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ON THE GLOBAL QUANTUM DUALITY PRINCIPLE

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Introduction

The quantum duality principle is known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be (Hopf) dual to each other, and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld (cf. [Dr], §7, and [Ga4]), states that any quantisation $F_h[[G]]$ of $F[[G]]$ yields also a quantisation of $U(\mathfrak{g}^*)$, and, conversely, any quantisation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ provides a quantisation of $F[[G^*]]$: here G^*, resp. $\mathfrak{g}^*$, is a Poisson group, resp. a Lie bialgebra, dual to G, resp. to $\mathfrak{g}$). Namely, Drinfeld defines two functors, inverse to each other, from the category of quantum enveloping algebras to the category of quantum formal series Hopf algebras and viceversa such that (roughly) $U_h(\mathfrak{g}) \mapsto F_h[[G^*]]$ and $F_h[[G]] \mapsto U_h(\mathfrak{g}^*)$.

In this paper I provide a *global* version of the above principle, improving Drinfeld's result and pushing as far as possible the treatment *in a Hopf algebra theoretical way*.

The general idea is the following. Quantisation of groups and Lie algebras is a matter of dealing with suitable Hopf algebras. In short, the Hopf algebras "of geometrical interest" are (simplifying a bit) the commutative and the connected cocommutative ones: the first are function algebras of affine algebraic groups (which are their maximal spectra), the second are restricted universal enveloping algebras of Lie algebras. A "quantisation" of such an object H_0 will be a Hopf algebra H depending on some parameter, say p, such that setting $p = 0$, i.e. taking the quotient of the algebra modulo p, one gets back the original Hopf algebra H_0. One must also remark that when a quantisation H is given the classical object H_0 inherits an additional structure, that of a Poisson algebra, if $H_0 = F[G]$, or that of a co-Poisson algebra, if $H_0 = U(\mathfrak{g})$; correspondingly, G is an affine *Poisson* group, $\mathfrak{g}$ is a *Lie bialgebra*, *and then also* its dual space $\mathfrak{g}^*$ is a Lie bialgebra; then we'll denote by G^* any affine Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$, and we say G^* is *dual* to G.

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In conclusion, one is lead to consider such "quantum groups", namely "p–depending" Hopf algebras which are either *commutative modulo p* or *cocommutative modulo p*.

In detail, I focus on the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free modules over a PID, say R; the role of the "quantisation parameter" then will be played by any prime element $p \in R$. For any such p, I introduce well-defined Drinfeld's-like functors from $\mathcal{HA}$ to itself, I show that their image is contained in a category of quantum groups — quantised function algebras in one case, quantised enveloping algebras in the other — and that when restricted to quantum groups these functors are inverse to each other and they exchange the type of the quantum group — switching "function" to "enveloping" — and the underlying group — switching G to G^*. Other details enter the picture to show that these functors endow $\mathcal{HA}$ with sort of a (inner) "Galois' correspondence", in which quantum groups — i.e. quantised enveloping algebras and quantised function algebras — play the role of Galois (sub)extensions, for they are exactly the objects which are fixed by the composition (in either order) of the two Galois maps.

From a purely algebraic point of view — and in characteristic zero, to make things easier — the quantum duality principle, coupled with the existence theorems for quantisations of Lie bialgebras or algebraic groups (given by [EK] and [E]), tells us (roughly speaking) that the category of *commutative* Hopf algebras and the category of *cocommutative* Hopf algebras are related in a very precise way via the "quantisation + Drinfeld's functors + specialisation" process: this requires passing through *general* (i.e neither commutative nor cocommutative) Hopf algebras, so we see that quantisation may be a way to rule special subclasses inside the whole category of Hopf algebras.

