

SINGULAR CURVES ON A $K3$ SURFACE AND LINEAR SERIES ON THEIR NORMALIZATIONS

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ABSTRACT. We study the Brill-Noether theory of the normalizations of singular, irreducible curves on a $K3$ surface. We introduce a *singular* Brill-Noether number ρ_{sing} and show that if $\text{Pic}(K3) = \mathbb{Z}[L]$, there are no \mathfrak{g}_d^r 's on the normalizations of irreducible curves in $|L|$, provided that $\rho_{sing} < 0$. We give examples showing the sharpness of this result. We then focus on the case of *hyperelliptic normalizations*, and classify linear systems $|L|$ containing irreducible nodal curves with hyperelliptic normalizations, for $\rho_{sing} < 0$, without any assumption on the Picard group.

INTRODUCTION

Smooth curves on $K3$ surfaces, and in particular their Brill-Noether theory, have played a fundamental role in algebraic geometry in the past decades (see e.g. [23], [22], [12], [24], [7], [8], [27], [2], [1], [9] and [28], just to mention a few). The Brill-Noether theory of these curves is both an important subject in its own right, especially because it is connected to the geometry of the surface and, at the same time, it is an important tool to prove results about smooth curves with general moduli.

Recall the following fundamental result of R. Lazarsfeld:

Theorem (Lazarsfeld [22]). *Let S be a $K3$ surface with $\text{Pic}(S) = \mathbb{Z}[L]$, with $L^2 = 2p - 2 > 0$. Let $\rho(p, r, d) := p - (r + 1)(p - d + r)$ be the Brill-Noether number.*

(i) *If $X \in |L|$ is smooth, and $\rho(p, r, d) < 0$, then*

$$W_d^r(X) := \{\mathcal{A} \in \text{Pic}^d(X) \mid h^0(X, \mathcal{A}) \geq r + 1\} = \emptyset,$$

(ii) *If $X \in |L|$ is a general member and $\rho(p, r, d) \geq 0$, then $W_d^r(X)$ is smooth outside $W_d^{r+1}(X)$, and of dimension $\rho(p, r, d)$.*

Similarly, in [12], Green and Lazarsfeld proved that all the smooth curves in a complete, base point free linear system $|L|$ on a $K3$ surface have the same Clifford index. Moreover, they proved that if the Clifford index is non-general, i.e. less than $\lfloor \frac{p-1}{2} \rfloor$, where $p = p_a(L)$ is the arithmetic genus of the curves in $|L|$, then it is induced by a line bundle on the surface. As a consequence, one can easily construct curves of any prescribed Clifford index and gonality on $K3$ surfaces [21]. This shows that curves on $K3$ surfaces with arbitrary Picard group may be far from being generic with respect to the Brill-Noether theoretical point of view. The hyperelliptic case is “classical” and dates back to Saint-Donat [25], who classified linear systems on S containing hyperelliptic curves :

Theorem (Saint-Donat [25]). *Let L be a globally generated line bundle with $L^2 > 0$ on a $K3$ surface S . The following conditions are equivalent:*

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- (i) the morphism φ_L defined by $|L|$, is not birational.
- (ii) There is a smooth, hyperelliptic curve in $|L|$.
- (iii) All the smooth curves in $|L|$ are hyperelliptic.
- (iv) Either $L^2 = 2$, or there exists a smooth, irreducible, elliptic curve E on S satisfying $E.L = 2$, or $L \sim 2B$ for a smooth, irreducible curve B with $B^2 = 2$.

A linear system $|L|$ satisfying these properties is said to be hyperelliptic.

