SOME RESULTS OF REGULARITY FOR SEVERI VARIETIES OF PROJECTIVE SURFACES*

FLAMINIO FLAMINI †

Dipartimento di Matematica, Universita' di "Roma Tre"
Largo San Leonardo Murialdo, 1 - 00146 Roma - Italy

e-mail: flamini@matrm3.mat.uniroma3.it

ABSTRACT

For a linear system $|C|$ on a smooth projective surface S, whose general member is a smooth, irreducible curve, the *Severi variety* $V_{|C|,\delta}$ is the locally closed subscheme of $|C|$ which parametrizes curves with only δ nodes as singularities. In this paper we give numerical conditions on the class of divisors and upper bounds on δ, ensuring that the corresponding Severi variety is smooth of codimension δ, Our result generalizes what is proven in [7] and [10]. We also consider examples of smooth Severi varieties on surfaces of general type in $\mathbb{P}^3$ which contain a line.

Introduction.

Nodal curves play a central role in the subject of singular curves. The definition of the *Severi variety* of irreducible, nodal curves on any smooth, projective surface is standard. For a given effective divisor $C \in Div(S)$, let $|C|$ denote the linear system associated to the line bundle $\mathcal{O}_S(C) \in Pic(S)$. If we suppose that the generic element of $|C|$ is a smooth, irreducible curve, it makes sense to consider the subscheme $V_{|C|,\,\delta}$ of $|C|$, which parametrizes all curves $C' \in |C|$ that are irreducible and have only δ nodes as singular points. It is well known that such a subscheme is locally closed in the projective space $|C|$.

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*2000 Mathematics Subject Classification: 14J29, 14H10.

†The author is a member of GNSAGA-CNR.

In [18], Anhang F, Severi studied some properties of the variety $V_{d,\,g}$, defined as the closure of the locus consisting of reduced and irreducible plane curves, of geometric genus g, in the projective space parametrizing plane curves of degree d. $V_{d,\,g}$ contains, as an open dense subscheme, the locus $V_{d,\,\delta}$ corresponding to irreducible curves having only δ nodes as singularities. $V_{d,\,g}$ is classically known as the Severi variety of plane curves of given genus and degree. He proved that, for every $d \geq 3$ and $0 \leq \delta \leq \binom{d-1}{2}$, $V_{|dL|,\,\delta}$ is non-empty and everywhere smooth of codimension δ in $|dL|$, where L denotes a line in $\mathbb{P}^2$. Only after more than 60 years, Harris completed, in [12], Severi's proof of the irreducibility of the Severi variety $V_{d,\,g}$ of the projective plane, by showing that the dense open subset $V_{d,\,\delta}$ is connected.

With abuse of terminology, we shall use in the sequel the word *Severi variety* for the locally closed subscheme $V_{|C|,\,\delta}$ of the linear system $|C|$ on a projective surface S.

In recent times, there have been many results on this subject and in many directions. In fact, one may study several problems concerning Severi varieties.

Existence problems are covered by recent investigations in the case of Del Pezzo surfaces ([9]) or K3 surfaces ([4]); on the other hand, we have a complete computation of the degree of Severi varieties in the plane, treated in [3]. There are some results even on the irreducibility problem, contained in [5], for general surfaces in $\mathbb{P}^3$ of degree $d \geq 8$.

A natural approach to the dimension problem is to use deformation theory of nodal curves. Apart from the Severi classical result, whose proof can be extended to some rational or ruled surfaces and to K3 surfaces, information about regularity of $V_{|C|,\,\delta}$ on a surface S of general type can be obtained by studying suitable rank 2 vector bundles on S. The first who used such approach were Chiantini and Sernesi. In [7] they found an upper bound on δ, ensuring that the family $V_{|C|,\,\delta}$ is smooth of codimension δ in the projective space $|C|$. Their proof focused on surfaces such that $|K_S|$ is ample and C is a divisor which is numerically equivalent to pK_S, $p \in \mathbb{Q}^+$ and $p \geq 2$.

An improvement of this result is given in [10]. The authors generalized this approach in two directions. In fact, they allowed arbitrary singularities and they weakened the assumption of K_S being ample, so that $S = \mathbb{P}^2$ is included. Their assumptions are: C, $C - K_S$ ample divisors and $C^2 \geq K_S^2$; moreover, some numerical hypotheses are made, which imply, in the case of nodes, that $(C - 2K_S)^2 > 0$ and $C(C - 2K_S) > 0$.

