A NOTE ON GONALITY OF CURVES ON GENERAL HYPERSURFACES

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Abstract. This short paper concerns the existence of curves with low gonality on smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$. After reviewing a series of results on this topic, we report on a recent progress we achieved as a product of the Workshop *Birational geometry of surfaces*, held at University of Rome "Tor Vergata" on January $11^{\text{th}} - 15^{\text{th}}$, 2016. In particular, we obtained that if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geqslant 2n+2$, the least gonality of a curve $C \subset X$ passing through a general point of X is $\text{gon}(C) = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$, apart from some exceptions we list.

Introduction

A natural way to investigate a projective variety X is to describe the geometry of its subvarieties, both when they are rigid in X, and when they deform in families. Basic questions concern for instance the existence of subvarieties of given invariants, especially when they have either dimension or codimension equal to one.

In this paper we are interested in studying the gonality of curves lying on general hypersurfaces $X \subset \mathbb{P}^{n+1}$ of sufficiently large degree. In Sections 1 and 2 we shall briefly review a series of results on this topic. Then, we shall report on a recent result governing the least gonality of a curve passing through the general point of X (cf. Theorem 3.3).

Our interest on this subject arose from discussions we had during the Workshop *Birational geometry of surfaces*, held on January $11^{\text{th}} - 15^{\text{th}}$, 2016 at University of Rome "Tor Vergata" among us, and with other partecipants. So, we are pleased to thank the Organizers for their invitation, and Concettina Galati, Margarida Melo, and Duccio Sacchi for helpful discussions.

Notation. We work throughout this paper over the field $\mathbb{C}$ of complex numbers. By *variety* we mean a complete reduced and irreducible algebraic variety X over $\mathbb{C}$, unless otherwise stated. By *curve* we intend a variety of dimension 1. We say that a property holds for a *general* (resp. *very general*) point $x \in X$ if it holds on a Zariski open non-empty subset of X (resp. outside the union of a countable collection of proper subvarieties of X).

1. Rational curves on hypersurfaces

The study of rational curves on projective varieties is an important issue in many areas of algebraic geometry, such as the birational classification of algebraic varieties with its incarnation in the minimal model program, and enumerative geometry in its modern manifestation of Gromov-Witten theory.

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We assume throughout the paper that $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d, so that its canonical bundle satisfies $K_X \cong \mathcal{O}_X(d-n-2)$. If $d < n+2$, then X is a Fano variety, and it is uniruled. On the other hand, X is no longer swept out by rational curves as soon as $d \geqslant n+2$ because of the nefness of K_X, although rational curves may still cover subvarieties of X having large dimension.

The study of rational curves on hypersurfaces has been carried out also with respect to the degrees of the curves. In this direction, there are classical results dating back to Morin, B. Segre and Predonzan regarding the existence of lines on complete intersections $Y \subset \mathbb{P}^{n+1}$ and determining for general Y the dimension of the Fano scheme $F_1(Y)$ in the Grassmannian $\mathbb{G}(1, n+1)$, which parametrizes lines on Y (see [39, 47, 45]). Actually, these results concern the existence of linear subvarieties of arbitrary dimension, and they have been extended in more recent times in various directions (see e.g. [8, 17]). In particular, for hypersurfaces one has (see [33, Theorem V.4.3])

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \leqslant 2n-1$. Then X contains lines and, if in addition X is assumed to be very general, then the Fano scheme $F_1(X) \subset \mathbb{G}(1, n+1)$ is smooth of dimension $2n-1-d$.*

Of course, there exist smooth hypersurfaces of any degree and dimension that contain lines, so in particular, they contain rational curves. On the other hand, a series of seminal results by Clemens, Ein, and Voisin ensures that, out of the range of Theorem 1.1, very general hypersurfaces do not contain rational curves (cf. [42]). Namely,

Theorem 1.2. *Let $X \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geqslant 2n$. Then X does not contain rational curves.*

The latter theorem has been proved by Clemens [16] for $d \geqslant 2n+1$. Then it has been extended by Ein [19, 20] to complete intersections in smooth arbitrary varieties, whereas the weaker hypothesis $d \geqslant 2n$ follows from the work of Voisin [49, 50]. In the boundary case $d = 2n-1$, it follows from [49] that if $n \geqslant 4$, there exist finitely many rational curves of fixed degree in X. Pacienza showed in [42] that for $n \geqslant 5$ the very general hypersurface of degree $2n-1$ does not contain rational curves other than lines (which exist by Theorem 1.1).

