

# A Formula for the Logarithm of the KZ Associator<sup>★</sup>

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Received October 03, 2006, in final form November 10, 2006; Published online November 13, 2006

Original article is available at <http://www.emis.de/journals/SIGMA/2006/Paper080/>

**Abstract.** We prove that the logarithm of a group-like element in a free algebra coincides with its image by a certain linear map. We use this result and the formula of Le and Murakami for the Knizhnik–Zamolodchikov (KZ) associator  $\Phi$  to derive a formula for  $\log(\Phi)$  in terms of MZV’s (multiple zeta values).

*Key words:* free Lie algebras; Campbell–Baker–Hausdorff series, Knizhnik–Zamolodchikov associator

*2000 Mathematics Subject Classification:* 17B01; 81R50

*To the memory of Vadim Kuznetsov.*

## 1 Logarithms of group-like elements

Let  $F_n$  be the free associative algebra generated by free variables  $x_1, \dots, x_n$ , let  $\mathfrak{f}_n \subset F_n$  be the free Lie algebra with the same generators, and let  $\widehat{\mathfrak{f}}_n, \widehat{F}_n$  be their degree completions (where  $x_1, \dots, x_n$  have degree 1). A group-like element of  $\widehat{F}_n$  is an element of the form  $X = 1 +$  (terms of degree  $> 0$ ), such that  $\Delta(X) = X \otimes X$ , where  $\Delta$  is the completion of the coproduct  $F_n \rightarrow F_n^{\otimes 2}$ , for which  $x_1, \dots, x_n$  are primitive. It is well-known that the exponential defines a bijection  $\exp: \widehat{\mathfrak{f}}_n \rightarrow \{\text{group-like elements of } \widehat{F}_n\}$  (also denoted  $x \mapsto e^x$ ). We denote by  $\log$  the inverse bijection.

We denote by  $\text{CBH}_n(x_1, \dots, x_n)$  the multilinear part (in  $x_1, \dots, x_n$ ) of  $\log(e^{x_1} \cdots e^{x_n})$ . Define a linear map

$$\text{cbh}_n: F_n \rightarrow \mathfrak{f}_n$$

by  $\text{cbh}_n(1) = 0$  and  $\text{cbh}_n(x_{i_1} \cdots x_{i_k}) := \text{CBH}_k(x_{i_1}, \dots, x_{i_k})$ . This map extends to a linear map  $\widehat{\text{cbh}}_n: \widehat{F}_n \rightarrow \widehat{\mathfrak{f}}_n$ .

**Proposition 1.** *If  $X \in \widehat{F}_n$  is group-like, then  $\log(X) = \widehat{\text{cbh}}_n(X)$ .*

**Proof.** It is known that  $F_n = U(\mathfrak{f}_n)$ , so that the symmetrization is an isomorphism  $\text{sym}: S(\mathfrak{f}_n) \rightarrow F_n$ . Denote by  $p_n: F_n \rightarrow \mathfrak{f}_n$  the composition of  $\text{sym}^{-1}$  with the projection  $S(\mathfrak{f}_n) = \bigoplus_{k \geq 0} S^k(\mathfrak{f}_n)$  onto  $S^1(\mathfrak{f}_n) = \mathfrak{f}_n$ . We first prove:

**Lemma 1.**  $p_n = \text{cbh}_n$ .

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<sup>★</sup>This paper is a contribution to the Vadim Kuznetsov Memorial Issue “Integrable Systems and Related Topics”. The full collection is available at <http://www.emis.de/journals/SIGMA/kuznetsov.html>

**Proof.** If  $\mathfrak{g}$  is a Lie algebra, let  $p_{\mathfrak{g}}: U(\mathfrak{g}) \rightarrow \mathfrak{g}$  be the composition of the inverse of the symmetrization  $S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  with the projection onto the first component of  $S(\mathfrak{g})$ . If  $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra morphism, then we have a commutative diagram

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{p_{\mathfrak{g}}} & \mathfrak{g} \\ U(\phi)\downarrow & & \downarrow \phi \\ U(\mathfrak{h}) & \xrightarrow{p_{\mathfrak{h}}} & \mathfrak{h} \end{array}$$

It follows from Lemma 3.10 of [4] that if  $k \geq 0$  and  $F_k$  is the free algebra with generators  $y_1, \dots, y_k$ , then  $p_k(y_1, \dots, y_k) = \text{CBH}_k(y_1, \dots, y_k)$ . If now  $\mathbf{i} = (i_1, \dots, i_k)$  is a sequence of elements of  $\{1, \dots, n\}$ , we have a unique morphism  $\phi_{\mathbf{i}}: \mathfrak{f}_k \rightarrow \mathfrak{f}_n$ , such that  $y_1 \mapsto x_{i_1}, \dots, y_k \mapsto x_{i_k}$ .

