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"The Crystal Duality Principle: from Hopf Algebras to Geometrical Symmetries"

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INTRODUCTION

"Yet these crystals are to Hopf algebras
but as is the body to the Children of Rees:
the house of its inner fire, that is within it
and yet in all parts of it, and is its life"

N. Barbecue, "Scholia"

Among all Hopf algebras over a field $\Bbbk$, there are two special families which are of relevant interest for their geometrical meaning. The function algebras $F[G]$ of algebraic groups G and the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$, if $\mathit{Char}\,(\Bbbk)=0$, or the restricted universal enveloping algebras $\mathbf{u}(\mathfrak{g})$ of restricted Lie algebras $\mathfrak{g}$, if $\mathit{Char}\,(\Bbbk)>0$. For brevity, we call both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ "enveloping algebras" and denote them by $\mathcal{U}(\mathfrak{g})$. Similarly by "restricted Lie algebra", when $\mathit{Char}\,(\Bbbk)=0$, we shall simply mean "Lie algebra". Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected, cocommutative and generated by their primitives.

In this paper we give functorial recipes to get out from any Hopf algebra two pairs of Hopf algebras of geometrical type, say $\big(F[G_+],\mathcal{U}(\mathfrak{g}_-)\big)$ and $\big(F[K_+],\mathcal{U}(\mathfrak{k}_-)\big)$. In addition, the algebraic groups obtained in this way are connected *Poisson* groups, and the (restricted) Lie algebras are (restricted) *Lie bialgebras*. Therefore, to each Hopf algebra — which encodes a general notion of "symmetry" — we can associate in a functorial way some symmetries of geometrical type, where the geometry involved is in fact Poisson geometry. Moreover, if $\mathit{Char}\,(\Bbbk)>0$ these Poisson groups have dimension 0 and height 1, which makes them very interesting for arithmetic geometry, hence for number theory too.

The construction of the pair $(G_+,\mathfrak{g}_-)$ uses pretty classical (as opposite to "quantum") methods: in fact, it might be part of the content of any basic textbook on Hopf algebras (and, surprisingly enough, it is not!). Instead, in order to obtain the pair $(K_+,\mathfrak{k}_-)$ one relies on the construction of the first pair, and uses the theory of quantum groups.

Let's describe our results in detail. Let $J:=\mathit{Ker}\,(\epsilon_H)$ be the augmentation ideal of the Hopf algebra H (where ϵ_H is the counit of H), and let $\underline{J}:=\big\{J^n\big\}_{n\in\mathbb{N}}$ be the associated J–adic filtration, $\widehat{H}:=G_{\underline{J}}(H)$ the associated graded vector space and $H^\vee:=H\Big/\bigcap_{n\in\mathbb{N}}J^n$. One proves that $\underline{J}$ is a Hopf algebra filtration, hence $\widehat{H}$ is a graded Hopf algebra: the latter happens to be connected, cocommutative and generated by its primitives, so $\widehat{H}\cong\mathcal{U}(\mathfrak{g}_-)$ for some restricted Lie algebra $\mathfrak{g}_-$. In addition, since $\widehat{H}$ is graded also $\mathfrak{g}_-$ itself is graded as a restricted Lie algebra. The fact that $\widehat{H}$ be cocommutative allows to define a Poisson cobracket on it (from the natural Poisson cobracket $\nabla := \Delta - \Delta^{\mathrm{op}}$ on H), which makes $\widehat{H}$ a graded *co-Poisson* Hopf algebra. Eventually, this implies that $\mathfrak{g}_-$ is a *Lie bialgebra*. So the right-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widehat{H}$.