I wish to stress the fact that, compared with Drinfeld's result, mine is "global" in several respects. First, I deal with functors applying to general Hopf algebras (not only quantum groups, i.e. I do not require them to be "commutative up to specialisation" or "cocommutative up to specialisation"). Second, I work with more global objects, namely (function algebras on) algebraic Poisson groups rather than (function algebras on) *formal* algebraic Poisson groups. Third, I do *not require* the geometric objects — Poisson groups and Lie bialgebras — to be finite dimensional. Fourth, the ground ring R is any PID, not necessarily $k[[h]]$ as in Drinfeld's approach: therefore one may have several points $(p) \in Spec\,(R)$, and to each of them the machinery applies: thus for any such (p) Drinfeld's functors are defined and for a given Hopf R–algebra H they yield two Hopf R–algebras, say $H'_{[p]}$ and $H^\vee_{[p]}$, such that the fibre over (p) of $H'_{[p]}$, resp. of $H^\vee_{[p]}$, is a quantum function algebra at (p), resp. a quantum enveloping algebra at (p), i.e. its reduction modulo (p) is the function algebra of a Poisson group, resp. the (restricted, if $Char\big(R/(p)\big) > 0$) universal enveloping algebra of a Lie bialgebra. In particular we have a method to get, out of any Hopf algebra over a PID, several "quantum groups", namely two of them (of the "enveloping algebra" and of the "function algebra" type) for each point of the spectrum of R. More in general, one can start from any Hopf algebra H over a field k and then take $H_x := k[x] \otimes_k H$, (x being an indeterminate): this is a Hopf algebra over the PID $k[x]$, to which Drinfeld's functors at any prime $p \in k[x]$ may be applied to give quantum groups.

In this note, I confine myself to state the result and to expound it on two examples: the case of semisimple groups and the so-called "Kostant-Kirillov structure" on any Lie algebra. All details, proofs and further examples can be found in [Ga5].

§ 1 Notation and terminology

1.1 The classical setting. Let k be a fixed field. We call "(affine) algebraic group" the maximal spectrum $G := Hom_{k\text{-}Alg}(H,k)$ of any commutative Hopf k–algebra H (in particular, we deal with *pro-affine* as well as *affine* algebraic groups); then H is called the algebra of regular function on G, denoted with $F[G]$. We say that G is connected if $F[G]$ has no non-trivial idempotents. If G is an algebraic group, we denote by $\mathfrak{m}_e$ the defining ideal of the unit element $e \in G$ (it is the augmentation ideal of $F[G]$), and by $\overline{{\mathfrak{m}_e}^2}$ the closure of ${\mathfrak{m}_e}^2$ w.r.t. the weak topology; the cotangent space of G at e is $\mathfrak{g}^\times := \mathfrak{m}_e \Big/ \overline{{\mathfrak{m}_e}^2}$, endowed with its weak topology; by $\mathfrak{g}$ we mean the tangent space of G at e, realized as the topological dual $\mathfrak{g} := \big(\mathfrak{g}^\times\big)^\star$ of $\mathfrak{g}^\times$: this is *the tangent Lie algebra of* G. By $U(\mathfrak{g})$ we mean the universal enveloping algebra of $\mathfrak{g}$: this is a connected cocommutative Hopf algebra, and there is a natural Hopf pairing between $F[G]$ and $U(\mathfrak{g})$. If $Char(k) > 0$ and $\mathfrak{g}$ is a restricted Lie algebra, we call $u(\mathfrak{g})$ the restricted universal enveloping algebra of $\mathfrak{g}$. Now assume G is a Poisson group: then $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[G]$ is a Poisson Hopf algebra Poisson structures. Then $\mathfrak{g}^\times$ and $\mathfrak{g}^\star$ (the topological dual of $\mathfrak{g}$ w.r.t. the weak topology) are Lie bialgebras too (maybe in a topological sense): in both cases, the Lie bracket is induced by the Poisson bracket of $F[G]$, and the non-degenerate natural pairings $\mathfrak{g} \times \mathfrak{g}^\times \longrightarrow k$ and $\mathfrak{g} \times \mathfrak{g}^\star \longrightarrow k$ are compatible with these Lie bialgebra structures: so $\mathfrak{g}$ and $\mathfrak{g}^\times$ or $\mathfrak{g}^\star$ are Lie bialgebras *dual to each other*. We denote by $G^\star$ any connected algebraic Poisson group with tangent Lie bialgebra $\mathfrak{g}^\star$, and say it is *dual* to G.

The quantum setting. Let R be a principal ideal domain, let $Q(R)$ be its quotient field, let $p \in R$ be a fixed prime element, and let $k_p := R\big/(p)$ be the corresponding residue field: for simplicity we shall assume throughout $Char(k_p) = 0$. Let $\mathcal{A}$ the category of torsion-free R–modules, and $\mathcal{HA}$ the subcategory of all Hopf algebras in $\mathcal{A}$. Let $\mathcal{A}_F$ the category of $Q(R)$–vector spaces, and $\mathcal{HA}_F$ be the subcategory of all Hopf algebras in $\mathcal{A}_F$. For any $M \in \mathcal{A}$, set $M_F := Q(R) \otimes_R M$.

