Curves in Hilbert modular varieties

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Lang-Vojta conjectures

Conjecture

Let $\tilde{X}$ be a projective manifold defined over a number field k and $D = \tilde{X} \setminus X$ a normal crossings divisor. If $(\tilde{X}, D)$ is of log-general type then for every ring of S-integers $\mathcal{O}_S$ the set of S-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.

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An algebraic function field analogue predicts

Conjecture

Let $\tilde{X}$ be a complex projective manifold, $D = \tilde{X} \setminus X$ a normal crossings divisor, $\tilde{C}$ a smooth projective curve and $S \subset \tilde{C}$ a finite subset. If $(\tilde{X}, D)$ is of log-general type then there exists a bound for the degree of the images of non-constant morphisms $\tilde{C} \setminus S \to X$.

Known results

Theorem (Bogomolov, Miyaoka)

Let X be a minimal complex projective surface of general type and $C \subset X$ an irreducible curve of geometric genus g. If $c_1^2 > c_2$ then

$$C.K_X \leq a(2g-2) + b$$

where a and b are effective functions of c_1^2 and c_2.

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Theorem (Chen, Pacienza-Rousseau)

Let $D \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq 2n+1$ and $f : \tilde{C} \to \mathbb{P}^n$ a finite morphism from a smooth projective curve such that $f(\tilde{C}) \not\subset D$. Then

$$(d-2n)\deg_C(f^*(\mathcal{O}(1)) \leq 2g(\tilde{C})-2+N_1(f^*D).$$

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- *$\tilde{X} = \mathbb{P}^2$, D a quartic, union of a smooth conic and two lines in general position (Corvaja-Zannier)*

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Theorem (Demailly)

Let X be complex projective Kobayashi hyperbolic manifold. Then there exists a constant $a > 0$ such that for every projective curve C and any finite morphism $f : C \to X$

$$\deg_C(f^* K_X) \leq a(2g-2).$$

Known results

Theorem (Autissier - Chambert-Loir - Gasbarri)

Let X be a compact quotient of a bounded symmetric domain. Then for any projective curve C and any finite morphism $g : C \to X$,

$$\deg_C(f^* K_X) \leq \dim(X)(2g-2).$$

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There is a projective resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma_K}$ with E the exceptional divisor, K_X the canonical line bundle of X such that if C is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$

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There exists a projective resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma}$ such that

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Proposition

Let $E_c \subset E$ be the hypersurface corresponding to the cusps resolution. Let $\mathcal{C}$ be a germ of analytic curve tangent to $\mathcal{F}_i$ passing through $p \in E_c$. Then $\mathcal{C}$ is contained in E_c.

Hilbert modular foliations

Proof.

The change of coordinates in the resolution of cusps is (locally) given by

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In particular the (locally defined) real valued function

$$g = \prod_j |u_j|^{a_{ij}}$$

is continuous, constant along the leaves of $\mathcal{F}_i$ and vanishes precisely on E_c. Therefore g necessarily vanishes along $\mathcal{C}$, hence $\mathcal{C}$ is contained in E_c. $\square$

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- If the leaf passes through an orbifold point: next lemma.

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Lemma

Leaves passing through an orbifold point are quotients of polydisk by finite groups, in particular they are Stein and hyperbolic.

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Indeed, $\Gamma_K = SL(2, \mathcal{O})$ acts on $\mathfrak{H}^n$ via the embedding of groups $SL(2, K) \hookrightarrow SL(2, \mathbb{R})^n$ and the projections $p_i : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})$ have restrictions to Γ_K which are injective.

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If two elements g and h of Γ_K are in the stabilizer of a leaf, it means that the projections $g_i := p_i(g)$ and $h_i := p_i(h)$ of g and h on the corresponding factor of $SL(2, \mathbb{R})^n$ have the same fixed point and so commute.

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Now suppose that g is in the stabilizer of the orbifold point, which is assumed to be non trivial. Then any other element h of the stabilizer of the leaf commutes with g which means that h is in fact in the stabilizer of the orbifold point:

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the stabilizer of the leaf and of the orbifold point coincide.

Proof of geometric Lang-Vojta for Hilbert modular varieties

Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma_K}$ with E the exceptional divisor, K_X the canonical line bundle of X such that if C is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

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We distinguish two cases: either $f(C)$ is contained in a leaf of a Hilbert modular foliation or not.

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We obtain

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- From the definition of the holomorphic sectional curvature one has, $\Theta \geq \frac{\omega}{2\pi . n}$.
- Therefore $\deg_C(f^*(K_X)) \leq n(2g(C) - 2)$.

S.M.T. for Hilbert modular varieties

The same proof gives the following Second Main Theorem:

Theorem

Consider a projective resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma}$ as above, E the exceptional divisor, K_X the canonical line bundle of X. Let $f : \mathbb{C} \to X$ be a non-constant entire curve such that $f(\mathbb{C})$ is not contained in E. Then

$$T_f(r, K_X) + T_f(r, E) \leq nN_1(r, f^*E) + S_f(r)\|,$$

where $S_f(r) = O(\log^+ T_f(r)) + o(\log r)$, and $\|$ means that the estimate holds outside some exceptional set of finite measure.