In this paper, we give a purely numerical criterion to prove the regularity of $V_{|C|,\,\delta}$ provided that δ is less than a suitable upper bound. More precisely, we prove the following

Theorem. *Let S be a smooth, projective surface and C be a smooth, irreducible divisor on S. Suppose that:*

1. $(C-2K_S)^2 > 0$ *and* $C(C-2K_S) > 0$;
2. $$\begin{array}{lll} K_S^2 > -4 & if & C(C-2K_S) \geq 8, \qquad or \\ K_S^2 \geq 0 & if & 0 < C(C-2K_S) < 8. \end{array}$$
3. $CK_S \geq 0$;
4. $H(C, K_S) < 4(C(C-2K_S)-4)$, *where* $H(C, K_S)$ *is the Hodge number of* C *and* S *(see def. 2);*
5. $$\begin{array}{lll} \delta \leq \frac{C(C-2K_S)}{4} - 1 & if & C(C-2K_S) \geq 8, \qquad or \\ \delta < \frac{C(C-2K_S)+\sqrt{C^2(C-2K_S)^2}}{8} & if & 0 < C(C-2K_S) < 8. \end{array}$$

Then, if $C' \in |C|$ is a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0-dimensional scheme of nodes in C', in the above hypotheses N imposes independent conditions to $|C|$, i.e. the Severi variety $V_{|C|,\,\delta}$ is smooth of codimension δ at the point $[C']$ parametrizing C'.

We shall give some examples which show that such result really generalizes the ones recalled before. Moreover, we can obtain some results on the Severi varieties of surfaces in $\mathbb{P}^3$ which are elements of a component of the Noether-Lefschetz locus, consisting of surfaces which contain a line. These are related to some results contained in [6], where the question of algebraic hyperbolicity for surfaces S in $\mathbb{P}^3$ and in $\mathbb{P}^4$ is treated.

In the sequel, we shall work in the category of $\mathbb{C}$–schemes. We will denote by $\equiv$ the linear equivalence of divisors, whereas $\equiv_{num}$ shall denote the numerical equivalence of divisors.

1 Preliminaries

Let S be a projective, non-singular algebraic surface and $|D|$ a complete linear system on S whose general member is a smooth, irreducible curve. If $p_a(D)$ denotes the *arithmetic genus* of D then, by the adjunction formula,

$$p_a(D) = \frac{D(D+K_S)}{2} + 1.$$

For a given $\delta \geq 1$, suppose that $V_{|D|,\,\delta}$ is non-empty. Let $C \in V_{|C|,\,\delta}$ and let N be the scheme consisting of the δ nodes of C. The *geometric genus* of C is $g = p_g(C) = p_a(C) - \delta$.

We know that the Zariski tangent space of $|D|$ at the point $[C]$, parametrizing C, is isomorphic to

$$H^0(S, \mathcal{O}_S(D))/ < C >,$$

whereas the Zariski tangent space to $V_{|D|,\,\delta}$ at $[C]$ is

$$T_{[C]}(V_{|D|,\,\delta}) \cong H^0(S, \mathcal{I}_N(D))/ < C >,$$

where $\mathcal{I}_N \subset \mathcal{O}_S$ denotes the ideal sheaf of the subscheme N of S (see, for example, [17]). The relative obstruction space is a subspace of $H^1(S, \mathcal{I}_N(D))$. In particular, N imposes independent conditions to $|D|$ if and only if

$$dim(V_{|D|,\,\delta}) = dim\ T_C(V_{|D|,\,\delta}) = dim(|D|) - \delta$$

at $[C]$. In this case, $V_{|D|,\,\delta}$ is smooth of the *expected dimension* at $[C]$. The component containing $[C]$ is called *regular* at $[C]$. Otherwise, it is said to be a *superabundant component.* The regularity property is very strong, since it implies that the nodes of C can be independently smoothed (see [5] or [7]).

Definition 1 Let S be a smooth, projective surface and $C \in Div(S)$. C is said to be a *nef divisor* if $CF \geq 0$, for each effective divisor F. A nef divisor C is called *big* if $C^2 > 0$.

We recall that, by the Kleiman criterion (see [13]), C is nef if and only if it is in the closure of the ample divisor cone of S.

Definition 2 Let S be a smooth, projective surface and $C \in Div(S)$. We shall denote by $H(C, K_S)$ the *Hodge number* of C and S, defined by

$$H(C, K_S) := (CK_S)^2 - C^2K_S^2.$$

The Index theorem (see [1], page 120) ensures that this number is non-negative when C (or K_S) is a nef divisor.