For any R–module M, we set $M_p := M\big/p\,M = k_p \otimes_R M$: this is a k_p–module (via scalar extension $R \rightarrow k_p$), which we call the *specialisation* of M at $p = 0$.

For any $H \in \mathcal{HA}$, let $I_H := {\epsilon_H}^{-1}(p\,R)$: set $I_H^\infty := \bigcap_{n=0}^{+\infty} I_H^{\,n}$, and $H_\infty := \bigcap_{n=0}^{+\infty} p^n H$.

Given $\mathbb{H}$ in $\mathcal{HA}_F$, a subset H of $\mathbb{H}$ is called *an R–integer form* (or simply *an R–form*) *of* $\mathbb{H}$ if: *(a)* H is an R–Hopf subalgebra of $\mathbb{H}$; *(b)* H is torsion-free as an R–module (hence $H \in \mathcal{HA}$); *(c)* $H_F := Q(R) \otimes_R H = \mathbb{H}$.

Definition. ("Global quantum groups" [or "algebras"]) Fix a prime $p \in R$.

(a) We call *quantized universal enveloping algebra* (in short, *QUEA*) any pair $(\mathbb{U}, U)$ such that $U \in \mathcal{HA}$, $\mathbb{U} \in \mathcal{HA}_F$, U is an R–integer form of $\mathbb{U}$, and $U_p := U \big/ pU$ is (isomorphic to) the universal enveloping algebra of a Lie algebra. We denote by $\mathcal{QUEA}$ the subcategory of $\mathcal{HA}$ whose objects are all the QUEA's.

(b) We call *quantized function algebra* (in short, *QFA*) any pair $(\mathbb{F}, F)$ such that $F \in \mathcal{HA}$, $\mathbb{F} \in \mathcal{HA}_F$, F is an R–integer form of $\mathbb{F}$, $F_\infty = I_F^\infty$ (notation of §1) and $F_p := F \big/ p\,F$ is (isomorphic to) the algebra of regular functions of a connected algebraic group. We call $\mathcal{QFA}$ the subcategory of $\mathcal{HA}$ whose objects are all the QFA's.

If $(\mathbb{U}, U)$ is a QUEA (at p), then U_p is a co-Poisson Hopf algebra, so $U_p \cong U(\mathfrak{g})$ for $\mathfrak{g}$ a *Lie bialgebra*; in this situation we shall write $\mathbb{U} = \mathbb{U}_p(\mathfrak{g})$, $U = U_p(\mathfrak{g})$. Similarly, if $(\mathbb{F}, F)$ is a QFA then F_p is a Poisson Hopf algebra, so $F_p \cong F[G]$ for G a *Poisson algebraic group*: thus we shall write $\mathbb{F} = \mathbb{F}_p[G]$, $F = F_p[G]$.

§ 2 The global quantum duality principle

Drinfeld's functors. (Cf. [Dr], §7) Let $H \in \mathcal{HA}$. Define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := id_H$, and $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ for every $n \in \mathbb{N}$, $n > 2$. Then set $\delta_n := (id_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}$. For a fixed prime $p \in R$, we define $H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in p^n H^{\otimes n} \;\forall\, n \in \mathbb{N} \,\big\}$ $(\subseteq H)$, $\quad H^\vee := \sum_{n \geq 0} p^{-n} I_H^{\,n}$ $(\subseteq H_F)$.

Theorem. ("The global quantum duality principle")

(a) The assignment $H \mapsto H'$, resp. $H \mapsto H^\vee$, defines a functor $(\ )' \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, resp. $(\ )^\vee \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, whose image lies in $\mathcal{QFA}$, resp. in $\mathcal{QUEA}$.

(b) For all $H \in \mathcal{HA}$, we have $H \subseteq \big(H^\vee\big)'$ and $H \supseteq \big(H'\big)^\vee$. Moreover, we have

$$H = \big(H^\vee\big)' \iff \big(H_F, H\big) \in \mathcal{QFA} \quad \text{and} \quad H = \big(H'\big)^\vee \iff \big(H_F, H\big) \in \mathcal{QUEA},$$

thus we have induced functors $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFA}$, $\big(\mathbb{H}, H\big) \mapsto \big(\mathbb{H}, H'\big)$ and $(\ )^\vee \colon \mathcal{QFA} \longrightarrow \mathcal{QUEA}$, $\big(\mathbb{H}, H\big) \mapsto \big(\mathbb{H}, H^\vee\big)$ which are inverse to each other.