The basic, important idea used to prove Theorem 1.2 and its variant for complete intersections consists in a fine analysis of positivity of suitable vector bundles on universal families of hypersurfaces, or complete intersections. The proof of the result announced in this paper also rely on the same ideas and techniques.

2. Irregular curves on algebraic varieties

Apart from rational curves, it is natural to wonder how a curve into a given variety is far from being rational. In this direction there are two natural invariants measuring the irrationality of a curve: its geometric genus and its gonality.

The existence of curves with low genus on particular classes of varieties has been investigated, as for instance, on Kummer [44], Prym [1, 41], and Jacobian varieties [36].

Turning to smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d, the case of plane curves is trivial, since the degree and the genus g are related by the well-known formula $g = \frac{(d-1)(d-2)}{2}$. The case of surfaces in $\mathbb{P}^3$ is more subtle. If $d \leqslant 3$, X carries curves of any genera, as it is rational. For $d = 4$,

X is a K3 surface and again any g can occur (cf. e.g. [13, Corollary 2.2]). The case of degree $d \geqslant 5$ is covered by the following result due to Xu ([51, Theorem 1]).

Theorem 2.1. *Let $X \subset \mathbb{P}^3$ be a very general surface of degree $d \geqslant 5$. Then, there is no curve on X with geometric genus $g \leqslant \frac{d(d-3)}{2} - 3$. Moreover, curves cut out on X by tri–tangent planes (and only them, if $d \geqslant 6$) achieve this bound.*

A similar result has been obtained by Chiantini and Lopez [9]. Moreover, a detailed study of the occurrence of the existence of genus g curves on a very general surface in $X \subset \mathbb{P}^3$ has been worked out by Ciliberto, Flamini and Zaidenberg in [13]. They proved that on a very general X, if the degree is $d \geqslant 5$, no curve of geometric genus $\frac{d^2-3d+4}{2} \leqslant g \leqslant d^2 - 2d - 9$ can occur. Furthermore, there exists a bound G_d such that X carries a curve of geometric genus g for any $g \geqslant G_d$. These results have been then extended in [14] to possible values for geometric genera of subvarieties Y of any smooth projective variety X of arbitrary dimension.

Concerning higher dimensional hypersurfaces $X \subset \mathbb{P}^{n+1}$, it is worth noting that the results [11, 19, 20, 49, 42] mentioned in the previous section on the existence of rational curves are more general. Indeed, they give sufficient conditions for the canonical bundle of the desingularization of any irreducible subvariety of X to be effective. As a consequence, they yield that the geometric genus of these subvarieties turns out to be sufficiently positive. We would also like to stress that these results are in the same line of the following earlier one by Clemens in [16], on families of *immersed* curves.

Theorem 2.2. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2$. Then X does not admit any irreducible family $\mathcal{C}$ of* immersed curves *(i.e. smooth curves C endowed with a finite morphism $C \xrightarrow{\nu} X$ which is everywhere of maximal rank) of (geometric) genus g which covers a subvariety of codimension less than D, where $D := \frac{2-2g}{\deg \nu} + d - (n+2)$.*

A notable extension by Pacienza (see [43]) gives sufficient conditions for a k-dimensional subvariety Y in a very general hypersurface $X \subset \mathbb{P}^{n+1}$ to be contained in the locus covered by the lines of X. In particular, if $n \geqslant 5$ and $d \geqslant 2n$, it turns out that any subvariety of X is of general type (cf. [43, Corollary 1.2]).

As we pointed out, another measure for the irrationality of a given curve C is the *gonality*, which is defined as

$$\operatorname{gon}(C) := \min \left\{ c \in \mathbb{N} \,\middle|\, \exists\, C \dashrightarrow \mathbb{P}^1 \text{ dominant of degree } c \right\}.$$

It is well known that gonality and genus are related by the inequality $\operatorname{gon}(C) \leqslant \left\lfloor \frac{g+3}{2} \right\rfloor$, and there is a stratification of the moduli space $\mathcal{M}_g$ of smooth curves of genus g given by gonality.