2 The main result

In the previous section we have recalled all definitions and properties needed to prove our principal result. We are now able to state the following

Theorem 1 *Let S be a smooth, projective surface and C be a smooth, irreducible divisor on S. Suppose that:*

1. $(C-2K_S)^2 > 0$ *and* $C(C-2K_S) > 0$;
2. $$\begin{array}{lllll} (i) & K_S^2 > -4 & if & C(C-2K_S) \geq 8, & or \\ (ii) & K_S^2 \geq 0 & if & 0 < C(C-2K_S) < 8. & \end{array}$$
3. $CK_S \geq 0$;
4. $H(C, K_S) < 4(C(C-2K_S)-4)$, *where* $H(C, K_S)$ *is the Hodge number of* C *and* S *(see Def.2);*
5. $$\begin{array}{lllll} (i) & \delta \leq \frac{C(C-2K_S)}{4} - 1 & if & C(C-2K_S) \geq 8, & or \\ (ii) & \delta < \frac{C(C-2K_S)+\sqrt{C^2(C-2K_S)^2}}{8} & if & 0 < C(C-2K_S) < 8. & \end{array}$$

Then, if $C' \in |C|$ is a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0-dimensional scheme of nodes in C', in the above hypotheses N imposes independent conditions to $|C|$, i.e. the Severi variety $V_{|C|,\,\delta}$ is smooth of codimension δ at the point $[C']$ parametrizing C'.

Proof: For the sake of simplicity we will write K, instead of K_S, to denote a canonical divisor of S. By contradiction, assume that N does not impose independent conditions to $|C|$. Let $N_0 \subset N$ be a minimal 0-dimensional subscheme of N for which this property holds and let $\delta_0 = |N_0|$. This means that $H^1(S,\, \mathcal{I}_{N_0}(C)) \neq 0$ and that N_0 satisfies the Cayley-Bacharach condition (see, for example, [11]). Therefore, a non-zero element of $H^1(\mathcal{I}_{N_0}(C))$ corresponds to a non-trivial rank 2 vector bundle $\mathcal{E} \in Ext^1(\mathcal{I}_{N_0}(C-K), \mathcal{O}_S)$; so one can consider the following exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{N_0}(C-K) \to 0. \tag{1}$$

This implies that

$$c_1(\mathcal{E}) = C - K, \ c_2(\mathcal{E}) = \delta_0 \leq \delta,$$

i.e.

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = (C-K)^2 - 4\delta_0. \tag{2}$$

Observe also that, from hypotheses 1. and 3., it immediately follows that $C(C-K) > 0$ and $C^2 > 0$. Indeed, $C(C-K) = C(C-2K) + CK > 0$ and $C^2 > CK \geq 0$. Since C is irreducible, this implies that C is a nef divisor.

We now want to compute (2) in cases 5.(i) and 5.(ii). In the first one,

$$(C-K)^2-4\delta_0 \geq (C-K)^2-4\delta = C^2-2CK-4+4+K^2-4\delta \geq K^2+4 > 0,$$

by 2(i). In the other case, using 5.(ii) and the Index Theorem (C is nef),

$$(C-K)^2-4\delta_0 \geq (C-K)^2-4\delta = C^2-2CK+K^2-4\delta > K^2 \geq 0,$$

since we supposed 2(ii).

In both cases, the vector bundle $\mathcal{E}$ is Bogomolov-unstable (see [2] or [15]), i.e. there exist $M,\ B \in Div(S)$ and a 0-dimensional scheme Z (possibly empty) such that

$$0 \to \mathcal{O}_S(M) \to \mathcal{E} \to \mathcal{I}_Z(B) \to 0 \tag{3}$$

holds and $(M-B) \in N(S)^+$. We recall that $N(S)^+$ denotes the ample divisor cone of S. This means that

$$\begin{aligned} (M-B)^2 &> 0, \\ (M-B)H &> 0, \quad \text{for all H ample divisor.} \end{aligned} \tag{4}$$

The exact sequence (3) ensures that $H^0(\mathcal{E}(-M)) \neq 0$. If we consider the tensor product of the exact sequence (1) by $\mathcal{O}_S(-M)$, we get

$$0 \to \mathcal{O}_S(-M) \to \mathcal{E}(-M) \to \mathcal{I}_{N_0}(C-K-M) \to 0. \tag{5}$$

We state that $H^0(\mathcal{O}_S(-M)) = 0$; otherwise, $-M$ would be an effective divisor, therefore $-MH > 0$ for each ample divisor H. From (3), it follows that $c_1(\mathcal{E}) = M + B$ so, by (1) and (4),

$$M - B = 2M - C + K \in N(S)^+. \tag{6}$$

Thus, for every ample divisor H,

$$MH > \frac{(C-K)H}{2}. \tag{7}$$

From (7) and from Kleiman's criterion, we get

$$MC \geq \frac{(C-K)C}{2}. \tag{8}$$

It follows that $-MC < 0$ so, since C is nef, $-M$ cannot be effective.