(c) ("Global Quantum Duality Principle") With notation of § 1, we have

$$U_p(\mathfrak{g})' \big/ p\, U_p(\mathfrak{g})' = F\big[G^\star\big], \qquad F_p[G]^\vee \big/ p\, F_p[G]^\vee = U(\mathfrak{g}^\times)$$

where the choice of the group $G^\star$ (among all the connected algebraic Poisson groups with tangent Lie bialgebra $\mathfrak{g}^\star$) depends on the choice of the QUEA $\big(\mathbb{U}_p(\mathfrak{g}), U_p(\mathfrak{g})\big)$. In other words, if $\big(\mathbb{U}_p(\mathfrak{g}), U_p(\mathfrak{g})\big)$ is a QUEA for the Lie bialgebra $\mathfrak{g}$, then $\big(\mathbb{U}_p(\mathfrak{g}), U_p(\mathfrak{g})'\big)$ is a QFA for the Poisson group $G^\star$, and if $\big(\mathbb{F}_p[G], F_p[G]\big)$ is a QFA for the Poisson group G, then $\big(\mathbb{F}_p[G], F_p[G]^\vee\big)$ is a QUEA for the Lie bialgebra $\mathfrak{g}^\times$. $\square$

Let me remark in particular that part *(c)* of the claim above shows, among other things, that the Hopf algebra (over $Q(R)$) $\mathbb{U}_p(\mathfrak{g})$ may be thought of — roughly — as a "quantum function algebra", as well as a "quantum universal enveloping algebra", for in fact at the same time it has an integer form which is a quantisation of a universal enveloping algebra and *also* an integer form which is a quantisation of a function algebra. Similarly, the Hopf algebra (over $Q(R)$) $\mathbb{F}_p[G]$ may be thought of as a "quantum universal enveloping algebra", as well as as a "quantum function algebra".

§ 3 First example: SL_2, SL_n, and the semisimple case

Let k be a field, and q be an indeterminate. Set $R := k\big[q,q^{-1}\big]$, $p := (q-1)$ (prime element in R). Let $\mathbb{U}_p(\mathfrak{sl}_2)$ be the associative unital $k(q)$–algebra with generators F, $K^{\pm 1}$, E, and relations $KK^{-1} = 1 = K^{-1}K$, $K^{\pm 1}F = q^{\mp 2}FK^{\pm 1}$, $K^{\pm 1}E = q^{\pm 2}EK^{\pm 1}$, $EF - FE = \big(q-q^{-1}\big)^{-1}\big(K-K^{-1}\big)$. This is a Hopf algebra, with Hopf structure given by $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$, $\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}$, $\Delta(E) = E \otimes 1 + K \otimes E$, $\epsilon(F) = 0$, $\epsilon\big(K^{\pm 1}\big) = 1$, $\epsilon(E) = 0$, $S(F) = -FK$, $S\big(K^{\pm 1}\big) = K^{\mp 1}$, $S(E) = -K^{-1}E$. Then let $U_p(\mathfrak{sl}_2)$ be the R–subalgebra of $\mathbb{U}_p(\mathfrak{sl}_2)$ generated by F, $H := p^{-1}(K-1)$, $\varGamma := \big(q-q^{-1}\big)^{-1}\big(K-K^{-1}\big)$, $K^{\pm 1}$, E. From the definition of $\mathbb{U}_p(\mathfrak{sl}_2)$ one gets an explicit presentation of $U_p(\mathfrak{sl}_2)$ as well, and sees it is a Hopf subalgebra. Further, one has $k(q) \otimes_R U_p(\mathfrak{sl}_2) = \mathbb{U}_p(\mathfrak{sl}_2)$, and also that $U_p(\mathfrak{sl}_2)$ is a free R–module. In fact, setting $p=0$ (i.e. $q=1$) the explicit presentation shows that $U_p(\mathfrak{sl}_2)\big/p\,U_p(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)$, thus $\big(\mathbb{U}_p(\mathfrak{sl}_2), U_p(\mathfrak{sl}_2)\big)$ is a QUEA with respect to the prime $p := (q-1) \in R$; this makes $\mathfrak{sl}_2$ a Lie bialgebra and SL_2 a Poisson group. A direct inspection proves that $U_p(\mathfrak{sl}_2)'$ is nothing but the unital R–subalgebra of $U_p(\mathfrak{sl}_2)$ generated by $\dot{F} := p\,F$, K, K^{-1}, $\dot{H} := p\,H$, $\dot{\varGamma} := p\,\varGamma$, $\dot{E} := p\,E$. As a consequence, one gets an explicit presentation of $U_p(\mathfrak{sl}_2)'$ which shows that it is a Hopf subalgebra, and that $U_p(\mathfrak{sl}_2)'\big/p\,U_p(\mathfrak{sl}_2)' = F\big[{}_aSL_2{}^*\big]$. Here ${}_aSL_2{}^*$ is the unique connected *adjoint* Poisson group dual to SL_2; a different choice of the initial QUEA leads us to the *simply connected* one, call it ${}_sSL_2{}^*$. Indeed, start from a "simply connected" version of $\mathbb{U}_p(\mathfrak{sl}_2)$, obtained from the previous one by adding a square root of K, call it L, and its inverse, and do the same when defining $U_p(\mathfrak{sl}_2)$. Then the new pair $\big(\mathbb{U}_p(\mathfrak{sl}_2), U_p(\mathfrak{sl}_2)\big)$ is again a quantisation of $U(\mathfrak{sl}_2)$, and $U_p(\mathfrak{sl}_2)'$ is like above but for the presence of the new generator L, and the same is when specializing q at 1: thus we get the function algebra of a Poisson group which is a double covering of ${}_aSL_2{}^*$, that is exactly ${}_sSL_2{}^*$. So changing the choice of the QUEA quantizing $\mathfrak{sl}_2$ we get *different* QFA's, one for each of the two connected Poisson algebraic groups dual of SL_2, i.e. having tangent Lie bialgebra $\mathfrak{sl}_2{}^*$; this shows the dependence of the dual group $G^\star$ (here denoted G^* since $\mathfrak{g}^\star = \mathfrak{g}^*$), mentioned in claim of the Theorem, on the choice of the QUEA.