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Consider a projective resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma}$ as above, E the exceptional divisor, K_X the canonical line bundle of X. Let $f : \mathbb{C} \to X$ be a non-constant entire curve such that $f(\mathbb{C})$ is not contained in E. Then

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where $S_f(r) = O(\log^+ T_f(r)) + o(\log r)$, and $\|$ means that the estimate holds outside some exceptional set of finite measure.

Remark

This generalizes results of Tiba on Hilbert modular surfaces.

S.M.T. for Hilbert modular varieties

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Corollary

Let X as above be a Hilbert modular variety of general type. Let $f : \mathbb{C} \to X$ be a non-constant entire curve which ramifies over E with order at least n, i.e. $f^*E \geq n \operatorname{supp} f^*E$. Then $f(\mathbb{C})$ is contained in E.

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If $f(\mathbb{C})$ is not contained in E then

$$nN_1(r, f^*E) \leq N(r, f^*E) \leq T_f(r, E) + O(1).$$

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Remark

For all K except a finite number, X_K is of general type (Tsuyumine).

The Green-Griffiths-Lang conjecture

Conjecture (Green-Griffiths-Lang)

Let X be a complex projective variety of general type. Then there exists a proper algebraic subvariety $Z \subsetneq X$ such that every (non-constant) entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Z$.

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Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties of dimension n satisfy the Green-Griffiths-Lang conjecture.

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Theorem (Rousseau-Touzet)

Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties of dimension n satisfy the Green-Griffiths-Lang conjecture.

Remark

Hilbert modular varieties provide counter-examples to the so-called "jet differentials" strategy developed (by Bloch, Green-Griffiths, Demailly, Siu...) to attack the Green-Griffiths-Lang conjecture (Diverio-Rousseau).

The Green-Griffiths-Lang conjecture

Let g denote the Bergman metric with Kähler form ω on $\mathfrak{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathfrak{H}^n/\Gamma$.

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Problem

Find conditions on F under which $||s||^{2b/ln}.g$ will extend as a pseudo-metric on X for some $b > 0$ suitably chosen.

The Green-Griffiths-Lang conjecture

Denote S_k^m the space of Hilbert modular form of weight k and order at least m, where the order is the vanishing order at the cusps.

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Let $F \in S_{2l}^{\nu l}$ and $b > 0$ then $||s||^{2b/ln}.g$ extends as a pseudo-metric over cusps vanishing on E_c if $\nu > \frac{n}{b}$.

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Proposition

Let $b > 0$. There is a constant c depending only on the order of the stabilizer of the elliptic fixed point such that if F is a Hilbert modular form of weight $2l$ vanishing with order $c.ln$ at elliptic fixed points then $||s||^{2b/ln}.g$ extends as a pseudo-metric over elliptic singularities vanishing on E_e.

The Green-Griffiths-Lang conjecture

Let F be a Hilbert modular form and $0 < \epsilon < 1/n$ such that $||s||^{2(1-n\epsilon)/ln}.g$ extends as a pseudo-metric on X vanishing on E.

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Proposition

There exists a constant $\beta > 0$ such that

$$\tilde{g} := \beta.||s||^{2(1-n\epsilon)/ln}.g$$

satisfies the following property: for any holomorphic map $f : \Delta \to X$ from the unit disc equipped with the Poincaré metric g_P, we have

$$f^*\tilde{g} \leq g_P.$$

The Green-Griffiths-Lang conjecture

Corollary

Let d_X be the Kobayashi pseudo-distance, β and F a Hilbert modular form as above. Then $\tilde{g} \leq d_X$. In particular, the degeneracy locus of d_X is contained in the base locus of these Hilbert modular forms.

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Corollary

Let X be a Hilbert modular variety such that there exists a Hilbert modular form as above. Then X satisfies the strong Green-Griffiths-Lang conjecture.

Existence of Hilbert modular forms

We use the following formula due to Tsuyumine

$$\dim S_k^{\nu k}(\Gamma_K) \geq (2^{-2n+1}\pi^{-2n}d_K^{3/2}\zeta_K(2) - 2^{n-1}\nu^n n^{-n} d_K^{1/2} hR)k^n + O(k^{n-1})$$

for even $k \geq 0$, where h, d_K, R, ζ_K denote the class number of K, the absolute value of the discriminant, the positive regulator and the zeta function of K.

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In particular, there is a modular form F with $\mathrm{ord}(f)/\,\mathrm{weight}(f) \geq \nu$, if

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Corollary

For n fixed, except for a finite number of K, there is a Hilbert modular form F such that $||s||^{2(1-n\epsilon)/ln}.g$ extends as a pseudo-metric over cusps. Moreover as d_K tends to infinity, the number of such forms grows at least with order $O(d_K^{3/2})$.

Estimation of elliptic fixed points

Corollary

If the number of elliptic fixed points is $O(d_K^{\epsilon})$ for $0 < \epsilon < 3/2$, then with finite exceptions, Hilbert modular varieties of dimension n satisfy the strong Green-Griffiths-Lang conjecture.

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For fixed n, the number of equivalence classes of elliptic fixed points is $O(d_K^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$.

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