If we consider the cohomology sequence associated to (5), we deduce that there exists a divisor $\Delta \in |C-K-M|$ s.t. $N_0 \subset \Delta$. If the irreducible nodal curve $C' \in |C|$, whose set of nodes is N, were component of Δ then $-M-K$ would be an effective divisor. By applying (8) and by using the fact that $C(C-K)>0$ and hypothesis 3., one determines

$$C'(-M-K) = C(-M-K) = -CK - CM \leq -CK - \frac{(C-K)C}{2} =$$

$$= -\frac{(C+K)C}{2} = -\frac{KC}{2} - \frac{C^2}{2} < -CK \leq 0,$$

which contradicts the effectiveness of $-M-K$, since C is nef.

Bezout's theorem implies that

$$C'\Delta = C'(C-K-M) \geq 2\delta_0. \tag{9}$$

On the other hand, taking M maximal, we may further assume that the general section of $\mathcal{E}(-M)$ vanishes in a 2-codimensional locus Z of S. Thus, $c_2(\mathcal{E}(-M)) = deg(Z) \geq 0$. By standard computations on Chern classes, we obtain

$$c_2(\mathcal{E}(-M)) = c_2(\mathcal{E}) + M^2 + c_1(\mathcal{E})(-M) = \delta_0 + M^2 - M(C-K),$$

which implies

$$\delta_0 \geq M(C-K-M). \tag{10}$$

By applying the Index theorem to the divisor pair $(C,\ 2M-C+K)$, we get

$$C^2(2M-C+K)^2 \leq (C(C-K) - 2C(C-K-M))^2. \tag{11}$$

From (9) and from the fact that $C(C-K)$ is positive, it follows that

$$C(C-K) - 2C(C-K-M) \leq C(C-K) - 4\delta_0. \tag{12}$$

We observe that the left side member of (12) is non-negative, since $C(C-K) - 2C(C-K-M) = C(2M-C+K)$, where C is effective and, by (6), $2M-C+K \in N(S)^+$. Thus, (12) still holds when we square both its members and, together with (11), this gives

$$C^2(2M-C+K)^2 \leq (C(C-K) - 4\delta_0)^2. \tag{13}$$

On the other hand, using (10), we get

$$(2M-C+K)^2 = 4(M - \frac{(C-K)}{2})^2 =$$

$$= (C-K)^2 - 4(C-K-M)M \geq (C-K)^2 - 4\delta_0,$$

i.e.

$$(2M - C + K)^2 \geq (C - K)^2 - 4\delta_0. \tag{14}$$

Putting together (13) and (14), we get

$$F(\delta_0) := 16\delta_0^2 - 4C(C - 2K)\delta_0 + (CK)^2 - C^2K^2 \geq 0. \tag{15}$$

Summarizing, the assumption on N, stated at the beginning, implies (15)[1]. We want to show that our numerical hypotheses hold if and only if the opposite inequality is satisfied. To this aim, observe that the discriminant of the equation $F(\delta_0) = 0$ is $16C^2(C-2K)^2$ so, by hypotheses 1. and 3., it is positive. The inequality $F(\delta_0) < 0$ is verified iff $\delta_0 \in (\alpha(C,K),\ \beta(C,\ K))$, where

$$\alpha(C,K) = \frac{C(C-2K) - \sqrt{C^2(C-2K)^2}}{8} \in \mathbb{R} \text{ and}$$

$$\beta(C,K) = \frac{C(C-2K) + \sqrt{C^2(C-2K)^2}}{8} \in \mathbb{R}\ ;$$

we have to show that, with our numerical hypotheses, $\delta_0 \in (\alpha(C,K),\ \beta(C,\ K))$.