Direct computation proves that $U_p(\mathfrak{sl}_2)'$ is generated by $\dot{F}$, $\dot{H}$, $\dot{\varGamma}$, and $\dot{E}$. This implies that $\big(U_p(\mathfrak{sl}_2)'\big)^\vee$ is generated by $p^{-1}\dot{F} = F$, $p^{-1}\dot{H} = H$, $p^{-1}\dot{\varGamma} = \varGamma$, $p^{-1}\dot{E} = E$, so it coincides with $U_p(\mathfrak{sl}_2)$. An entirely similar analysis works in the "adjoint" case as well.

A like study applies to the general case of the other quantized enveloping algebras associated to semisimple Lie algebras by Jimbo and Lusztig: the outcome, based on the results in [Ga1], is essentially the same. Furthermore, these arguments apply as well to the case of $\mathfrak{g}$ *an untwisted affine Kac-Moody algebra*, basing on the analysis in [Ga2].

Now we consider a QFA for SL_n. Let $F_p[SL_n]$ be the unital associative R–algebra generated by $\big\{ \rho_{ij} \,\big|\, i,j = 1,\dots,n \big\}$ with relations $\rho_{ij}\rho_{ik} = q\,\rho_{ik}\rho_{ij}$, $\rho_{ik}\rho_{hk} = q\,\rho_{hk}\rho_{ik}$ $(\forall\, j<k,\ i<h)$, $\rho_{il}\rho_{jk} = \rho_{jk}\rho_{il}$, $\rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \big(q-q^{-1}\big)\,\rho_{il}\rho_{jk}$ $(\forall\, i<j,\ k<l)$, $det_q(\rho_{ij}) := \sum_{\sigma\in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\rho_{2,\sigma(2)}\cdots\rho_{n,\sigma(n)} = 1$. This is a Hopf algebra, with comultiplication, counit and antipode given by $\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{ik}\otimes\rho_{kj}$, $\epsilon(\rho_{ij}) = \delta_{ij}$, $S(\rho_{ij}) = (-q)^{j-i} det_q\Big(\big(\rho_{hk}\big)_{h\neq j}^{k\neq i}\Big)$ for $i,j = 1,\dots,n$. Let $\mathbb{F}_p[SL_n] := k(q)\otimes_R F_p[SL_n]$. Then $\Big(\mathbb{F}_p[SL_n], F_p[SL_n]\Big)$ is a QFA, with $F_p[SL_n] \xrightarrow{\,q\rightarrow 1\,} F[SL_n]$.