Then

$$\begin{aligned} p_n(x_{i_1} \cdots x_{i_k}) &= p_{\mathfrak{f}_n} \circ U(\phi_{\mathbf{i}})(y_1 \cdots y_k) = \phi_{\mathbf{i}} \circ p_{\mathfrak{f}_k}(y_1 \cdots y_k) \\ &= \phi_{\mathbf{i}}(\text{CBH}_k(y_1, \dots, y_k)) = \text{CBH}_k(x_{i_1}, \dots, x_{i_k}) = \text{cbh}_n(x_{i_1} \cdots x_{i_k}), \end{aligned}$$

which proves the lemma. ■

**End of proof of Proposition 1.** We denote by  $\widehat{p}_n: \widehat{F}_n \rightarrow \widehat{\mathfrak{f}}_n$  the map similarly derived from the isomorphism  $\widehat{F}_n \simeq \widehat{\bigoplus_{k \geq 0} S^k(\widehat{\mathfrak{f}}_n)}$  (where  $\widehat{\bigoplus}$  is the direct product). Then  $p_n = \text{cbh}_n$  implies  $\widehat{p}_n = \widehat{\text{cbh}}_n$ .

If now  $X \in \widehat{F}_n$  is group-like, let  $\ell := \log(X)$ . We have  $X = 1 + \ell + \ell^2/2! + \cdots$ ; here  $\ell^k \in S^k(\widehat{\mathfrak{f}}_n)$ , so  $\widehat{p}_n(X) = \ell$ . Hence  $\widehat{\text{cbh}}_n(X) = \ell = \log(X)$ . ■

## 2 Corollaries

The KZ associator is defined as follows. Let  $A_0, A_1$  be noncommutative variables. Let  $F_2$  be the free associative algebra generated by  $A_0$  and  $A_1$ , let  $\mathfrak{f}_2 \subset F_2$  be its (free) Lie subalgebra generated by  $A_0$  and  $A_1$ . Let  $\widehat{F}_2$  and  $\widehat{\mathfrak{f}}_2$  be the degree completions of  $F_2$  and  $\mathfrak{f}_2$  ( $A_0$  and  $A_1$  have degree 1).

The KZ associator  $\Phi$  is defined [1] as the renormalized holonomy from 0 to 1 of the differential equation

$$G'(z) = \left( \frac{A_0}{z} + \frac{A_1}{z-1} \right) G(z), \tag{1}$$

i.e.,  $\Phi = G_1 G_0^{-1}$ , where  $G_0, G_1 \in \widehat{F}_2 \widehat{\otimes} \mathcal{O}_{]0,1[}$  are the solutions of (1) with  $G_0(z) \sim z^{A_0}$  as  $z \rightarrow 0^+$  and  $G_1(z) \sim (1-z)^{A_1}$  as  $z \rightarrow 1^-$ ; here  $\mathcal{O}_{]0,1[}$  is the ring of analytic functions on  $]0,1[$ , and  $\widehat{F}_2 \widehat{\otimes} V$  is the completion of  $F_2 \otimes V$  w.r.t. the topology defined by the  $F_2^{\geq n} \otimes V$  (here  $F_2^{\geq n}$  is the part of  $F_2$  of degree  $\geq n$ ).