From 5., it immediately follows that $\delta_0 < \beta(C,K)$, since, as we shall see in the sequel, the bound in 5.(i) is smaller than $\beta(C,K)$. Note also that $\alpha(C,K) \geq 0$. Indeed, if $\alpha(C,K) < 0$, then $C(C-2K) < \sqrt{C^2(C-2K)^2}$, which contradicts the Index theorem, since $C(C-2K) > 0$.

Observe that $\alpha(C,K) < 1$ if and only if

$$C(C-2K) - 8 < \sqrt{C^2(C-2K)^2} \tag{16}$$

To simplify the notation, we put $t = C(C-2K)$ so that (16) becomes

$$t - 8 < \sqrt{t^2 - 4H(C,K)}. \tag{17}$$

Two cases can occur.

If $t-8 < 0$, there is nothing to prove since the right side member of (17) is always positive. Note, before proceeding to consider the other case, that, in this situation, we want that $\beta(C,K) > 1$ in order to have at least an effective positive integral value for the number of nodes; but $\beta(C,K) > 1$ if and only if $(0 <)8-t < \sqrt{t^2 - 4H(C,K)}$. By squaring both members of the previous inequality, we get $4H(C,K) < 16t - 64$, which is our hypothesis 4.; thus, the upper-bound for δ is surely greater than 1. Moreover, the

[1]We remark that, in the case of nodes, this condition is the same of [10]; moreover, their hypotheses (1.2) and (1.3) coincide in the case of nodes and become $F(\delta_0) < 0$.

expression for such bound is the one in 5.(ii) and it can not be written in a better non-trivial form.

For the other case, if $t-8\geq 0$, by squaring both members of (17), we get $H(C,K) < 4(C(C-2K)-4)$, which is our hypothesis 4.. Therefore, $\alpha(C,K) < 1$; moreover, the condition $\beta(C,K) > 1$ is trivially satisfied, since it is equivalent to $t-8 > -\sqrt{C^2(C-2K)^2}$. From (16), we can write

$$\frac{C(C-2K)+C(C-2K)-8}{8} < \frac{C(C-2K)+\sqrt{C^2(C-2K)^2}}{8},$$

so we can replace the bound $\delta < \beta(C,K)$ with the more "accessible" one $\delta \leq \frac{C(C-2K)}{4}-1$, which is the bound in 5.(i).

Observe that

$$\frac{C(C-2K)+C(C-2K)-8}{8} < \frac{C(C-2K)+\sqrt{C^2(C-2K)^2}}{8} \leq \frac{C(C-2K)}{4}.$$

Therefore, 5.(i) is the right approximation.

In conclusion, our numerical hypotheses contradict (15), therefore the assumption $h^1(\mathcal{I}_N(C)) \neq 0$ leads to a contradiction. □

Remark 1 (1) The previous theorem gives purely numerical conditions to deduce some information about Severi varieties of smooth projective surfaces. In the next section, we shall discuss some interesting examples of projective surfaces to which our theorem easily applies. More precisely, we will consider smooth surfaces in $\mathbb{P}^3$ which are general elements of a component of the Noether-Lefschetz locus; for example, surfaces of general type, of degree $d \geq 5$, which contain a line.

Our result obviously generalizes the one of Chiantini and Sernesi. In their case, since $C \equiv_{num} pK_S$, $p \in \mathbb{Q}$ and $p \geq 2$, we always have $\alpha(C,K_S) = 0$ and $\beta(C,K_S) = \frac{p(p-2)}{4}K_S^2$; this depends on the fact that $H(pK_S,K_S)=0$, for every p. With the further hypotheses that $p \in \mathbb{Z}^+$, p odd, and that the Neron-Severi group of S is $NS(S) \cong \mathbb{Z}[K_S]$, they proved that one can take $\delta = \frac{(p-1)^2}{4}K_S^2$. These bounds are sharp, at least for the general quintic surface in $\mathbb{P}^3$.

Furthermore, as recalled in Remark 1.2 in [7], in the case of rational or ruled surfaces (for which $CK_S < 0$) or $K3$ surfaces (for which $CK_S = 0$) if $|C|$ is base point free the argument for $S=\mathbb{P}^2$ can be repeated without changes, since the line bundle N_φ, on the normalization of the nodal curve

$C' \in |C|$, is non-special. Our result focuses on cases in which $CK_S \geq 0$ (see hypothesis 3.), where the previous approach fails.
(2) One can immediately deduce that when the Hodge number is zero, i.e. when we are considering a divisor pair such that $(CK)^2 = C^2K^2$, then in the previous proof we find $\alpha(C,K) = 0$ and $\beta(C,K) = \frac{C(C-2K)}{4}$.
(3)Theorem 1 generalizes, in the case of nodes, the result in [10]. This will be clear after having considered the following examples.