In his groundbreaking paper [34], Lazarsfeld proved among other things that smooth curves C in $|\mathcal{O}_X(1)|$, with X a smooth K3 surface of genus $p \geqslant 3$ and $\operatorname{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$, behave generically from the point of view of Brill–Noether theory (despite the fact that these curves have general moduli only if $p \leqslant 11$ and $p \neq 10$; cf. e.g. [24] for an overview). Thus, any such a curve C has geometric genus p and gonality $\operatorname{gon}(C) = \left\lfloor \frac{p+3}{2} \right\rfloor$.

In [15, 22, 23] the authors consider the problem of extending Lazarsfeld's results to singular curves in $|\mathcal{O}_X(1)|$, computing the gonality of their normalizations. The results in [15] improve those in [22, 23], by giving a necessary condition on p, g and k for the existence of an irreducible, possibly singular, curve $C \in |\mathcal{O}_X(1)|$ of geometric genus g whose normalization has a $\mathfrak{g}^1_k$. For all admissible cases, a family of expected dimension of nodal curves in $|\mathcal{O}_X(1)|$ with geometric genus g, carrying a $\mathfrak{g}^1_k$ on their normalizations, has been exhibited.

Similar problems have been investigated in different settings, for instance on rational and ruled surfaces [3, 10, 37, 48], Enriques surfaces [31, 32], elliptic K3's [46], toric surfaces [29], as well as abelian ones [44, 30].

Concerning the gonality of curves on hypersurfaces of $\mathbb{P}^{n+1}$, not much was known till very recently, except in particular cases. Basili [4] has proven that for a smooth complete intersection of two surfaces in $\mathbb{P}^3$, the gonality is computed by the linear series cut out on the curve by a pencil of planes off the intersection with a base line, which is a maximal multisecant line to the curve. The same result holds for any curve on smooth quadrics in $\mathbb{P}^3$ (see [2, 37]), while a result on curves on a certain quartic surface in $\mathbb{P}^3$ is due to Farkas [21]. An extended review on curves whose gonality is computed by multisecant lines can be found in [28].

In the case of very general hypersurfaces of general type, the most complete result is the following (cf. [7, Proposition 3.8]):

Theorem 2.3. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n$.*
If $C \subset X$ is an irreducible curve, then

$$\mathrm{gon}(C) \geqslant d - 2n + 1.$$

More generally, if X contains an irreducible subvariety $S \subseteq X$ of dimension $s > 0$ covered by a family of irreducible curves whose general element has gonality c, then

$$c \geqslant d - 2n + s.$$

3. Covering gonality of general hypersurfaces

Given an n-dimensional projective variety X, we are interested in determining the least gonality of curves passing through a general point $x \in X$. This is a birational invariant introduced in [7], called the *covering gonality* of X, and it can be equivalently defined in terms of families of curves covering X as

$$\mathrm{cov.\,gon}(X) := \min \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists \text{ a family } \mathcal{C} \stackrel{\pi}{\longrightarrow} T \text{ of irreducible curves } C_t := \pi^{-1}(t) \\ \text{and a dominant morphism } f\colon \mathcal{C} \longrightarrow X \text{ such that} \\ \text{for general } t \in T,\ \mathrm{gon}(C_t) = k \text{ and } f_{|C_t} \text{ is birational} \end{array} \right\}.$$

A family of curves as above is said to be a *covering family* of k-gonal curves.

General results on this invariant have been recently proved (cf. [7]). Furthermore, the covering gonality of specific classes of varieties has been computed (see e.g. [35] concerning smooth surfaces in $\mathbb{P}^3$, and [5] on second symmetric products of curves).

In the case of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of large degree, the covering gonality satisfies the following bound (cf. [7, Theorem A]).

