easily finds — much like in the proof of Lemma 4.7 — that

$$\widehat{\pi^\vee_G}\,\Big((q-1)^{-1} D^{1\,\dots\,\widehat{j}\,\dots\,r\,i}\Big) \;=\; \Big((q-1)^{-1} \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)}\, x_{1\,\sigma(1)} \cdots x_{r\,\sigma(r)}\Big) \;=$$
$$=\; \widehat{\pi^\vee_G}\,\Big((q-1)^{-1} \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)} \prod\nolimits_{k=1}^{r} \big(\delta_{k\,\sigma(k)} + (q-1)\,\chi_{k\,\sigma(k)}\big)\Big)$$

The only term in $(q-1)$ in the expansion of $D^{1\,\dots\,\widehat{j}\,\dots\,r\,i}$ comes from the product

$$\big(1+(q-1)\chi_{1\,1}\big) \cdots \big(1+(q-1)\chi_{r\,r}\big)(q-1)\chi_{i\,j} \equiv (q-1)\chi_{i\,j} \mod (q-1)^2 \mathcal{O}\big(G/P\big)$$

Therefore, from the previous analysis we get

$$\widehat{\pi^\vee_G}\,\Big((q-1)^{-1}\, D^{1\,\dots\,\widehat{j}\,\dots\,r\,i}\Big) \;=\; \widehat{\pi^\vee_G}\,\big(\chi_{i,j}\big) \;=\; \mathrm{E}_{i,j}$$
$$\widehat{\pi^\vee_G}\,\big(D_0\big) \;=\; \widehat{\pi^\vee_G}(1) \;=\; 1 \quad , \qquad \widehat{\pi^\vee_G}\,\big(D_0^{-1}\big) \;=\; \widehat{\pi^\vee_G}(1) \;=\; 1$$

so in the end $\widehat{\pi^\vee_G}\big(\mu_{ij}\big) = (-1)^{r-j}\, \mathrm{E}_{i,j}$, for all $1 \leq j \leq r < i \leq n$.

The outcome is that $\pi^\vee_{G/P}\Big(\mathcal{O}_q\big(G/P\big)^\vee\Big) \;=\; U(\mathfrak{h})$, where

$$\mathfrak{h} \;:=\; \mathit{Span}\,\Big(\big\{\, \mathrm{E}_{i,j} \,\big|\, r+1 \leq i \leq n \,,\; 1 \leq j \leq r \,\big\}\Big) \quad .$$

On the other hand, from the very definitions and our description of ${\mathfrak{gl}_n}^*$ one easily finds that $\mathfrak{h} = \mathfrak{p}^\perp$, for $\mathfrak{p} := \mathit{Lie}\,(P)$. The claim follows. $\square$

Proposition 4.9 claims that $\mathcal{O}_q\big(G/P\big)^\vee$ is a quantization of $U\big(\mathfrak{p}^\perp\big)$, i.e. it is a unital $\Bbbk_q$–algebra whose semiclassical limit is $U\big(\mathfrak{p}^\perp\big)$. Now, the fact that $U\big(\mathfrak{p}^\perp\big)$ describes (infinitesimally) a homogeneous space for G^* is encoded in algebraic terms by the fact that it is a (left) coideal of $U(\mathfrak{g}^*)$; in other words, $U\big(\mathfrak{p}^\perp\big)$ is a (left) $U(\mathfrak{g}^*)$–comodule w.r.t. the restriction of the coproduct of $U(\mathfrak{g}^*)$. Thus, for $\mathcal{O}_q\big(G/P\big)^\vee$ to be a quantization of $U\big(\mathfrak{p}^\perp\big)$ *as a homogeneous space* we need also a quantization of this fact: namely, we would like $\mathcal{O}_q\big(G/P\big)^\vee$ to be a left coideal of $\mathcal{O}_q(G)^\vee$, our quantization of $U(\mathfrak{g}^*)$. But this makes no sense at all, as $\mathcal{O}_q\big(G/P\big)^\vee$ is not even a subset of $\mathcal{O}_q(G)^\vee$!