Examples: 1) Let $S \subset \mathbb{P}^3$ be a smooth quartic surface; thus, $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$. Let H denote the plane section and D be the generic element of $|2H|$. From Bertini's theorem, D is smooth and irreducible. If $\pi : \tilde{S} \to S$ denotes the blow-up of S in a point $P \in S$ and E the associated exceptional divisor, then $K_{\tilde{S}} \equiv E$, i.e. the canonical divisor of the blown-up surface is linearly equivalent to the exceptional divisor. Thus, $C \equiv 2\pi^*(H)$ can not be ample, since $CK_{\tilde{S}} = 0$; so, the first hypothesis in [10] does not hold.

Nevertheless, observe that the generic element of $|C|$ is smooth and irreducible. Moreover, $C - 2K_{\tilde{S}} \equiv 2\pi^*(H) - 2E$ so that $(C - 2K_{\tilde{S}})^2 = 12$, $C(C - 2K_{\tilde{S}}) = 16$, $CK_{\tilde{S}} = 0$, $K_{\tilde{S}}^2 = -1$, $H(C, K_{\tilde{S}}) = 16$ and $4(C(C - 2K_{\tilde{S}}) - 4) = 48$. Since we are in the situation 5.(i), we get $\delta \leq \frac{16}{4} - 1 = 3$, i.e. on $\tilde{S}$, if $V_{|2\pi^*(H)|,\delta} \neq \emptyset$ and if $\delta \leq 3$, then it is everywhere smooth of the expected dimension.

2) Let S be a smooth quintic surface in $\mathbb{P}^3$ which contains a line L. Denote by $\Gamma \subset S$ a plane quartic which is coplanar to L, so that $\Gamma \equiv H - L$ (H denotes the plane section). Thus,

$$H^2 = 5,\ HL = 1,\ L^2 = -3,\ H\Gamma = 4,\ \Gamma^2 = 0 \text{ and } \Gamma L = 4.$$

Choose $C \equiv 3H + L$, so that $|C|$ contains the curves which are residue to Γ in the complete intersection of S with the smooth quartic surfaces of $\mathbb{P}^3$ containing Γ. $|3H + L|$ is base point free and not composed with a pencil, since $(3H + L)L = 0$ and $3H$ is an ample divisor. By Bertini's theorems, its general member is smooth and irreducible; but C and $C - K_S$ can not be both either ample or, even, nef divisors. In fact, $CL = 0$ and $(C - K_S)L = (2H + L)L = -1$. Therefore, the result in [10] can not be applied.

Neverthless, $CK_S = C(C - 2K_S) = H(C, K_S) = 16$, $(C - 2K_S)^2 = 4$, $K_S^2 = 5$, $4(C(C-2K_S)-4) = 48$ and, since $C(C-2K_S) > 8$, $\delta \leq \frac{16}{4} - 1 = 3$. Thus, if $|3H + L|$ contains some nodal, irreducible curves, then, if $\delta \leq 3$, $V_{|3H+L|,\delta}$ is everywhere smooth of the expected dimension.

3 Some results on surfaces in $\mathbb{P}^3$ which contain a line

We now consider a class of examples to which our result can be easily applied. We shall focus on surfaces of $\mathbb{P}^3$ containing a line. Such approach can be generalized to surfaces belonging to other components of the Noether-Lefschetz locus.

First, let $S \subset \mathbb{P}^3$ be a smooth quintic and $L \subset S$ a line. Since $p_a(L) = p_g(L) = 0$, by the adjunction formula and by the fact that $K_S \equiv H$ we get $L^2 = -3$. As before,

$$K_S^2 = 5, \; LH = 1, \; L^2 = -3.$$

We are interested on some results of regularity for Severi varieties of curves on S, which are residue to the line L in the complete intersection of S with a general surface of degree a passing through the line. Thus $C \equiv aH - L$ on S. By straightforward computations, we get

$$deg(C) = (aH - L)H = 5a - 1,$$

$$p_a(C) = \frac{5a^2 + 3a}{2} - 1.$$

We want to find conditions on a in order to apply our result.
(i) $|C|$ has a smooth and irreducible general member: for the smoothness, we have to prove that $|aH - L|$ is base point free and not composed with a pencil. Since $a \geq 1$, $aH - L = (a-1)H + H - L$. If $a \geq 2$, the linear system $|(a-1)H|$ can not have fixed intersection points on L. We can restrict ourselves to study the behaviour of $|H - L|$ on L. If $|H - L|$ admitted fixed points on L, each of those points should be a tangent point for S and the general plane of $\mathbb{P}^3$ passing through the line. This would imply that S is a singular surface in such points, which contradict the hypothesis. Moreover, $|H - L|$ can not be composed with a pencil, since $|H - L| + L \subset |H|$.