On the other hand, one considers a second filtration — increasing, whereas $\underline{J}$ is decreasing — namely $\underline{D}$ which is defined in a dual manner to $\underline{J}$. For each $n \in \mathbb{N}$, let δ_n be the composition of the n–fold iterated coproduct followed by the projection onto $J^{\otimes n}$ (note that $H = \Bbbk \cdot 1_H \oplus J$); then $\underline{D} := \big\{ D_n := \mathit{Ker}\,(\delta_{n+1}) \big\}_{n \in \mathbb{N}}$. Let now $\widetilde{H} := G_{\underline{D}}(H)$ be the associated graded vector space and $H' := \bigcup_{n \in \mathbb{N}} D_n$. Again, one shows that $\underline{D}$ is a Hopf algebra filtration, hence $\widetilde{H}$ is a graded Hopf algebra: moreover, the latter is commutative, so $\widetilde{H} = F[G_+]$ for some algebraic group G_+. One proves also that $\widetilde{H} = F[G_+]$ has no non-trivial idempotents, thus G_+ is connected; in addition, since $\widetilde{H}$ is graded, G_+ as a variety is just an affine space. A deeper analysis shows that in the positive characteristic case G_+ has dimension 0 and height 1. The fact that $\widetilde{H}$ be commutative allows to define on it a Poisson bracket (from the natural Poisson bracket on H given by the commutator) which makes $\widetilde{H}$ a graded *Poisson* Hopf algebra. This means that G_+ is an algebraic *Poisson group*. So the left-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widetilde{H}$.

The relationship among H and the "geometrical" Hopf algebras $\widehat{H}$ and $\widetilde{H}$ can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(H^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{H} \qquad (\bigstar)$$

Here the one-way arrows are Hopf algebra morphisms and the two-ways arrows are regular 1-parameter deformations of Hopf algebras, realized through the *Rees* Hopf algebras $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$ associated to the filtration $\underline{D}$ of H and to the filtration $\underline{J}$ of $H^\vee$.

The construction of the pair $(K_+, \mathfrak{k}_-)$ uses quantum group theory, the basic ingredients being $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$. In the present framework, by quantum group we mean, loosely speaking, a Hopf $\Bbbk[t]$–algebra (t an indeterminate) H_t such that either *(a)* $H_t \big/ t\,H_t \cong F[G]$ for some connected Poisson group G — then we say H_t is a QFA — or *(b)* $H_t \big/ t\,H_t \cong \mathcal{U}(\mathfrak{g})$, for some restricted Lie bialgebra $\mathfrak{g}$ — then we say H_t is a QrUEA. Formula $(\bigstar)$ says that $H'_t := \mathcal{R}^t_{\underline{D}}(H)$ is a QFA, with $H'_t \big/ t\,H'_t \cong \widetilde{H} = F[G_+]$, and that $H^\vee_t := \mathcal{R}^t_{\underline{J}}(H)$ is a QrUEA, with $H^\vee_t \big/ t\,H^\vee_t \cong \widehat{H} = \mathcal{U}(\mathfrak{g}_-)$. Now, a general result — the "Global Quantum Duality Principle", in short GQDP, see [Ga1–2] — teaches us how to construct from the QFA H'_t a QrUEA, call it $\big(H'_t\big)^\vee$, and how to build out of the QrUEA $H^\vee_t$ a QFA, say $\big(H^\vee_t\big)'$. Then $\big(H'_t\big)^\vee \big/ t\,\big(H'_t\big)^\vee = \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, and $\big(H^\vee_t\big)' \big/ t\,\big(H^\vee_t\big)' = F[K_+]$ for some connected Poisson group K_+. This gives the pair $(K_+, \mathfrak{k}_-)$. The very construction implies that $\big(H'_t\big)^\vee$ and $\big(H^\vee_t\big)'$ yield another frame of regular 1-parameter deformations for H' and $H^\vee$, namely

$$\mathcal{U}(\mathfrak{k}_-) \xleftrightarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} F[K_+] \qquad (\maltese)$$

which is the analogue of $(\bigstar)$. In addition, when $\mathit{Char}(\Bbbk) = 0$ the GQDP also claims that the two pairs $(G_+, \mathfrak{g}_-)$ and $(K_+, \mathfrak{k}_-)$ are related by Poisson duality: namely, $\mathfrak{k}_-$ *is the cotangent Lie bialgebra of* G_+, and $\mathfrak{g}_-$ *is the cotangent Lie bialgebra* of K_+; in short, we write $\mathfrak{k}_- = \mathfrak{g}^\times$ and $K_+ = G_-^\star$. Therefore the four "Poisson symmetries" G_+, $\mathfrak{g}_-$, K_+ and $\mathfrak{k}_-$, attached to H are actually encoded simply by the pair (G_+, K_+).