Theorem 3.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant n + 2$. Then*

$$\mathrm{cov.\,gon}(X) \geqslant d - n. \tag{3.1}$$

When $n = 1$, X is a plane curve of degree $d \geqslant 3$, hence the gonality and the covering gonality coincide. By a classical result already known to M. Noether, we have $\operatorname{cov.gon}(X) = d - 1$, and any map $X \dashrightarrow \mathbb{P}^1$ of degree $d-1$ is the projection from some point $x \in X$ (see [12, Teorema 3.14]; cf. also [27, Theorem 2.1]).

When $X \subset \mathbb{P}^3$ is a smooth surface of degree $d \geqslant 4$, the situation is more complicated, but the problem is still totally understood, i.e. $\operatorname{cov.gon}(X) = d-2$. Indeed, if X is a smooth quartic in $\mathbb{P}^3$, then X is a K3 surface, and Bogomolov-Mumford's theorem states that any such a surface contains at most countably many rational curves and it is covered by singular elliptic curves (cf. [38, p.351]), thus $\operatorname{cov.gon}(X) = d-2 = 2$. The case $d \geqslant 5$ has been treated by Lopez and Pirola in [35], where they proved that $\operatorname{cov.gon}(X) = d-2$ and classified all the covering families attaining this value (cf. [35, Corollary 1.7]). In particular, all but one of the constructions of families computing the covering gonality rely on the existence of some rational or elliptic curve on X, which do not occur when X is assumed to be a very general surface in $\mathbb{P}^3$ (cf. Theorem 2.1). Thus the following holds (see [35, Corollary 1.8]).

Theorem 3.2. *Let $X \subset \mathbb{P}^3$ be a very general surface of degree $d \geqslant 5$. Then*

$$\operatorname{cov.gon}(X) = d-2,$$

and any covering family $\mathcal{C} \xrightarrow{\pi} T$ of $(d-2)$-gonal curves consists of tangent plane sections, i.e. a general fibre $C_t := \pi^{-1}(t)$ is a plane curve cut out on X by the tangent plane $T_xX \subset \mathbb{P}^3$ at some point $x \in X$.

In the above setting, the $(d-2)$-gonal map $C_t \dashrightarrow \mathbb{P}^1$ is the projection of the plane curve C_t from its singular point $x \in C_t$.

As far as higher dimensional hypersurfaces are concerned, we are able to prove the following:

Theorem 3.3. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$. Then*

$$d - \left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor \leqslant \operatorname{cov.gon}(X) \leqslant d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor, \qquad (3.2)$$

and the equality to the right hand side holds for any $n \in \mathbb{N} \smallsetminus \left\{ 4\alpha^2 + 3\alpha,\ 4\alpha^2 + 5\alpha + 1 \,\middle|\, \alpha \in \mathbb{N} \right\}$.

The second part of the statement simply follows from the fact that $\left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor = \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$ if and only if $n \notin \left\{ 4\alpha^2 + 3\alpha,\ 4\alpha^2 + 5\alpha + 1 \,\middle|\, \alpha \in \mathbb{N} \right\}$. On the other hand, if $n = 4\alpha^2 + 3\alpha$ or $n = 4\alpha^2 + 5\alpha + 1$ for some $\alpha \in \mathbb{N}$, then the two bounds in (3.2) differ by one.

Note that this is a substantial improvement of Theorem 3.1 when X is assumed to be very general.

Remark 3.4. Actually, the hypothesis of Theorem 3.1 can be weakened by assuming $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant n+2$ having only canonical singularities (see [7, Corollary 1.11]). Moreover, under this less restrictive assumption, the inequality (3.1) is sharp for any dimension n. Indeed, a hypersurface $X \subset \mathbb{P}^{n+1}$ with an ordinary singular point of multiplicity n does satisfy $\operatorname{cov.gon}(X) = d - n$ (cf. [7, Example 1.7 (v)]).

The complete proof of Theorem 3.3 will appear in a forthcoming paper. Here we give a short sketch of it. We combine various techniques coming from several of the aforementioned papers, such as [6, 7, 11, 19, 20, 35, 42, 49].