The set of ordered monomials $M := \Big\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} \rho_{hk}^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \,\Big|\, N_{st}\in \mathbb{N}\ \forall\, s,t\,;\ min\big\{N_{1,1},\dots,N_{n,n}\big\} = 0 \Big\}$ is an R–basis of $F_p[SL_n]$ and a $k(q)$–basis of $\mathbb{F}_p[SL_n]$ (cf. [Ga2], Theorem 7.4). From this one argues that $F_p[G]^\vee$ is just the unital R–subalgebra of $\mathbb{F}_p[SL_n]$ generated by $\Big\{ r_{ij} := p^{-1}\big(\rho_{ij}-\delta_{ij}\big) \,\Big|\, i,j = 1,\dots,n \Big\}$. Then one easily gets an explicit presentation of $F_p[G]^\vee$, which shows that it is a Hopf subalgebra (of $\mathbb{F}_p[SL_n]$), and that $F_p[SL_n]^\vee \xrightarrow{\,q\rightarrow 1\,} U\big(\mathfrak{sl}_n{}^*\big)$ as predicted by the Theorem.

We sketch the case of $n=2$ (see also [FG]). Using notation $\mathrm{a} := \rho_{1,1}$, $\mathrm{b} := \rho_{1,2}$, $\mathrm{c} := \rho_{2,1}$, $\mathrm{d} := \rho_{2,2}$, we have the relations $\mathrm{a\,b} = q\,\mathrm{b\,a}$, $\mathrm{a\,c} = q\,\mathrm{c\,a}$, $\mathrm{b\,d} = q\,\mathrm{d\,b}$, $\mathrm{c\,d} = q\,\mathrm{d\,c}$, $\mathrm{b\,c} = \mathrm{c\,b}$, $\mathrm{a\,d} - \mathrm{d\,a} = \big(q-q^{-1}\big)\mathrm{b\,c}$, $\mathrm{a\,d} - q\,\mathrm{b\,c} = 1$ holding in $F_p[SL_2]$ and in $\mathbb{F}_p[SL_2]$, with $\Delta(\mathrm{a}) = \mathrm{a}\otimes\mathrm{a} + \mathrm{b}\otimes\mathrm{c}$, $\Delta(\mathrm{b}) = \mathrm{a}\otimes\mathrm{b} + \mathrm{b}\otimes\mathrm{d}$, $\Delta(\mathrm{c}) = \mathrm{c}\otimes\mathrm{a} + \mathrm{d}\otimes\mathrm{c}$, $\Delta(\mathrm{d}) = \mathrm{c}\otimes\mathrm{b} + \mathrm{d}\otimes\mathrm{d}$, $\epsilon(\mathrm{a}) = 1$, $\epsilon(\mathrm{b}) = 0$, $\epsilon(\mathrm{c}) = 0$, $\epsilon(\mathrm{d}) = 1$, $S(\mathrm{a}) = \mathrm{d}$, $S(\mathrm{b}) = -q^{+1}\mathrm{b}$, $S(\mathrm{c}) = -q^{-1}\mathrm{c}$, $S(\mathrm{d}) = \mathrm{a}$. Then $F_p[SL_2]^\vee$ is generated by the elements $H_+ := r_{1,1} = p^{-1}\big(\mathrm{a}-1\big)$, $E := r_{1,2} = p^{-1}\mathrm{b}$, $F := r_{2,1} = p^{-1}\mathrm{c}$ and $H_- := r_{2,2} = p^{-1}\big(\mathrm{d}-1\big)$ with relations $H_+E = qEH_+ + E$, $H_+F = qFH_+ + F$, $EH_- = q\,H_-E + E$, $FH_- = qH_-F + F$, $EF = FE$, $H_+H_- - H_-H_+ = \big(q-q^{-1}\big)EF$, $H_- + H_+ = p\,\big(q\,EF - H_+H_-\big)$, and Hopf operations given by $\Delta(H_+) = H_+\otimes 1 + 1\otimes H_+ + p\,\big(H_+\otimes H_+ + E\otimes F\big)$, $\epsilon(H_+) = 0$, $S(H_+) = H_-$, $\Delta(E) = E\otimes 1 + 1\otimes E + p\,\big(H_+\otimes E + E\otimes H_-\big)$, $\epsilon(E) = 0$, $S(E) = -q^{+1}E$, $\Delta(F) = F\otimes 1 + 1\otimes F + p\,\big(F\otimes H_+ + H_-\otimes F\big)$, $\epsilon(F) = 0$, $S(F) = -q^{-1}F$, $\Delta(H_-) = H_-\otimes 1 + 1\otimes H_- + p\,\big(H_-\otimes H_- + F\otimes E\big)$, $\epsilon(H_-) = 0$, $S(H_-) = H_+$, from which one gets $F_p[SL_2]^\vee \xrightarrow{\,q\rightarrow 1\,} U(\mathfrak{sl}_2{}^*)$ as co-Poisson Hopf algebras.