In particular, when $H^\vee = H = H'$ from $(\bigstar)$ and $(\maltese)$ together we find

$$\begin{array}{ccccc} \mathcal{U}(\mathfrak{g}_-) & \xleftrightarrow[H_t^\vee]{0 \leftarrow t \rightarrow 1} & H^\vee & \xleftrightarrow[(H_t^\vee)']{1 \leftarrow t \rightarrow 0} & F[K_+] \quad \Big(= F\big[G_-^\star\big] \ \text{if} \ \mathit{Char}(\Bbbk) = 0 \Big) \\ & & \| & & \\ & & H & & \\ & & \| & & \\ F[G_+] & \xleftrightarrow[H_t']{0 \leftarrow t \rightarrow 1} & H' & \xleftrightarrow[(H_t')^\vee]{1 \leftarrow t \rightarrow 0} & \mathcal{U}(\mathfrak{k}_-) \quad \Big(= U\big(\mathfrak{g}_+^\times\big) \ \text{if} \ \mathit{Char}(\Bbbk) = 0 \Big) \end{array}$$

This gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type, i.e. Lie bialgebras or Poisson algebraic groups.

When the Hopf algebra H we start from is already of geometric type, the result involves Poisson duality. Namely, if $\mathit{Char}(\Bbbk) = 0$ and $H = F[G]$, then $\mathfrak{g}_- = \mathfrak{g}^*$ (where $\mathfrak{g} := \mathit{Lie}(G)$), and if $H = \mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})$, then $\mathit{Lie}(G_+) = \mathfrak{g}^*$, i.e. G_+ has $\mathfrak{g}$ as cotangent Lie bialgebra. If instead $\mathit{Char}(\Bbbk) > 0$, we have only a slight variation on this result.

The construction of $\widehat{H}$ and $\widetilde{H}$ needs only "half the notion" of a Hopf algebra. In fact, we construct $\widehat{A}$ for any augmented algebra A (roughly, an algebra with an augmentation, or counit, i.e. a character), and $\widetilde{C}$ for any coaugmented coalgebra C (a coalgebra with a coaugmentation, or unit, i.e. a coalgebra morphism from $\Bbbk$ to C). In particular this applies to bialgebras, for which both $\widehat{B}$ and $\widetilde{B}$ are (graded) Hopf algebras. We can also perform a second construction using $\big(B_t'\big)^\vee$ and $\big(B_t^\vee\big)'$ (via a stronger version of the GQDP), and get from these a second pair of bialgebras $\Big(\big(B_t'\big)^\vee\Big|_{t=0}, \big(B_t^\vee\big)'\Big|_{t=0}\Big)$. Then again $\big(B_t'\big)^\vee\Big|_{t=0} \cong \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, while $\big(B_t^\vee\big)'\Big|_{t=0}$ is commutative with no non-trivial idempotents, but it's not, in general, a Hopf algebra. So the spectrum of $\big(B_t^\vee\big)'\Big|_{t=0}$ is a connected algebraic Poisson monoid, but not necessarily a Poisson group.

It is worthwhile pointing out that everything in fact follows from the GQDP, which — in the stronger formulation — deals with augmented algebras and coaugmented coalgebras over 1-dimensional domains. The content of this paper can in fact be obtained as a corollary of the GQDP as follows. Pick any augmented algebra or coaugmented coalgebra over $\Bbbk$, and take its scalar extension from $\Bbbk$ to $\Bbbk[t]$: the latter ring is a 1-dimensional domain, hence we can apply the GQDP, and (almost) every result in the present paper will follow.

In the last section we apply these results to the case of group algebras and their duals. Another interesting application — based on a non-commutative version of the function algebra of the group of formal diffeomorphism on the line, also called "Nottingham group" — is illustrated in detail in a separate paper (see [Ga3]).

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