This problem leads us to enlarge a bit our quantizations $\mathcal{O}_q\big(G/P\big)^\vee$ and $\mathcal{O}_q(G)^\vee$: we take their $(q$–$1)$–adic completions, namely $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ and $\widehat{\mathcal{O}_q(G)}^\vee$. While not affecting their behavior at $q=1$ (i.e., their semiclassical limits are the same), this operation solves the problem. Indeed, $\widehat{\mathcal{O}_q(G)}^\vee$ is big enough to contain $\mathcal{O}_q\big(G/P\big)^\vee$, by Corollary 4.8*(b)*. Then, as $\widehat{\mathcal{O}_q(G)}^\vee$ is a topological Hopf algebra, inside it we must look at the closure of $\mathcal{O}_q\big(G/P\big)^\vee$. Thanks to Corollary 4.8*(c)* (which means, roughly, that an Artin-Rees lemma holds), the latter is nothing but $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$. Finally, next result tells us that $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ is a left coideal of $\widehat{\mathcal{O}_q(G)}^\vee$, as expected.

PROPOSITION 4.10. *$\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ is a left coideal of $\widehat{\mathcal{O}_q(G)}^\vee$.*

Proof. Recall that the coproduct $\widehat{\Delta}$ of $\widehat{\mathcal{O}_q(G)}^\vee$ takes values in the *topological* tensor product $\widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \widehat{\mathcal{O}_q(G)}^\vee$, which by definition is the $(q-1)$–adic completion of the *algebraic* tensor product $\widehat{\mathcal{O}_q(G)}^\vee \otimes \widehat{\mathcal{O}_q(G)}^\vee$. Our purpose then is to show that this coproduct $\widehat{\Delta}$ maps $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ in the topological tensor product $\widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$.

By construction, the coproduct of $\mathcal{O}_q(G)^\vee$, hence of $\widehat{\mathcal{O}_q(G)}^\vee$ too, is induced by that of $\mathcal{O}_q(G)$, say $\Delta : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$. Now, the latter can be uniquely (canonically) extended to a coassociative algebra morphism

$$\widetilde{\Delta} : \mathcal{O}_q(G)\big[D_{I_0}^{-1}\big] \longrightarrow \mathcal{O}_q(G)\big[D_{I_0}^{-1}\big] \,\widetilde{\otimes}\, \mathcal{O}_q(G)\big[D_{I_0}^{-1}\big]$$

where $\widetilde{\otimes}$ is the $J_\otimes$–adic completion of the algebraic tensor product, with

$$J_\otimes := J \otimes \mathcal{O}_q(G) + \mathcal{O}_q(G) \otimes J \ , \qquad J := Ker\big(\epsilon_{\mathcal{O}_q(G)}\big) \ .$$

In fact, since $\Delta(D_0) = D_0 \otimes D_0 + \sum\limits_{K \neq I_0} D_K^{I_0} \otimes D^K$, one easily computes

$$\begin{aligned}
\widetilde{\Delta}(D_0) \ &= \ \Big(1 + \sum_{K \neq I_0} D_K^{I_0}\, D_0^{-1} \otimes D^K D_0^{-1}\Big)\big(D_0 \otimes D_0\big) \\
\widetilde{\Delta}\big(D_0^{-1}\big) \ &= \ \big(D_0 \otimes D_0\big)^{-1}\Big(1 + \sum_{K \neq I_0} D_K^{I_0}\, D_0^{-1} \otimes D^K D_0^{-1}\Big)^{-1} \\
&= \Big(D_0^{-1} \otimes D_0^{-1}\Big) \sum_{n \geq 0} (-1)^n \left(\sum_{K \neq I_0} D_K^{I_0}\, D_0^{-1} \otimes D^K D_0^{-1}\right)^n
\end{aligned}$$