Whereas the Brill-Noether theory of *smooth* curves on $K3$ surfaces is quite well understood, almost nothing is known about the Brill-Noether theory of *singular* curves; by this we mean the Brill-Noether theory of *normalizations* of singular curves on $K3$ surfaces. Singular curves on $K3$ surfaces with nodes as the only singularities behave in the expected way in the following sense, as a consequence of a result of Xi Chen and of the theory of Severi varieties (cf. Remark 0.4 for details):

Theorem (Xi Chen [5]). *If $S \subset \mathbb{P}^p$, $p \geq 3$, is a general, primitive $K3$ surface of genus p , whose hyperplane divisor is H , then, for each integer $m \geq 1$, the Severi varieties $V_{|mH|, \delta}(S) \subset |mH|$, parametrizing the universal family of irreducible curves in $|mH|$ having δ nodes as the only singularities, are nonempty and hence of the expected dimension $p_a(mH) - \delta$, for each $\delta \leq p_a(mH)$.*

It is therefore natural to try to extend the study of linear series on smooth curves lying on a $K3$ surface to the case of curves which are the normalizations of singular curves on a $K3$ surface, at least in the nodal case. Beside of the interest on its own, the problem and the results we obtain in this paper are related to the study of the Hilbert scheme of points $S^{[k]}$ of a $K3$ surface. Indeed, a \mathfrak{g}_k^1 on the normalization C of a curve X on S gives rise to a rational curve in $S^{[k]}$ and, as a consequence of a result of D. Huybrechts and S. Boucksom ([17, Prop. 3.2] and [4, Th. 1.2]), we know that rational curves span the Mori cone of $S^{[k]}$. (When $k = 2$, see [16] for a conjecture about the numerical and geometrical properties of the rational curves generating the extremal rays.) We plan to return to this topic in a future work.

In this paper we draw the attention to a number which seems to be an important invariant for the study of the Brill-Noether theory of normalizations of singular curves on $K3$ surfaces.

Let X be a reduced, irreducible, possibly singular curve in $|L|$ and let g be its *geometric genus* (i.e. the genus of its normalization). Set $p_a := p_a(X)$, and define the *singular Brill-Noether number* as follows :

$$(0.1) \quad \rho_{sing}(p_a, r, d, g) := \rho(g, r, d) + p_a - g.$$

Our starting point is a result which we derive from the interesting work of T. L. Gómez [11] concerning rank-one, torsion-free sheaves on singular curves on smooth $K3$ surfaces.

Theorem 1. *Let S be a $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}[L]$ with $L^2 > 0$. Let X be a reduced, irreducible curve in the linear system $|L|$, of arithmetic genus $p_a := p_a(X) = p_a(L)$ and having δ singular points of given multiplicities as the only singularities. Let $C \xrightarrow{\nu} X$ be the normalization of X , whose geometric genus is denoted by g . Then, if $\rho_{sing}(p_a, r, d, g) < 0$ one has $W_d^r(C) = \emptyset$.*

Note that, when X has δ nodes, then $p_a - g = \delta$. In particular, for $r = 1$ and $d = 2$, we have

$$\rho_{sing}(p_a, 1, 2, p_a - \delta) < 0 \Leftrightarrow \delta \leq (p_a(X) - 3)/2.$$

We show the sharpness of Theorem 1 by providing examples (cf. Examples 2.8 and 2.10) of nodal curves on a $K3$ surface having hyperelliptic normalizations, with $\rho_{sing} \geq 0$.

Examples 2.8 and 2.10 also show that the theorem of Green and Lazarsfeld on the constancy of the Clifford index of *smooth* curves in a linear system $|L|$ cannot be extended to the curves in the Severi varieties $V_{|L|,\delta}(S)$ of $|L|$ (cf. Remark 2.11).

Our second main result is a classification, via a Bogomolov-Reider type approach, of linear systems $|L|$ on a $K3$ surface S with *arbitrary* Picard group, containing δ -nodal curves $X \in |L|$ with hyperelliptic normalizations and with $\delta \leq (p_a(X) - 3)/2$ (that is $\rho_{sing}(p_a(X), 1, 2, p_a(X) - \delta) < 0$).

In order to achieve the classification, we have to recall the classical notion of neutral node for irreducible, δ -nodal curves having hyperelliptic normalizations (cf. §3). More precisely, if C is the normalization of such a curve X and if $y_1, y_2 \in C$ are the preimages of a node $x \in X$ then, if y_1, y_2 are conjugated by the hyperelliptic involution existing on C , $x \in X$ is called a *neutral node*, otherwise $x \in X$ is a *non-neutral node* (cf. Definition 3.2).

