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“A Brauer algebra theoretic proof of Littlewood’s restriction rules”

INTRODUCTION

“Non potrai dir che quest’è cosa dura:
usando la dualità di Brauer
dimostrazione dar, novella e pura”

N. Barbecue, “Scholia”

Let $U$ be a complex vector space, endowed with an orthogonal or symplectic form, and let $G$ be either $O(U)$ or $Sp(U)$ respectively. Consider a simple polynomial $GL(U)$–module $V_\lambda$ (associated in a standard way to a partition $\lambda$), and restrict it to $G$. If $\lambda'_1 + \lambda'_2 \leq \dim(U)$, in the orthogonal case ($\lambda'$ being the dual partition to $\lambda$), or $\lambda'_1 \leq \dim(U)/2$, in the symplectic case, then its decomposition into simple $G$–modules is described by the Littlewood’s restriction rule (cf. [L]), which gives a formula for the multiplicity in $V_\lambda$ of each simple $G$–module. The main aim in this article is to prove this formula.

It is well known (cf. e.g. [W], [H]) that one can realize a copy of $V_\lambda$ inside the tensor power $U^{\otimes f}$, where $f$ is the sum of parts of $\lambda$ (i.e. $\lambda$ is a partition of $f$). By the general theory of centralizer algebras, a bijection $V_\lambda \leftrightarrow M_\lambda$ exists between simple $GL(U)$–modules and simple modules over $End_{GL(U)}(U^{\otimes f})$ — the centralizer algebra of the $GL(U)$–action on $U^{\otimes f}$ — occurring in $U^{\otimes f}$, which interchanges dimensions and multiplicities. Similarly, a bijection $W_\mu \leftrightarrow N_\mu$ exists between simple $G$–modules and simple modules over $End_G(U^{\otimes f})$ — the centralizer algebra of the $G$–action on $U^{\otimes f}$ — occurring in $U^{\otimes f}$ (which is now thought of as a $G$–module), which interchanges dimensions and multiplicities. Then we have an identity $[V_\lambda : W_\mu] = [N_\mu : M_\lambda]$, thus to get the multiplicity $[V_\lambda : W_\mu]$ we can compute the above right-hand-side term instead: in other words, instead of studying $V_\lambda|_{GL(U)} \big|_G$ we study $N_\mu|_{End_{GL(U)}(U^{\otimes f})}^{End_G(U^{\otimes f})}$. Therefore, if

$$[V_\lambda : W_\mu] = C_\mu^\lambda$$

is the identity given in Littlewood’s restriction formula, our aim is to prove that

$$[N_\mu : M_\lambda] = C_\mu^\lambda$$

Now, one has that $End_{GL(U)}(U^{\otimes f}) = \mathbb{C}[S_f]$, with $S_f$ acting on $U^{\otimes f}$ by index permutation. On the other hand, $End_G(U^{\otimes f})$ is a quotient of the Brauer algebra $B_f(eN)$,
where \( N = \text{dim}_C(U) \) and \( \epsilon \) is the “sign” of the form on \( U \) (“+” for orthogonal and “−” for symplectic case); the kernel of \( \pi_U : \mathcal{B}_f^{(eN)} \to \text{End}_G(U \otimes f) \) is also known, essentially from the Second Fundamental Theorem of Invariant Theory (for the group \( G \)). In the stable case (i.e. when \( f \leq N/2 \) in the symplectic case and \( f \leq N \) in the orthogonal case) \( \pi_U \) is an isomorphism, and Littlewood’s formula can be proved as a corollary of a suitable description of \( V \otimes f \) (cf. [GP]). In the general case a different approach is necessary.

To describe \( \mathcal{B}_f^{(x)} \) we can display an explicit basis \( D_f \) — whose elements are certain graphs — and assign the multiplication rules for elements in this basis — based on “composition” of graphs. Then from the previously mentioned description of \( \text{Ker}(\pi_U) \) we take out an explicit set of linear generators of this kernel.

In addition, the simple \( G \)-modules \( N_\mu \) are quotients of certain \( \mathcal{B}_f^{(eN)} \)-modules \( N'_\mu \) which have a nice combinatorial description (in terms of graphs related to those of \( D_f \)); moreover, we prove that the kernel of the epimorphism \( N'_\mu \to N_\mu \) is just \( \text{Ker}(\pi_U).N'_\mu \). Now, the multiplicity \([N'_\mu : M_\lambda]\) is exactly equal to the right-hand-side part of (⋆); then it is enough for us to show that in \( \text{Ker}(\pi_U).N'_\mu \), as a \( \mathbb{C}[S_f] \)-module, there are no components of type \( M_\lambda \) for \( \lambda \) such that \( \lambda^1_1 + \lambda^2_2 \leq \text{dim}(U) \) (in the orthogonal case) or \( \lambda^1_1 \leq \text{dim}(U)/2 \) (in the symplectic case). We deduce this fact from the previous description of \( \text{Ker}(\pi_U) \).

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References