A crucial aspect involved in the proof is the description of the geometry of k-gonal curves $C \subset X$ passing through a general point of X. In this direction, we argue as in [5, Section 4] rephrasing the problem in terms of correspondences with null trace on X and using Mumford's technique of induced differentials [40]. Then, [6, Theorem 2.5] assures that if $k := \mathrm{gon}(C) \leqslant d-2$, the fibers of a degree k map $\phi\colon C \dashrightarrow \mathbb{P}^1$ consist of collinear points. Furthermore, by combining the approaches of [6] and [7], we prove that there exists a point $x \in X$ such that the curve C lies on the cone $T_x^{d-k}X \subset T_xX$ of lines having multiplicity of intersection with X at x of order at least $d-k$. In particular, the map $\phi\colon C \dashrightarrow \mathbb{P}^1$ is obtained by projection from $x \in T_x^{d-k}X$ to some rational curve lying on a general hyperplane section of $T_x^{d-k}X$. Vice versa, given an integer $h \leqslant d-2$, a point $x \in X$, and a rational curve R contained in a general hyperplane section of $T_x^{d-h}X$, the cone over R with vertex at x cuts out on X an irreducible curve C satisfying $\mathrm{gon}(C) \leqslant h$.

Recall that the cone $T_x^{d-h}X$ is scheme theoretically defined as the complete intersection of the polars of degrees $1, \dots, d-h-1$ of x with respect to X. So, we slightly extend Predonzan's result [11, Theorem 2.1] on the existence of linear spaces in complete intersections of $\mathbb{P}^{n+1}$, and deduce that, for $h \geqslant d - \frac{\sqrt{16n+1}-1}{2}$, there exists a $(n-1)$-dimensional family of cones $T_x^{d-h}X$ each containing a line ℓ_x not belonging to the ruling. Thus, the linear span of x and ℓ_x intersects X along a plane curve C admitting a degree h map $C \dashrightarrow \ell_x \cong \mathbb{P}^1$ given by projection from x. Therefore this construction gives rise to a covering family of curves with gonality at most h, where we may set $h = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$, so that the upper bound in (3.2) is achieved.

The lower bound in Theorem 3.3 is in turn obtained by following the argument of Pacienza [42], which is based on the vector bundle approach developed by Ein and Voisin in [19, 20, 49]. We assume k to be the covering gonality of X and, in the universal family over the Grassmannian $\mathbb{G}(1, n+1)$ of lines in $\mathbb{P}^{n+1}$, we consider the locus $\widetilde{\Delta}_{d-k} \subset \mathbb{P}^{n+1} \times \mathbb{G}(1, n+1)$ of pairs $(x, [\ell])$ such that the line $\ell \subset \mathbb{P}^{n+1}$ is tangent to X at x with intersection multiplicity at least $d-k$. By the above discussion, any covering family of k-gonal curves on X induces a $(n-1)$-dimensional subvariety of $\widetilde{\Delta}_{d-k}$ swept out by rational curves. By studying the positivity properties of the canonical bundle of $\widetilde{\Delta}_{d-k}$, we determine necessary conditions on k for the existence of such a subvariety. Namely, we obtain $k \geqslant d - \frac{\sqrt{16n+9}-1}{2}$, which leads to the lower bound.

To conclude, we would like to point out a couple of open problems which naturally arise from the above analysis. The first is to determine the covering gonality of very general hypersurfaces in $\mathbb{P}^{n+1}$ for the remaining values of n, that is $n \in \left\{ 4\alpha^2 + 3\alpha,\ 4\alpha^2 + 5\alpha + 1 \,\middle|\, \alpha \in \mathbb{N} \right\}$. In analogy with the cases where the covering gonality was computed, we conjecture that $\mathrm{cov.gon}(X) = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$.

Secondly, there is the problem of characterizing the families of curves for which the covering gonality bound is achieved. In particular, it would be interesting to understand whether for $n \gg 0$, these families consist only of planar sections.

References

[1] A. Alzati, G.P. Pirola, On abelian subvarieties generated by symmetric correspondences, *Math. Z.* **205** (1990), 333–342.

[2] E. Ballico, On the gonality of curves in $\mathbb{P}^n$, *Comment. Math. Univ. Carolin.* **38** (1997), 177–186.