Now for the identity $\big(F_p[G]^\vee\big)' = F_p[G]$. From $\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{i,k}\otimes\rho_{k,j}$, we get $\Delta^N(\rho_{ij}) = \sum\limits_{k_1,\dots,k_{N-1}=1}^{n} \rho_{i,k_1}\otimes\rho_{k_1,k_2}\otimes\cdots\otimes\rho_{k_{N-1},j}$, by repeated iteration. Using this and the explicit R–basis of $F_p[SL_n]$ we mentioned above, one proves that $\big(F_p[SL_n]^\vee\big)'$ is the unital R–subalgebra of $\mathbb{F}_p[SL_n]$ generated by $\big\{ p\,r_{ij} \,\big|\, i,j = 1,\dots,n \big\}$; since $p\,r_{ij} = \rho_{ij} - \delta_{ij}$, the latter algebra does coincide with $F_p[SL_n]$, as expected.

The case of any semisimple group G can be dealt with in a different way (cf. [Ga5]).

§ 4 Second example: the Kostant-Kirillov structure

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k, and $\mathfrak{g}^*$ be its dual space. Giving $\mathfrak{g}^*$ the trivial Lie bracket, $\mathfrak{g}$ becomes a Lie bialgebra, and $\mathfrak{g}^*$ a Poisson algebraic (Abelian) group; in particular $F[\mathfrak{g}^*] \cong S(\mathfrak{g})$ (the symmetric algebra), and its Poisson bracket is given by the Lie bracket of $\mathfrak{g}$ extended to the whole of $S(\mathfrak{g})$ via the Leibnitz' rule.

Set $R := k[h]$. Let $\mathfrak{g}_h := \mathfrak{g}[h] = k[h] \otimes_k \mathfrak{g}$, endow it with the unique R–linear Lie bracket $[\ ,\ ]_h$ given by $[x,y]_h := h\,[x,y]$ for all $x, y \in \mathfrak{g}$, and set $H := U(\mathfrak{g}_h)$, endowed with its natural structure of Hopf algebra. Then H is a free R–algebra, so that $H \in \mathcal{HA}$ and $H_F := k(h) \otimes_R H \in \mathcal{HA}_F$ (in the sense of §1); its fibres at $(h-1) \in \mathit{Spec}(R)$, $(h) \in \mathit{Spec}(R)$ (in other words, its specialisations at $h=1$ and at $h=0$) are $H_{(h-1)} = U(\mathfrak{g})$ as a *co-Poisson* Hopf algebra, $H_{(h)} = S(\mathfrak{g}) = F[\mathfrak{g}^*]$ as a *Poisson* Hopf algebra; in a more suggesting way, we can also express this with notation like $H \xrightarrow{h\to 1} U(\mathfrak{g})$, $H \xrightarrow{h \to 0} F[\mathfrak{g}^*]$. Therefore, H is a QUEA at $(h-1)$ *and* it is a QFA at (h); thus now we go and consider Drinfeld's functors for H w.r.t. the prime $(h-1)$ and w.r.t. the prime (h).