Our result shows that any element in $V_{|L|,\delta}$ with hyperelliptic normalization cannot have too few non-neutral nodes:

Theorem 2. *Let S be a $K3$ surface and $L \in \text{Pic}(S)$ be globally generated and nonhyperelliptic. Let δ be a positive integer such that $p_a(L) - \delta \geq 2$.*

Inside the Severi variety $V_{|L|,\delta}$, consider the subscheme $V_{|L|,\delta}^{hyper}$ parametrizing those elements having hyperelliptic normalizations. Let $[X] \in V_{|L|,\delta}^{hyper}$ and let $\delta_0 \leq \delta$ be the number of non-neutral nodes of X .

Then either $\delta_0 \geq (p_a(L) - 5)/2$ or $(\delta_0, p_a(L)) = (2, 10)$.

Moreover, if $\delta_0 \leq (p_a(L) - 3)/2$ or $(\delta_0, p_a(L)) = (2, 10)$ then, for any pair of (possibly coinciding) points on X belonging to the image of the \mathfrak{g}_2^1 of C , there exists an effective divisor D on S passing through all the non-neutral nodes of X and through the pair of points and satisfying one of the following conditions:

- (i) $L = \mathcal{O}_S(3D)$, $\delta_0 = 2$, $p_a(L) = 10$;
- (ii) $L = \mathcal{O}_S(2D)$ and either $\delta_0 = (p_a(L) - 3)/2$ and $D^2 = \delta_0 + 1 \geq 4$ (so $\delta_0 \geq 3$ is odd), or $\delta_0 = (p_a(L) - 5)/2$ and $D^2 = \delta_0 + 2$ (so δ_0 is even);
- (iii) D and X are linearly independent in $\text{Pic}(S)$, $D \cdot X = 2\delta_0 + 2$, $(p_a(L) - 5)/2 \leq \delta_0 \leq (p_a(L) - 3)/2$ and $D^2 = \delta_0$ (so δ_0 is even).

A similar statement can also be obtained for curves with normalizations possessing \mathfrak{g}_k^1 's for $k > 2$, but to avoid longer classification results and proofs - which are more difficult only from a technical point of view - we choose to keep the exposition clearer and state it only for $k = 2$.

Theorem 2 has the following two corollaries:

Corollary 3. *Let L be a globally generated, nonhyperelliptic line bundle on a $K3$ surface and δ be a positive integer such that $p_a(L) - \delta \geq 2$.*

If $V_{|L|,\delta}^{hyper} \neq \emptyset$, then either $\delta \geq (p_a(L) - 5)/2$ or $(\delta, p_a(L)) = (2, 10)$

Corollary 4. *Let S be a $K3$ surface with $\text{Pic}(S) = \mathbb{Z}[L]$, for some line bundle L on S with $L^2 \geq 2$, and $n \geq 1$ an integer.*

If $V_{|nL|,\delta}^{\text{hyper}} \neq \emptyset$, for some $\delta \leq \frac{p_a(nL)-3}{2} = \frac{1}{4}n^2L^2 - 1$ then either

- (a) $(n, L^2) = (3, 2)$ and $\delta = 2$ or 3 , or
- (b) $\sqrt{\frac{\delta+1}{L^2}} \in \mathbb{Z}$ and $n = 2\sqrt{\frac{\delta+1}{L^2}}$, or
- (c) $\sqrt{\frac{\delta+2}{L^2}} \in \mathbb{Z}$ and $n = 2\sqrt{\frac{\delta+2}{L^2}}$.

In particular, in any case we have $n \geq 2$.

The paper is organized as follows. After some preliminaries, we prove Theorem 1 in Section 1. Section 2 is devoted to discussing two examples that prove the sharpness of Theorem 1. In Section 3 we briefly recall the notion of neutral and non-neutral nodes and give some results on the matter. In the last section, we translate the fact of having hyperelliptic normalization into a non-separation statement for $|L|$ and then apply a Bogomolov-Reider type technique to prove Theorem 2 and its two corollaries.