[3] E. Ballico, Gonality of reduced curves on a smooth surface with generically spanned anticanonical line bundle, *Int. J. Pure Appl. Math.* **55** (2009), 105–108.

[4] B. Basili, Indice de Clifford des intersections complètes de l'espace, *Bull. Soc. Math. France* **124** (1996), 61–95.

[5] F. Bastianelli, On symmetric products of curves, *Trans. Amer. Math. Soc.* **364** (2012), 2493–2519.

[6] F. Bastianelli, R. Cortini, and P. De Poi, The gonality theorem of Noether for hypersurfaces, *J. Algebraic Geom.* **23** (2014), 313–339.

[7] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, B. Ullery, Measures of irrationality for hypersurfaces of large degree, arXiv:1511.01359 (2016).

[8] C. Borcea, Deforming varieties of k-planes of projective complete intersections, *Pacific J. Math.* **143** (1990), 25–36.

[9] L. Chiantini, A. F. Lopez, Focal loci of families and the genus of curves on surfaces, *Proc. Amer. Math. Soc.* **127** (1999), 3451–3459.

[10] Y. Choi, S. Kim, Gonality and Clifford index of projective curves on ruled surfaces, *Proc. Amer. Math. Soc.* **140** (2012), 393–402.

[11] C. Ciliberto, Osservazioni su alcuni classici teoremi di unirazionalità per ipersuperficie e complete intersezioni algebriche proiettive, *Ricerche Mat.* **29** (1980), 175–191.

[12] C. Ciliberto, Alcune applicazioni di un classico procedimento di Castelnuovo, *Seminari di geometria, 1982-1983 (Bologna, 1982/1983)*, Univ. Stud. Bologna, Bologna, 1984, 17–43.

[13] C. Ciliberto, F. Flamini, M. Zaidenberg, Genera of curves on a very general surface in $\mathbb{P}^3$, *Int. Math. Res. Not. IMRN* **22** (2015), 12177–12205.

[14] C. Ciliberto, F. Flamini, M. Zaidenberg, Gaps for geometric genera, *Arch. Math. (Basel)* **106** (2016), 531–541.

[15] C. Ciliberto, A. L. Knutsen, On k-gonal loci in Severi varieties on general K3 surfaces and rational curves on hyperkähler manifolds, *J. Math. Pures Appl.* **101** (2014), 473–494.

[16] H. Clemens, Curves on generic hypersurfaces, *Ann. Sci. École Norm. Sup.* **19** (1986), 629–636.

[17] O. Debarre L. Manivel, Sur la varieté des espaces lineaires contenus dans une intersection complete, *Math. Ann.* **312** (1998), 549–574.

[18] P. De Poi, On first order congruences of lines of $\mathbb{P}^4$ with a fundamental curve, *Manuscripta Math.* **106** (2001), 101–116.

[19] L. Ein, Subvarieties of generic complete intersections, *Invent. Math.* **94** (1988), 163–169.

[20] L. Ein, Subvarieties of generic complete intersections II, *Math. Ann.* **289** (1991), 465–471.

[21] G. Farkas, Brill-Noether loci and the gonality stratification of M_g, *J. Reine Angew. Math.* **539** (2001), 185–200.

[22] F. Flamini, A.L. Knutsen, G. Pacienza, Singular curves on a K3 surface and linear series on their normalizations, *Internat. J. Math.* **18** (2007), 671–693.

[23] F. Flamini, A.L. Knutsen, G. Pacienza, On families of rational curves in the Hilbert square of a surface (with an appendix by Edoardo Sernesi), *Michigan Math. J.* **58** (2009), 639–682.

[24] F. Flamini, A.L. Knutsen, G. Pacienza, E. Sernesi, Nodal curves with general moduli on K3 surfaces, *Comm. Algebra* **36** (2008), 3955–3971.

[25] J. Harris, B. Mazur, R. Pandharipande, Hypersurfaces of low degree, *Duke Math. J.* **95** (1998), 125–160.

[26] J. Harris, M. Roth, J. Starr, Rational curves on hypersurfaces of low degree *J. Reine Angew. Math.* **571** (2004), 73–106.