Let's now look at the restriction $\widetilde{\Delta}_r$ of $\widetilde{\Delta}$ to $\mathcal{O}_q^{loc}\big(G/P\big)$. We have

$$\widetilde{\Delta}_r(t_{ij}) \ = \ \widetilde{\Delta}_r\big(D^{1 \dots \widehat{j} \dots r\, i}\, D_0^{-1}\big) \ = \ \widetilde{\Delta}\big(D^{1 \dots \widehat{j} \dots r\, i}\big) \cdot \widetilde{\Delta}\big(D_0\big)^{-1} \ =$$

$$= \Big(\sum_L D_L^{1\dots\widehat{j}\dots r\, i} \, D_0^{-1} \otimes D^L \, D_0^{-1} \Big) \cdot \sum_{n \geq 0} (-1)^n \, \Big(\sum_{K \neq I_0} D_K^{I_0} \, D_0^{-1} \otimes D^K D_0^{-1} \Big)^n$$

Now, by Proposition 3.5 we know that each product $D^L \, D_{I_0}^{-1}$ is a combination of the t_{ij}'s. Hence the formula above shows that $\widetilde{\Delta}_r$ maps $\mathcal{O}_q^{loc}\big(G/P\big)$ into $\mathcal{O}_q(G)\big[D_0^{-1}\big] \,\widetilde{\otimes}\, \mathcal{O}_q^{loc}\big(G/P\big)$.

By scalar extension, $\widetilde{\Delta}$ uniquely extends to a map defined on the $\Bbbk(q)$–vector space $\Bbbk(q) \otimes_{\Bbbk_q} \mathcal{O}_q(G)\big[D_0^{-1}\big]$, which we still call $\widetilde{\Delta}$. Its restriction to the similar scalar extension of $\mathcal{O}_q^{loc}\big(G/P\big)$ clearly coincides with the scalar extension of $\widetilde{\Delta}_r$, hence we call it $\widetilde{\Delta}_r$ again. Finally, the restriction of $\widetilde{\Delta}$ to $\mathcal{O}_q(G)^\vee\big[D_0^{-1}\big]$ and of $\widetilde{\Delta}_r$ to $\mathcal{O}_q\big(G/P\big)^\vee$ both coincide — by construction — with the proper restrictions of the coproduct of $\widehat{\mathcal{O}_q(G)}^\vee$ (cf. Corollary 4.8).

In the end, we are left to compute $\widetilde{\Delta}_r(\mu_{ij})$. The computation above gives

$$\widehat{\Delta}(\mu_{ij}) \; = \; \widetilde{\Delta}_r(\mu_{ij}) \; = \; (q-1)^{-1} \, \widetilde{\Delta}_r(t_{ij}) \; =$$
$$= (q-1)^{-1} \sum_L D_L^{1\dots\widehat{j}\dots r\, i} D_0^{-1} \otimes D^L D_0^{-1} \cdot \sum_{n \geq 0} (-1)^n \Big(\sum_{K \neq I_0} D_K^{I_0} D_0^{-1} \otimes D^K D_0^{-1} \Big)^n$$

Now, each left-hand side factor above belongs to $\widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$, because either $D^L \in J_{G/P}^{loc}$ (if $L \neq I_0$, with notation of §4.3), or $D_L^{1\dots\widehat{j}\dots r\, i} \in J$ (if $L = I_0$, with $J := Ker\big(\epsilon_{\mathcal{O}_q(G)}\big)$). On right-hand side instead we have

$$D^K \in J_{G/P}^{loc} \subseteq (q-1)\, \mathcal{O}_q\big(G/P\big)^\vee \quad , \qquad D_K^{I_0} \in J \subseteq (q-1)\, \mathcal{O}_q(G)^\vee$$

whence — as $D_0^{-1} \in \widehat{\mathcal{O}_q(G)}^\vee$ and $D_0^{-1} \in \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ — we get

$$\sum_{K \neq I_0} D_K^{I_0} \, D_0^{-1} \otimes D^K D_0^{-1} \; \in \; (q-1)^2 \, \widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$$

so that $\displaystyle\sum_{n \geq 0} (-1)^n \Big(\sum_{K \neq I_0} D_K^{I_0} \, D_0^{-1} \otimes D^K D_0^{-1} \Big)^n \in \widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$

The final outcome is $\widehat{\Delta}(\mu_{ij}) \in \widehat{\mathcal{O}_q(G)}^\vee \,\widehat{\otimes}\, \mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ for all i and all j . As the μ_{ij}'s topologically generate $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$, this proves the claim. $\square$

In the end, we get the main result of this paper.

