Nicola CICCOLI, Fabio GAVARINI

# *"Quantum duality principle for coisotropic subgroups and Poisson quotients"*

in: N. Bokan, M. Djoric, A. T. Fomenko, Z. Rakic, B. Wegner, J. Wess (eds.),
*Contemporary Geometry and Related Topics, Proceedings of the Workshop (Belgrade, June 26–July 2, 2005)*, 2006, pp. 99–118

Published by Faculty of Mathematics, University of Belgrade, Serbia, 2006, pp. 99–118

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http://www.emis.de/proceedings/CGRT2005/Articles/cgrt07.pdf

## INTRODUCTION

In the study of quantum groups, the natural semiclassical counterpart is the theory of deformation (or quantization) of Poisson groups: actually, Drinfeld himself introduced Poisson groups as the semiclassical limits of quantum groups. Therefore, it should be not surprising that the geometry of quantum groups turns more clear and comprehensible when its connection with Poisson geometry is more transparent. The same situation occurs when dealing with Poisson homogeneous spaces of Poisson groups.

In particular, in the study of Poisson homogeneous spaces a special role is played by *Poisson quotients*. By this we mean Poisson homogeneous spaces whose symplectic foliation has at least one zero-dimensional leaf: therefore, they can be seen as pointed Poisson homogeneous spaces, just like Poisson groups themselves are pointed by the identity element.

Poisson quotients form a natural subclass of Poisson homogeneous $G$–spaces ($G$ a Poisson group) which is best adapted to the standard relation between homogeneous $G$–spaces and subgroups of $G$: to a given Poisson quotient, one associates the stabilizer subgroup of its distinguished point (the fixed zero-dimensional symplectic leaf). What characterizes such subgroups is *coisotropy*, with respect to the Poisson structure on $G$ (see the definition in Section 2 later on). On the other hand, if a (closed) subgroup $K$ of $G$ is coisotropic, then the homogeneous $G$–space $G/K$ is a Poisson quotient. So the two notions of Poisson quotient and coisotropic subgroup must be handled in couple. In particular, the quantization process for a Poisson $G$–quotient corresponds to a similar procedure for the attached coisotropic subgroup of $G$.

If one looks at quantizations of a Poisson homogeneous space, their existence is guaranteed only if the space is a quotient [8]; thus the notion of Poisson quotient shows up naturally as a necessary condition. On the other hand, let $K$ be a subgroup of $G$, and assume that $G$ has a quantization, inducing on it a Poisson group structure. If $K$ itself also admits a quantization, which is "consistent" (in a natural sense) with the one of $G$, then $K$ is automatically coisotropic in $G$. So also the related notion of coisotropic subgroup shows to be a necessary condition for the existence of quantizations. Of course an analogous description can be entirely carried out at an infinitesimal level, with conditions at the level of Lie bialgebras.

When dealing with quantizations of Poisson groups (or Lie bialgebras), a precious tool is the quantum duality principle (QDP). Roughly speaking, it claims that any quantized enveloping algebra can be turned — via a functorial recipe — into a quantum function algebra for the dual Poisson group; conversely, any quantum function algebra can be turned into a quantization of the enveloping algebra of the dual Lie bialgebra. To be precise, let $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ respectively be the category of all quantized universal enveloping algebras (QUEA) and the category of all quantized formal series Hopf algebras (QFSHA), in Drinfeld's sense. Then the QDP establishes — [6], [11] — a category equivalence between $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ via two functors, $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee \colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$. Moreover, starting from a QUEA over a Lie bialgebra (resp. from a QFSHA over a Poisson group) the functor $(\ )'$ (resp. $(\ )^\vee$ ) gives a QFSHA (resp. a QUEA) over the dual Poisson group (resp. the dual Lie bialgebra). In short, $U_\hbar(\mathfrak{g})' = F_\hbar[[G^*]]$ and $F_\hbar[[G]]^\vee = U_\hbar(\mathfrak{g}^*)$ for any Lie bialgebra $\mathfrak{g}$ and Poisson group $G$ with $Lie(G) = \mathfrak{g}$. So from a quantization of any Poisson group this principle gets out a quantization of the dual Poisson group too.

In this paper we establish a similar quantum duality principle for (closed) coisotropic subgroups of a Poisson group $G$, or equivalently for Poisson $G$–quotients, sticking to the formal approach (hence dealing with quantum groups à la Drinfeld). The starting point is that any formal coisotropic subgroup $K$ of a Poisson group $G$ has two possible algebraic descriptions via objects related to $U(\mathfrak{g})$ or $F[[G]]$, and similarly for the formal Poisson quotient $G\big/K$; thus the datum of $K$ or equivalently of $G\big/K$ is described algebraically in four possible ways. By *quantization* of such a datum we mean a quantization of any one of these four objects, which has to be "consistent" — in a natural sense — with given quantizations $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ of $G$. Our "QDP" now is a bunch of functorial recipes to produce, out of a quantization of $K$ or $G\big/K$ as before, a similar quantization of the so-called *complementary dual* of $K$, that is the coisotropic subgroup $K^\perp$ of $G^*$ whose tangent Lie bialgebra is just $\mathfrak{k}^\perp$ inside $\mathfrak{g}^*$, or of the associated Poisson $G^*$–quotient, namely $G^*\big/K^\perp$. The basic idea is quite simple. The quantizations of coisotropic subgroups — or Poisson quotients — are sub-objects of quantizations of Poisson groups, and the recipes of the original QDP (for Poisson groups) apply to the latter objects. Then we simply "restrict", somehow, such recipes to the previously mentioned sub-objects.

In recent times, the general problem of quantizing coisotropic manifolds of a given Poisson manifold, in the context of deformation quantization, has raised quite some interest (cf. [1], [2]). It is then important to point out that ours is by no means an existence result: instead, it can be thought of as a *duplication result*, because it yields a new quantization — for a complementary dual object — out of one given from scratch (much like the QDP for quantum groups). On the other hand, we would better stress that our result is really *effective*, and calling for applications. A sample of application is presented in the extended version of this work [3]; see also Subsection 5.6 in the present paper.

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