[27] R. Hartshorne, Clifford index of ACM curves in $\mathbb{P}^3$, *Milan J. Math.* **70** (2002), 209–221.

[28] R. Hartshorne, E. Schlesinger, Gonality of a general ACM curve in $\mathbb{P}^3$, *Pacific J. Math.* **251** (2011), 269–313.

[29] R. Kawaguchi, The gonality and the Clifford index of curves on a toric surface, *J. Algebra* **449** (2016), 660–686.

[30] A.L. Knutsen, M. Lelli–Chiesa, G. Mongardi, Severi Varieties and Brill-Noether theory of curves on abelian surfaces, *J. Reine Angew. Math.*, to appear.

[31] A.L. Knutsen, A.F. Lopez, Brill–Noether theory for curves on Enriques surfaces, I: the positive cone and gonality, *Math. Z.* **261** (2009), 659–690.

[32] A.L. Knutsen, A.F. Lopez, Brill-Noether theory of curves on Enriques surfaces, II. The Clifford index, *Manuscripta Math.* **147** (2015), 193–237.

[33] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, vol. 32, Springer–Verlag, Berlin, 1996.

[34] R. Lazarsfeld, Brill-Noether-Petri without degenerations, *J. Differential Geom.* **23** (1986), 299-307.

[35] A.F. Lopez, P. Pirola, On the curves through a general point of a smooth surface in $\mathbb{P}^3$, *Math. Z.* **19** (1995), 93–106.

[36] V.O. Marcucci, On the genus of curves in a Jacobian variety, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **12** (2013), 735–754.

[37] G. Martens, The gonality of curves on a Hirzebruch surface, *Arch. Math. (Basel)* **67** (1996), 349–352.

[38] S. Mori, S. Mukai, The uniruledness of the moduli space of curves of genus 11, in *Algebraic geometry (Tokyo/Kyoto, 1982)*, Lecture Notes in Math., vol. 1016, Springer–Verlag, Berlin, 1983, 334–353.

[39] U. Morin, Sull'insieme degli spazi lineari contenuti in una ipersuperficie algebrica, *Atti Accad. Naz. Lincei, Rend., Cl. Sci. Fis. Mat. Nat.* **24**, 1936, 188–190.

[40] D. Mumford, Prym varieties I, in *Contributions to analysis* (L. V. Alfors et al. eds.), Academic Press, New York, 1974, 325–350.

[41] J. C. Naranjo, G. P. Pirola, On the genus of curves in the generic Prym variety, *Indag. Math. (N.S.)* **5** (1994), 101–105.

[42] G. Pacienza, Rational curves on general projective hypersurfaces, *J. Algebraic Geom.* **12** (2003), 245–267.

[43] G. Pacienza, Subvarieties of general type on a general projective hypersurface, *Trans. Amer. Math. Soc.* **356** (2004), 2649–2661.

[44] G.P. Pirola, Curves on generic Kummer varieties, *Duke Math. J.* **59** (1989), 701–708.

[45] A. Predonzan, Intorno agli S_k giacenti sulla varietà intersezione completa di più forme, *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* **5**, (1948), 238–242.

[46] M. Ramponi, Gonality and Clifford index of curves on elliptic K3 surfaces with Picard number two, *Arch. Math. (Basel)* **106** (2016), 355–362.

[47] B. Segre, Intorno agli S_k che appartengono alle forme generali di dato ordine, *Atti Accad. Naz. Lincei, Rend., Cl. Sci. Fis. Mat. Nat.* **4**, (1948), 261–265, 341-346.

[48] T. Takahashi, Galois morphism computing the gonality of a nonsingular projective curve on a Hirzebruch surface, *J. Pure Appl. Algebra* **216** (2012), 12–19.

[49] C. Voisin, On a conjecture of Clemens on rational curves on hypersurfaces, *J. Differential Geom.* **44** (1996), 200–213.

[50] C. Voisin, A correction on "On a conjecture of Clemens on rational curves on hypersurfaces", *J. Differential Geom.* **49** (1998), 601–611.

[51] G. Xu, Subvarieties of general hypersurfaces in projective space, *J. Differential Geom.* **39** (1994), 139–172.

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