We recall Le and Murakami's formula for  $\Phi$  [3]. We say that a sequence  $(a_1, \dots, a_n) \in \{0, 1\}^n$  is admissible if  $a_1 = 1$  and  $a_n = 0$ . If  $(a_1, \dots, a_n)$  is admissible, we set

$$\omega_{a_1, \dots, a_n} = \int_0^1 \omega_{a_1} \circ \cdots \circ \omega_{a_n},$$

where  $\omega_0(t) = dt/t$ ,  $\omega_1(t) = dt/(t-1)$  and  $\int_a^b \alpha_1 \circ \cdots \circ \alpha_n = \int_{a \leq t_1 \leq \dots \leq t_n \leq b} \alpha_1(t_1) \wedge \cdots \wedge \alpha_n(t_n)$ . Up to sign, the  $\omega_{a_1, \dots, a_n}$  are MZV's (multiple zeta values).

If  $(i_1, \dots, i_n)$  is an arbitrary sequence in  $\{0, 1\}^n$ , and  $(a_1, \dots, a_n)$  is an admissible sequence, define integers  $C_{i_1, \dots, i_n}^{a_1, \dots, a_n}$  by the relation

$$\begin{aligned} & \sum_{(i_1, \dots, i_n) \in \{0, 1\}^n} C_{i_1, \dots, i_n}^{a_1, \dots, a_n} A_{i_n} \cdots A_{i_1} \\ &= \sum_{\substack{S \subset \{\alpha | a_\alpha = 0\}, \\ T \subset \{\beta | a_\beta = 1\}}} (-1)^{\text{card}(S) + \text{card}(T)} A_1^{\text{card}(T)} A(a_1, \dots, a_n)^{S, T} A_0^{\text{card}(S)}, \end{aligned}$$

where for any  $S \subset \{\alpha | a_\alpha = 0\}$ ,  $T \subset \{\beta | a_\beta = 1\}$ ,  $A(a_1, \dots, a_n)^{S, T} := \prod_{\alpha \in [1, n] \setminus (S \cup T)} A_{a_\alpha}$  (the product is taken in decreasing order of the  $\alpha$ 's).

**Theorem 1 ([3]).**

$$\Phi = 1 + \sum_{n \geq 1} \sum_{\substack{(a_1, \dots, a_n) \text{ admissible} \\ (i_1, \dots, i_n) \in \{0, 1\}^n}} \omega_{a_1, \dots, a_n} C_{i_1, \dots, i_n}^{a_1, \dots, a_n} A_{i_n} \cdots A_{i_1}.$$

Since  $\Phi \in \widehat{F}_2$  is a group-like element, Proposition 1 implies that  $\log(\Phi) = \widehat{\text{cbh}}_2(\Phi)$ , therefore:

**Corollary 1.**

$$\log(\Phi) = \sum_{n \geq 1} \sum_{\substack{(a_1, \dots, a_n) \text{ admissible} \\ (i_1, \dots, i_n) \in \{0, 1\}^n}} \omega_{a_1, \dots, a_n} C_{i_1, \dots, i_n}^{a_1, \dots, a_n} \text{CBH}_n(A_{i_n}, \dots, A_{i_1}).$$

Using the explicit formula of [2], one computes similarly the logarithm of the analogue  $\Psi$  of the KZ associator of the equation  $G'(z) = (A/z + \sum_{\zeta | \zeta^n = 1} b_\zeta / (z - \zeta)) G(z)$ .

Proposition 1 also implies:

**Lemma 2.** *Let  $\mathfrak{g}$  be a nilpotent Lie algebra,  $G$  be the associated Lie group, let  $a < b \in \mathbb{R}$ . Fix  $h(z) \in C^0([a, b], \mathfrak{g})$  and let  $H$  be the holonomy from  $a$  to  $b$  of the differential equation  $H'(z) = h(z)H(z)$ , where  $H(z) \in C^1([a, b], G)$ . Then*

$$\log(H) = \sum_{n \geq 1} \int_{a \leq z_1 \leq \dots \leq z_n \leq b} \text{CBH}_n(h(z_n), \dots, h(z_1)) dz_1 \cdots dz_n.$$

## Acknowledgements

We first established the formula for  $\log(\Phi)$  in Corollary 1 by analytic computations (using a direct proof of Lemma 2). It was the referee who remarked its formal similarity with the formula of Le and Murakami (Theorem 1); this remark can be expressed as the equality  $\log(\Phi) = \widehat{\text{cbh}}_2(\Phi)$ . This led us to try and understand whether this formula followed from the group-likeness of  $\Phi$ , which is indeed the case (Proposition 1). C. Reutenauer then pointed out that a part of our argument is a result in his book.

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