PRELIMINARIES

We work in the category of algebraic \mathbb{C} -schemes. Terminology and notation are the standard ones (cf. [15]).

By a $K3$ surface is meant a smooth, algebraic surface S (reduced and irreducible) with $h^1(\mathcal{O}_S) = 0$ and $K_S = 0$. The *sectional genus* $p_a(L)$ of a line bundle L (with sections) is defined by $L^2 = 2p_a(L) - 2$. By adjunction, $p_a(L)$ is the arithmetic genus of any member of $|L|$. Recall that by Saint-Donat's classical results [25], on a $K3$ surface $|L|$ is base point free if and only if it has no fixed components.

We use the following standard definitions:

Definition 0.2. *A line bundle L on a surface S is called primitive if $L \not\sim nL'$ for any $n > 1$ and $L' \in \text{Pic}(S)$.*

A polarized surface (resp. primitively polarized surface) is a pair (S, L) , where S is a surface and $L \in \text{Pic}(S)$ is globally generated and ample (resp. primitive, globally generated and ample).

Let S be a smooth, algebraic surface and let $|L|$ be a complete linear system on S , whose general element is a smooth, irreducible curve. For any integer $0 \leq \delta \leq p_a(L)$, one denotes by

$$(0.3) \quad V_{|L|,\delta}(S)$$

the locally closed and functorially defined subscheme of $|L|$ parametrizing the universal family of irreducible curves in $|L|$ having δ nodes as the only singularities. These are classically called *Severi varieties* of irreducible, δ -nodal curves in $|L|$ on S .

Remark 0.4. Recall that if S is a $K3$ surface, there are several properties concerning Severi varieties on S which are well-known. Indeed, let L be a globally generated line bundle on S such that $p_a(L) \geq 2$. Assume that, for a given $\delta \leq p_a(L)$, $V_{|L|,\delta}(S) \neq \emptyset$. Let $[X] \in V_{|L|,\delta}(S)$ and let $N = \text{Sing}(X)$ be the subscheme of its nodes. Since S is a $K3$ surface, then N always imposes independent conditions on the linear system $|L|$, i.e. $H^1(S, \mathcal{J}_{N/S} \otimes L) = (0)$ (see e.g. [6]). By the description of the tangent space

at $[X]$ to $V_{|L|,\delta}(S)$, the condition $H^1(S, \mathcal{J}_{N/S} \otimes L) = (0)$ is equivalent to the fact that $[X]$ is an unobstructed point of $V_{|L|,\delta}(S)$ (see e.g. [6] or [10]).

Therefore, on $K3$ surfaces, if $V_{|L|,\delta}(S) \neq \emptyset$, then each of its irreducible components is *regular*, i.e. smooth and of the expected dimension $p_a(L) - \delta$. Furthermore, the non-emptiness of $V_{|L|,\delta}(S)$, for some δ , implies, for each $\delta' < \delta$, that $V_{|L|,\delta'}(S) \neq \emptyset$ and, moreover, that each irreducible component of $V_{|L|,\delta'}(S)$ is also regular.

For what concerns existence results on $K3$ surfaces we refer to Chen's result (cf. [5]) and its consequence quoted in the introduction. On the other hand, we stress that it is still not known whether the Severi varieties $V_{|mH|,\delta}(S)$ are irreducible.

Definition 0.5. *Let S be a smooth, algebraic surface and let $|L|$ be a complete linear system on S , whose general element is smooth and irreducible. Consider $V_{|L|,\delta}(S)$. We denote by*

$$V_{|L|,\delta}^{hyper}(S)$$

the subscheme of $V_{|L|,\delta}(S)$ parametrizing curves whose normalizations are hyperelliptic curves. Curves in $V_{|L|,\delta}^{hyper}(S)$ will be called δ -hyperelliptic nodal curves.

When no confusion arises, we will omit S and simply write $V_{|L|,\delta}$ and $V_{|L|,\delta}^{hyper}$.

1. AN EXTENSION OF LAZARSFELD'S RESULT TO THE SINGULAR CASE

The aim of this section is to prove Theorem 1 stated in the introduction.

