

TEORIA delle RAPPRESENTAZIONI 1

- 8 CFU -

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1 - FOUNDATIONS of the THEORY of LIE ALGEBRAS: Algebras, associative algebras, Lie algebras. Subalgebras, ideals. Morphism of algebras. Operations between subalgebras and between ideals. The classical Lie algebras. Correspondence Theorems (for morphisms between algebras). Fundamental Theorem of Homomorphism. Isomorphism Theorems.

2 - SPECIAL CLASSES of LIE ALGEBRAS: Simple Lie algebras. Derived subalgebra; the derived series, the lower central series. Nilpotent Lie algebras; Engel's Theorem. Solvable Lie algebras; Lie's Theorem. The radical of a Lie algebra. Semisimple Lie algebras. Jordan-Chevalley decomposition for semisimple Lie algebras. The Killing form; Cartan's criterion (for solvability); criteria for semisimplicity. Structure of a semisimple Lie algebra.

3 - MODULES and REPRESENTATIONS: Modules (or "representations"), submodules, quotients, morphisms; Fundamental Theorem of Homomorphism, Isomorphism Theorems; direct products and direct sums; modules of morphisms, dual module. Tensor product and direct sums between modules. Simple or semisimple modules. Finite-dimensional modules for $\mathfrak{sl}(2)$: structure and classification.

4 - UNIVERSAL ENVELOPING ALGEBRAS: The universal enveloping algebra of a Lie algebra. The Poincaré-Birkhoff-Witt Theorem. The functorial identification between L -modules and $U(L)$ -modules. The isomorphism between $U(L' \oplus L'')$ and $U(L') \otimes U(L'')$. The structure of Hopf algebra for $U(L)$ and the tensor structure in the category of L -modules.

5 - STRUCTURE of SEMISIMPLE LIE ALGEBRAS: Cartan subalgebras (in a semisimple Lie algebra). Roots, root spaces and Cartan decomposition; $\mathfrak{sl}(2)$ -triples. Properties of root vectors and coroots. The root system associated with a semisimple Lie algebra and a Cartan subalgebra of it. Invariance of Killing form with respect to isomorphisms. The Conjugacy Theorem for Cartan subalgebras. An isomorphism between semisimple Lie algebras induces an isomorphism between the associated root systems. The decomposition into direct sum of simple ideals corresponds to the decomposition of the root system into union of irreducible (sub)systems.

6 - ABSTRACT (FINITE) ROOT SYSTEMS: Abstract (finite) root systems. Weyl group of a root system. Root strings. Simple roots; bases of simple roots. Existence Theorem of Bases. Weyl chambers. The Weyl group action on the set of bases and the set of Weyl chambers. The Cartan matrix and Dynkin diagram of a root system. Irreducible root systems. Classification of irreducible root systems.

7 - CONSTRUCTION of SEMISIMPLE LIE ALGEBRAS from ROOT SYSTEMS: Constructions of algebraic systems by generators and relations. Standard sets of generators for a semisimple Lie algebra; relations enjoyed by a standard set of generators. Serre's Theorem: construction of a semisimple Lie algebra with a specified Lie semisimple root system. Classification of semisimple Lie algebras.

8 - MODULES & REPRESENTATIONS of a SEMISIMPLE LIE ALGEBRA: Finite-dimensional modules of a semisimple Lie algebra: weight spaces, complete reducibility. Highest weight modules: structure, existence and uniqueness. Classification of all finite-dimensional modules of a semisimple Lie algebra.