Let $(\ )^{\vee(h)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )^{\prime(h)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ be the Drinfeld's functors at $(h)\,\big(\in \mathit{Spec}(k[h])\big)$. Direct computation shows that $H^{\vee(h)} = U\big({\mathfrak{g}_h}^{\vee(h)}\big)$. As a first consequence, $\big(H^{\vee(h)}\big)_{[h]} \cong U\big(\mathfrak{g}_h / h\,\mathfrak{g}_h\big) \cong U(\mathfrak{g})$, as *co-Poisson* Hopf algebras, according to the second half of part *(c)* of the Theorem. Second, since $H^{\vee(h)} = U\big({\mathfrak{g}_h}^{\vee(h)}\big)$, and $\delta_n(\eta) = 0$ for all $\eta \in U\big({\mathfrak{g}_h}^{\vee(h)}\big)$ such that $\partial(\eta) < n$ one finds $\big(H^{\vee(h)}\big)^{\prime(h)} = U\big(h\,{\mathfrak{g}_h}^{\vee(h)}\big) = U\big(h\,h^{-1}\mathfrak{g}_h\big) = U(\mathfrak{g}_h) = H$, so $\big(H^{\vee(h)}\big)^{\prime(h)} = H$, which agrees with part *(b)* of the Theorem. On the other hand, it is easy to see that $H^{\prime(h)} = U(h\,\mathfrak{g}_h)$, whence $\big(H^{\prime(h)}\big)_{[h]} = \big(U(h\,\mathfrak{g}_h)\big)_{[h]} \cong S(\mathfrak{g}_{ab})\ \big(\cong U(\mathfrak{g}_{ab})\big)$ *where* $\mathfrak{g}_{ab}$ *is* $\mathfrak{g}$ *endowed with the trivial Lie bracket*, so $\big(H^{\prime(h)}\big)_{[h]} \cong S(\mathfrak{g}_{ab})\ \big(\cong U(\mathfrak{g}_{ab})\big)$ has trivial Poisson bracket. Iterating, we find that $\Big(\big(\cdots\big((H)^{\prime(h)}\big)^{\prime(h)}\cdots\big)^{\prime(h)}\Big)_{[h]} \cong S(\mathfrak{g}_{ab})$.

Now look at Drinfeld's functors $(\ )^{\vee(h-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )^{\prime(h-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ at $(h-1)\ \big(\in \mathit{Spec}(k[h])\big)$. Set ${\mathfrak{g}_h}^{\prime(h-1)} := (h-1)\,\mathfrak{g}_h$, let $\ : \mathfrak{g}_h \xrightarrow{\cong} {\mathfrak{g}_h}^{\prime(h-1)}$ be the $k[h]$–module isomorphism given by $z \mapsto z' := (h-1)\,z \in {\mathfrak{g}_h}^{\prime(h-1)}$, and push over via it the Lie bialgebra structure of $\mathfrak{g}_h$ to an isomorphic Lie bialgebra structure on ${\mathfrak{g}_h}^{\prime(h-1)}$, whose Lie bracket will be denoted by $[\ ,\ ]_*$. Then ${\mathfrak{g}_h}^{\prime(h-1)} \big/ (h-1)\,{\mathfrak{g}_h}^{\prime(h-1)} \cong \mathfrak{g}_h \big/ (h-1)\mathfrak{g}_h \cong \mathfrak{g}$ as Lie bialgebras. Then direct computation yields $H^{\prime(h-1)} = U\big((h-1)\,\mathfrak{g}_h\big) = U\big({\mathfrak{g}_h}^{\prime(h-1)}\big)$, where ${\mathfrak{g}_h}^{\prime(h-1)}$ is considered as a Lie $k[h]$–subalgebra of $\mathfrak{g}_h$. Now, if $x', y' \in {\mathfrak{g}_h}^{\prime(h-1)}$ we have $x'\,y' - y'\,x' = (h-1)\,\big[x',y'\big]_*$, therefore $\big(H^{\prime(h-1)}\big)_{(h-1)} = S\big({\mathfrak{g}_h}^{\prime(h-1)} \big/ (h-1)\,{\mathfrak{g}_h}^{\prime(h-1)}\big)$ as *Poisson Hopf* algebras. Finally, since ${\mathfrak{g}_h}^{\prime(h-1)} \big/ (h-1)\,{\mathfrak{g}_h}^{\prime(h-1)} \cong \mathfrak{g}$ as Lie algebras we have $\big(H^{\prime(h-1)}\big)_{(h-1)} = S(\mathfrak{g}) = F[\mathfrak{g}^*]$ as Poisson Hopf algebras, as claimed in part *(c)* of the Theorem. Second, since $H^{\prime(h-1)} = U\big({\mathfrak{g}_h}^{\prime(h-1)}\big)$, we have that $\big(H^{\prime(h-1)}\big)^{\vee(h-1)}$ is just the unital $k[h]$–subalgebra of H_F generated by $(h-1)^{-1}{\mathfrak{g}_h}^{\prime(h-1)} = (h-1)^{-1}(h-1)\,\mathfrak{g}_h = \mathfrak{g}_h$, that is to say $\big(H^{\prime(h-1)}\big)^{\vee(h-1)} = U(\mathfrak{g}_h) = H$, according to part *(b)* of the Theorem.

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