Remark 1.1. Before proving the theorem, we want to add some comments on the hypothesis $\text{Pic}(S) = \mathbb{Z}[L]$. This could seem restrictive, but one cannot hope to weaken this hypothesis too much if one uses the Lazarsfeld and the Gómez procedure (cf. the final Remark in [22]). However, going through the paper of Lazarsfeld, one sees that the crucial point is to make sure that a certain vector bundle has no nontrivial endomorphisms (cf. [22, Prop. 2.2]). This is the case under the following hypothesis:

$$(1.2) \quad L \text{ has no decomposition } L \sim M + N \text{ with } h^0(M) \geq 2 \text{ and } h^0(N) \geq 2,$$

if one carefully looks through the proof of [22, Lemma 1.3].

Therefore, the hypothesis $\text{Pic}(S) = \mathbb{Z}[L]$ in Theorem 1, as well in Lazarsfeld's theorem quoted in the introduction, can be weakened to (1.2).

To prove Theorem 1 we first need to introduce some notation and some more general facts.

Let Y be an integral curve (not necessarily smooth), of arithmetic genus p_a ; one can consider the complete scheme $\overline{\mathcal{J}}^d(Y)$, defined as the scheme parametrizing rank one torsion-free sheaves on Y of degree d (recall that, if $\mathcal{A} \in \overline{\mathcal{J}}^d(Y)$, by definition $d = \deg(\mathcal{A}) := \chi(\mathcal{A}) + p_a - 1$, cf. e.g. [11, § 2]). When Y lies on a smooth surface, $\overline{\mathcal{J}}^d(Y)$ is integral and contains, as an open, dense subscheme, the generalized Jacobian $J^d(Y)$, defined as the variety parametrizing line bundles of degree d on Y . Therefore $\overline{\mathcal{J}}^d(Y)$ is a compactification of $J^d(Y)$. One can consider the complete subscheme of $\overline{\mathcal{J}}^d(Y)$, called the *generalized Brill-Noether locus*,

$$(1.3) \quad \overline{W}_d^r(Y)$$

parametrizing those sheaves \mathcal{A} which are special and such that $h^0(\mathcal{A}) \geq r + 1$.

Proposition 1.4. *Let S be a K3 surface and L be a globally generated line bundle on S such that $L^2 > 0$. Let X be a reduced, irreducible curve in $|L|$, of arithmetic genus $p_a := p_a(X) = p_a(L)$ and having δ singular points of given multiplicities as the only singularities.*

Let $C \xrightarrow{\nu} X$ be the normalization of X , whose geometric genus is denoted by g .

Assume furthermore that C possesses a line bundle $\mathcal{A} \in \text{Pic}^d(C)$ such that $|\mathcal{A}| = \mathfrak{g}_d^r$; then $\nu_(\mathcal{A}) \in \overline{W}_{d+p_a-g}^r(X)$.*

Proof. It is well-known that $\nu_*(\mathcal{A})$ is a rank-one, torsion-free, coherent sheaf on X (see e.g. [15, II, 5.20]). Therefore we have that

$$(1.5) \quad \deg(\nu_*(\mathcal{A})) = \chi(X, \nu_*(\mathcal{A})) + p_a(X) - 1.$$

By Leray's isomorphism,

$$(1.6) \quad \chi(X, \nu_*(\mathcal{A})) = \chi(C, \mathcal{A}) = d - g + 1 \quad \text{and} \quad h^0(X, \nu_*(\mathcal{A})) = h^0(C, \mathcal{A}) = r + 1.$$

This means that $\deg(\nu_*(\mathcal{A})) = d - g + 1 + p_a(X) - 1 = d + (p_a(X) - g)$. \square

By recalling Formula (0.1) in the introduction, we can state the following:

Proposition 1.7. *With notation and assumptions as in Theorem 1, we have:*

$$(1.8) \quad \overline{W}_{d+p_a-g}^r(X) = \emptyset \Leftrightarrow \rho_{\text{sing}}(p_a, r, d, g) < 0.$$