Theorem 4.11. $\mathcal{O}_q\widehat{\big(G/P\big)}^\vee$ *is a quantum homogeneous* G^**–space, which is indeed an infinitesimal quantization of the homogeneous* G^**–space* $\mathfrak{p}^\perp$.

Proof. Just collect the previous results. By Proposition 4.9 and by the fact that $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}\Big|_{q=1} = \mathcal{O}_q\big(G/P\big)^{\vee}\Big|_{q=1}$ we have that the specialization of $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$ is $U\big(\mathfrak{p}^{\perp}\big)$. Moreover we saw that $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$ is a subalgebra, and left coideal, of $\widehat{\mathcal{O}_q(G)}^{\vee}$. Finally, we have

$$\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee} \bigcap \, (q-1)\,\widehat{\mathcal{O}_q(G)}^{\vee} \;=\; (q-1)\,\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$$

as an easy consequence of Corollary 4.8 *(c)*. Therefore, $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$ is a quantum homogeneous space, in the usual sense. As $\widehat{\mathcal{O}_q(G)}^{\vee}$ is a quantization of $\mathfrak{g}^*$, we have that $\widehat{\mathcal{O}_q\big(G/P\big)}^{\vee}$ is in fact a quantum homogeneous space for G^*; of course, this is a quantization of *infinitesimal* type. $\square$

Remark 4.12. All these computations can be repeated, step by step, taking $G = SL_n$ and $P = P_1$.

References

[1] V. Chari, A. Pressley, *Quantum Groups*, Cambridge Univ. Press, Cambridge (1994).

[2] N. Ciccoli, R. Fioresi, F. Gavarini, *Quantum Duality Principle for Projective Homogeneous Spaces*, J. Noncommut. Geom. **2** (2008), 449–496.

[3] N. Ciccoli, F. Gavarini, *A quantum duality principle for coisotropic subgroups and Poisson quotients*, Adv. Math. **199** (2006), 104–135.

[4] C. De Concini, D. Eisenbud, C. Procesi, *Young Diagrams and Determinantal Varieties*, Invent. Math. **56** (1980), 129–165.

[5] V. G. Drinfeld, *Quantum groups*, Proc. Intern. Congress of Math. (Berkeley, 1986) (1987), 798–820.

[6] R. Fioresi, *Quantum deformation of the Grassmannian manifold*, J. Algebra **214** (1999), 418–447.

[7] R. Fioresi, *A deformation of the big cell inside the Grassmannian manifold* $G(r,n)$, Rev. Math. Phys. **11** (1999), 25–40.

[8] F. Gavarini, *Quantum function algebras as quantum enveloping algebras*, Comm. Algebra **26** (1998), 1795–1818.

[9] F. Gavarini, *The quantum duality principle*, Ann. Inst. Fourier (Grenoble) **52** (2002), 809–834.

[10] F. Gavarini, *The global quantum duality principle: theory, examples, and applications*, 120 pages, see `http://arxiv.org/abs/math.QA/0303019` (2003).

[11] F. Gavarini, *The global quantum duality principle*, J. Reine Angew. Math. **612** (2007), 17–33.

[12] V. Lakshmibai, N. Reshetikhin, *Quantum flag and Schubert schemes*, Contemp. Math. **134**, Amer. Math. Soc., Providence, RI (1992), 145–181.

[13] E. Taft, J. Towber, *Quantum deformation of flag schemes and Grassmann schemes, I. A q-deformation of the shape-algebra for* GL(n), J. Algebra **142** (1991), 1–36.