For the irreducibility, we can use the fact that C and L are linked in $\mathbb{P}^3$ (see [14]). In fact, this implies that C is projectively normal in $\mathbb{P}^3$, i.e. if we consider the exact sequence

$$0 \to \mathcal{I}_{C/\mathbf{P}^3}(\rho) \to \mathcal{O}_{\mathbf{P}^3}(\rho) \to \mathcal{O}_C(\rho) \to 0,$$

then $H^1(\mathcal{I}_{C/\mathbf{P}^3}(\rho)) = 0$, for each $\rho \in \mathbb{Z}$. By choosing $\rho = 0$, we get $H^0(\mathcal{I}_{C/\mathbf{P}^3}) = H^1(\mathcal{I}_{C/\mathbf{P}^3}) = 0$ so $H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_{\mathbf{P}^3})$. This proves that C is a connected curve; since we have already proven its smoothness, then the general member is also irreducible.

(ii) Numerical hypotheses: By simple computations, one observes that all the numerical conditions in Theorem 1 simultaneously hold if $a \geq 4$.

We can completely generalize the previous procedure to the case of a smooth surface of degree $d \geq 6$ which contains a line L. For the detailed computations, the reader is referred to [8].

Let $S \subset \mathbb{P}^3$ be such a surface and $C \equiv aH - L$, so that $deg(C) = ad - 1$ and

$$p_a(C) = \frac{ad(a+d) - 2a - d(4a+1) + 3}{2}.$$

Moreover, $L^2 = 2 - d$, since $K_S \equiv (d-4)H$ and $LH = 1$.

For the smoothness and the irreducibility of the general member of $|C|$ one can use the previous argument. Now, $K_S^2 = (d-4)^2 d \geq 24$, because $d \geq 6$. It is not difficult to compute (see [8] for details) that, for $6 \leq d \leq 7$, all the numerical hypotheses in Theorem 1 simultaneously hold if $a \geq 2d - 6$ (note that, for $d = 5$ we obtained $a \geq 4$ so that $d = 5,\ 6,\ 7$ behave in the same way). On the other hand, for $d \geq 8$, the condition on the Hodge number (i.e. hypothesis 4. in Theorem 1) determines a bound on a which is bigger than the one determined by the other conditions, i.e. $2d - 6$. Indeed, when $d \geq 8$, condition 4. holds if and only if

$$4a^2 d - 8a(d^2 - 4d + 1) - (d^4 - 10d^3 + 33d^2 - 44d + 56) > 0.$$

By solving this inequality, we find

$$a > d - 4 + \frac{1}{d} + \frac{1}{2}\sqrt{d^3 - 6d^2 + d + 28 + \frac{24}{d} + \frac{4}{d^2}}. \tag{18}$$

It is a straightforword computation to find that the right side member of (18) is bigger than $2d - 6$ when $d \geq 8$. Therefore, in this case, all the numerical conditions of Theorem 1 simultaneously hold if (18) holds.

In order to find a better expression for such a lower-bound on a, we observe that

$$\sqrt{d^3 - 6d^2 + d + 28 + \frac{24}{d} + \frac{4}{d^2}} < \sqrt{d^3 - 6d^2 + d + 32},$$

since $d \geq 8$.

We are looking for a real number b such that $\sqrt{d^3 - 6d^2 + d + 32} \leq \sqrt{(d\sqrt{d} - b)^2}$. For such a value, we have

$$2b\sqrt{d} \leq 6d - 1 + \frac{b^2 - 32}{d}. \tag{19}$$

Moreover, (18) becomes

$$a \geq d - 3 + \frac{d}{2}\sqrt{d} - \frac{b}{2}. \tag{20}$$

Obviously, the right side member of (20) must be greater than $2d-6$ for $d \geq 8$. Observe that this happens if and only if

$$d\sqrt{d} > 2d + b - 6. \qquad (21)$$

Therefore, putting $\varphi(d) := d\sqrt{d} - 2d + 6$, from (21) we have $b < \varphi(d)$. The function $\varphi(d)$ is monotone increasing for $d \geq 2$ so, to find a uniform bound on b for all the cases in $d \geq 8$, it is sufficient to consider $b < \varphi(8)$, i.e. $b \leq 12$. By taking into account (19), we find that in all cases a good choice is $b = 9$. Thus, (18) can be replaced by $a \geq d - 3 + \frac{d\sqrt{d}-9}{2}$.