Proof. (\Rightarrow) Assume $\rho_{\text{sing}}(p_a, r, d, g) \geq 0$. By the very definition of ρ_{sing} and by a straightforward computation, we have that

$$(1.9) \quad \rho_{\text{sing}}(p_a, r, d, g) = \rho(p_a, r, d + p_a - g).$$

Since $p_a = p_a(X) = p_a(L)$, thus for the general $D \in |L|$ we have $\dim(W_{d+m}^r(D)) = \rho(p_a, r, d + p_a - g)$ by Lazarsfeld's theorem quoted in the introduction. By upper semi-continuity, as in [11, Proof of Theorem I, p. 749], we have that $\overline{W}_{d+p_a-g}^r(X)$ is nonempty and $\dim(\overline{W}_{d+p_a-g}^r(X)) \geq \rho(p_a, r, d + p_a - g) \geq 0$.

(\Leftarrow) Assume $\overline{W}_{d+p_a-g}^r(X) \neq \emptyset$; by [11, Prop. 2.5 and proof of Thm. I at p. 749-750] it is possible to deform the couple (X, \mathcal{A}) , where $\mathcal{A} \in \overline{W}_{d+p_a-g}^r(X)$, to a couple (D, \mathcal{L}) , where D is a smooth, irreducible curve in $|L|$ and $\mathcal{L} \in W_{d+p_a-g}^r(D)$. By Lazarsfeld's theorem quoted in the introduction, since $p_a = p_a(X) = p_a(L)$ and D is smooth, then $\rho(p_a, r, d + p_a - g) \geq 0$. One can conclude by using the equality in (1.9). \square

Proof of Theorem 1. It directly follows from Propositions 1.4 and 1.7. \square

Suppose now that X is a reduced, irreducible curve on S having δ nodes as the only singularities. Recall that in this case we have that $p_a(X) - g = \delta$, since each singular point is an ordinary double point. In particular, when $r = 1$ and $d = 2$, the condition $\rho_{\text{sing}}(p_a(X), 1, 2, p_a(X) - \delta) < 0$ is equivalent to the upper-bound on the number of nodes $\delta \leq (p_a(X) - 3)/2$.

Thus, recalling Definition 0.5, an immediate consequence of Theorem 1 is the following.

Corollary 1.10. *Let S be a K3 surface such that $\text{Pic}(S) = \mathbb{Z}[L]$, with L of sectional genus $p = p_a(L) \geq 3$. If*

$$(1.11) \quad \delta \leq (p - 3)/2$$

then $V_{|L|, \delta}^{\text{hyper}} = \emptyset$.

Remark 1.12. To conclude this section, we remark that even if non-existence results easily follow from Lazarsfeld's and Gómez's procedures, the same does not occur for what concerns existence results in the complementary range of values $\rho_{sing}(p_a, r, d, p_a - \delta) \geq 0$. To see this, suppose for simplicity that X is nodal. Let N be the scheme of its δ nodes. Let $W_d^r(X, N) \subset \overline{W}_d^r(X)$ be the subscheme parametrizing the elements of $\overline{W}_d^r(X)$ which are not locally free exactly at N . Then $W_d^r(X, N) \cong W_{d-\delta}^r(C)$ (cf. e.g. [3]).

Proposition 1.7 yields that, if $\rho_{sing} \geq 0$, then $\overline{W}_d^r(X) \neq \emptyset$, but this does not insure that $W_d^r(X, N) \subset \overline{W}_d^r(X)$ is nonempty. We shall discuss in §2 two examples of existence when $\rho_{sing} \geq 0$. These examples will show the sharpness of Theorem 1.

2. EXISTENCE OF δ -HYPERELLIPTIC NODAL CURVES WITH $\rho_{sing} \geq 0$

In this section X will always denote a reduced, irreducible curve on S having δ nodes as the only singularities.

