INTRODUCTION

"Dualitas dualitatum et omnia dualitas"

N. Barbecue, "Scholia"

The quantum duality principle is known in literature under at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be considered to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld (cf. [Dr]), states that any quantization of the universal enveloping algebra of a Poisson group can also be understood — in some sense — as a quantization of the dual formal Poisson group, and, conversely, any quantization of a formal Poisson group also “serves” as a quantization of the universal enveloping algebra of the dual Poisson group: this is the point of view we are interested in. I am now going to describe this result more in detail.

Let $k$ be a field of zero characteristic. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $k$, $U(\mathfrak{g})$ its universal enveloping algebra: then $U(\mathfrak{g})$ has a natural structure of Hopf algebra. Let $F[[\mathfrak{g}]]$ be the (algebra of regular functions on the) formal group associated to $\mathfrak{g}$: it is a complete topological Hopf algebra (the coproduct taking values in a suitable topological tensor product of the algebra with itself), which has two realizations. The first one is as follows: if $G$ is an affine algebraic group with tangent Lie algebra $\mathfrak{g}$, and $F[G]$ is the algebra of regular functions on $G$, then $F[[\mathfrak{g}]]$ is the $m_e$-completion of $F[G]$ at the maximal ideal $m_e$ of the identity element $e \in G$, endowed with the $m_e$-adic topology. The second one is $F[[\mathfrak{g}]] := U(\mathfrak{g})^*$, the linear dual of $U(\mathfrak{g})$, endowed with the weak topology. In any case, $U(\mathfrak{g})$ identifies with the topological dual of $F[[\mathfrak{g}]]$, i.e. the set of all $k$-linear continuous maps from $F[[\mathfrak{g}]]$ to $k$, where $k$ is given the discrete topology; similarly $F[[\mathfrak{g}]] = U(\mathfrak{g})^*$ is also the topological dual of $U(\mathfrak{g})$ if we take on the latter space the discrete topology: in particular, a (continuous) biduality theorem relates $U(\mathfrak{g})$ and $F[[\mathfrak{g}]]$, and evaluation yields a natural Hopf pairing among them. Now assume $\mathfrak{g}$ is a Lie bialgebra: then $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[[\mathfrak{g}]]$ is a topological Poisson Hopf algebra, and the above pairing is compatible with these additional co-Poisson and Poisson structures. Further, the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well, so we can consider also $U(\mathfrak{g}^*)$ and $F[[\mathfrak{g}^*]]$.

Let $\mathfrak{g}$ be a Lie bialgebra. A quantization of $U(\mathfrak{g})$ is, roughly speaking, a topological Hopf $k[[h]]$-algebra which for $h = 0$ is isomorphic, as a co-Poisson Hopf algebra, to $U(\mathfrak{g})$: these objects form a category, called $QUEA$. Similarly, a quantization of $F[[\mathfrak{g}]]$ is, in short, a topological Hopf $k[[h]]$-algebra which for $h = 0$ is isomorphic, as a topological Poisson Hopf algebra, to $F[[\mathfrak{g}]]$: we call $QFSHA$ the category formed by these objects.

The quantum duality principle (after Drinfeld) states that there exist two functors, namely $(\ )': QUEA \to QFSHA$ and $(\ )^\vee: QFSHA \to QUEA$, which are inverse of
each other, and if $U_h(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ and $F_h[[\mathfrak{g}]]$ is a quantization of $F[[\mathfrak{g}]]$, then $U_h(\mathfrak{g})'$ is a quantization of $F[[\mathfrak{g}^*]]$, and $F_h[[\mathfrak{g}]]''$ is a quantization of $U(\mathfrak{g}^*)$.

This paper provides an explicit thorough proof — seemingly, the first one in literature — of this result, in a slightly stronger version, too. I also point out some further details and what is true when $k$ has positive characteristic, and sketch a plan for generalizing all this to the infinite dimensional case.

Note that several properties of the objects I consider have been discovered and exploited in the works by Etingof and Kazhdan (see [EK1], [EK2]), by Enriquez (cf. [E]) and by Kassel and Turaev (cf. [KT]), who deal with some special cases of quantum groups, arising from a specific construction, and also applied Drinfeld’s results. The analysis in the present paper shows that that those properties are often direct consequences of more general facts.

I point out that Drinfeld’s result is essentially local in nature, as it deals with quantizations over the ring of formal series and ends up only with infinitesimal data, i.e. objects attached to Lie bialgebras; a global version of the principle, dealing with quantum groups over a ring of Laurent polynomials, which give information on the global data of the underlying Poisson groups will be provided in a forthcoming paper (cf. [Ga2]): this is useful in applications, e.g. it yields a quantum duality principle for Poisson homogeneous spaces, cf. [CG].

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References