Analogous computations show that, when $d \geq 5$, only condition 2.(i) can occur, i.e. $C(C - 2K_S) \geq 8$. Therefore the expression for the bound on the number of nodes is the one in 5.(i), which is

$$\delta \leq \frac{a^2d - 2a(d^2 - 4d + 1) + 2d - 15}{4}.$$

Now, by summarizing all we have observed up to now, we are able to state the following

Proposition 1 *Let S be a smooth surface in $\mathbb{P}^3$ of degree $d \geq 5$, which contains a line L. Consider on S the linear system $|aH - L|$, with*

1. *$a \geq 2d - 6$, if $5 \leq d \leq 7$;*
2. *$a \geq \lceil d - 3 + \frac{d\sqrt{d}-9}{2} \rceil$, if $d \geq 8$.*

(We denote by $\lceil x \rceil$ the round-up *of the real number x, i.e. the smallest integer which is bigger than or equal to x). Suppose, also, that for a given integer δ the Severi variety $V_{|aH-L|,\delta}$ is non-empty. Then, if*

$$\delta \leq \frac{a^2d - 2a(d^2 - 4d + 1) + 2d - 15}{4},$$

the Severi variety is everywhere smooth of the expected dimension.

Remark 2 We want to point out that our results, in a certain sense, agree with what is stated in [6]. In fact the authors proved the following result.

Theorem. *Let D be a reduced curve in $\mathbb{P}^3$ and s, d be two integers such that $d \geq s + 4$. Moreover, suppose that:*
i)there exists a surface $Y \subset \mathbb{P}^3$ of degree s which contains D;
ii) the general element of the linear system $|\mathcal{O}_Y(dH - D)|$ is smooth and irreducible.

Denote by S a general surface in $\mathbb{P}^3$ of degree d and containing D. Thus, S does not contain reduced, irreducible curves $C \neq D$ of geometric genus $g < 1 + deg(C)\frac{(d-s-5)}{2}$. In particular, if $d \geq s+6$ and $p_g(D) \geq 2$, S is algebraically hyperbolic.

In the case of our proposition, S is a surface of degree $d \geq 5$ and $D = L$, such that $L^2 = 2-d$. Thus, we can consider $s = 1$, i.e. Y is a plane containing the line L and $|\mathcal{O}_Y(dH-L)| = |\mathcal{O}_{\mathbf{P}^2}(d-1)|$ which has a smooth and irreducible general element. Therefore, if there exists a curve C of a given degree, then

$$p_g(C) \geq 2 + \frac{(d-6)}{2}deg(C) = 2 + \frac{(d-6)}{2}CH.$$

If, moreover, C is a nodal curve, then

$$\delta = p_a(C) - p_g(C) \leq \frac{C^2 + CK_S}{2} + 1 - 2 - \frac{(d-6)}{2}CH =$$
$$\frac{C^2}{2} + \frac{(d-4)}{2}CH - \frac{(d-6)}{2}CH - 1 = \frac{C^2}{2} + CH - 1.$$

On the other hand, since in such cases, when all our hypotheses are satisfied, $C(C-2K_S) \geq 8$, $5(i)$ determines

$$\delta \leq \frac{C(C-2dH+8H)}{4} - 1 = \frac{C^2}{4} - \frac{(d-4)}{2}CH - 1.$$

Observe that $\frac{C^2}{4} - \frac{(d-4)}{2}CH - 1 \leq \frac{C^2}{2} + CH - 1$ if and only if $\frac{C^2}{4} + CH(\frac{d}{2} - 1) \geq 0$. Since $d \geq 5$ and since C is big and nef (consequence of 1. and 3.), this latter inequality is always strictly verified. This means that our bounds on δ are in the range of values, for the number of nodes, that are necessary for the existence of such a curve.

ACKNOWLEDGMENTS

This work is part of my Ph.D. thesis. I wish to thank many people who helped me during its preparation, especially L. Chiantini, D. Franco, M. Franciosi, A. F. Lopez, G. Pareschi, E. Sernesi and A. Verra. My special thanks to my Ph.D. advisor E. Sernesi for having introduced me in such an interesting research area.

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