Let S be as in Corollary 1.10 and suppose further that it is a general, primitive $K3$. Then, by Chen's theorem quoted in the introduction, $V_{|L|, \delta} \neq \emptyset$, for each $\delta \leq p$. On the other hand, (1.11) gives an upper-bound on δ , which ensures that the subschemes $V_{|L|, \delta}^{hyper}$ are empty. However, these subschemes are not always empty. Indeed, for $\delta = p - 2$, one clearly has

$$V_{|L|, p-2} = V_{|L|, p-2}^{hyper},$$

since the normalizations of the curves they parametrize have genus 2. These can be considered as trivial examples of δ -hyperelliptic nodal curves (cf. Definition 0.5). Therefore nontrivial questions on $V_{|L|, \delta}^{hyper}$ make sense only for

$$(2.1) \quad p \geq 3 \text{ and } \frac{p-2}{2} \leq \delta \leq p-3.$$

We shall now discuss in details the unique "nontrivial" examples of δ -hyperelliptic nodal curves on a general complete intersection $K3$ surface of type $(2, 3)$ in \mathbb{P}^4 and of type $(2, 2, 2)$ in \mathbb{P}^5 (see Examples 2.8 and 2.10 below). We start with the following remark and result.

Remark 2.2. Recall from [25] that if L is a globally generated line bundle on a $K3$ surface S with $p := p_a(L) \geq 3$ and such that $|L|$ is non-hyperelliptic, i.e. the morphism φ_L defined by $|L|$ is birational (cf. Saint-Donat's Theorem quoted in the introduction). Thus, φ_L maps S birationally onto a projectively normal surface $\varphi_L(S) \subseteq \mathbb{P}^p$ with at worst rational double points as singularities. These points are exactly the images of the contracted, smooth rational curves on S , i.e. curves Γ such that $\Gamma \cdot L = 0$. Moreover φ_L is an isomorphism outside these contracted curves.

It follows that every irreducible curve $X \in |L|$ is mapped isomorphically to a hyperplane section of $\varphi_L(S)$.

Proposition 2.3. *Let S be a $K3$ surface and L be a globally generated line bundle on S with $p := p_a(L) \geq 3$ and such that $|L|$ is non-hyperelliptic. Let $[X] \in V_{|L|, \delta}$, for some $\delta \leq p - 3$, and let $N = \text{Sing}(X)$. Let $\nu : C \rightarrow X$ be its normalization, where $g = p - \delta$ is the geometric genus of C . Identify X with its image in $\varphi_L(S) \subset \mathbb{P}^p$ by Remark 2.2.*

Then C is hyperelliptic if and only if the rational map

$$\pi_{\Lambda_N|_X} : X \dashrightarrow \mathbb{P}^{g-1},$$

induced on $X \subset \mathbb{P}^{p-1}$ by the projection from the linear subspace $\Lambda_N := \text{Span}_{\mathbb{P}^{p-1}}(N)$, maps X 2 : 1 onto a smooth, rational normal curve in \mathbb{P}^{g-1} .

Proof. Let Δ_C be the degree-two pencil on C . Then, $\omega_C \cong \Delta_C^{\otimes(g-1)}$, i.e. the canonical morphism

$$\Phi_C : C \rightarrow \mathbb{P}^{g-1}$$

maps C 2 : 1 onto a rational normal curve $\Gamma \subset \mathbb{P}^{g-1}$, where Φ_C is the morphism given by $|\omega_C|$.

We have $X \cong \varphi_L(X) = \varphi_L(S) \cap H$, for some $H = \mathbb{P}^{p-1}$. Consider Λ_N the linear subspace of H given by the span of the δ nodes of $X \subset H$. By our assumption $\text{Card}(N) = \delta \leq p - 3$, we have that $\dim(\Lambda_N) = \delta - 1$ if and only if the δ nodes are in general linear position in $H \cong \mathbb{P}^{p-1}$, i.e. if and only if $h^1(\mathcal{J}_{N/S}(L)) = 0$. This vanishing holds by Remark 0.4.

Therefore, for each $\delta \leq p - 3$, we have

$$(2.4) \quad \dim(\Lambda_N) = \delta - 1.$$

Let \tilde{N} be the 2δ points given by the pull-back of N to C . Since C is the normalization of X , by adjunction on the surface \tilde{S} (given by the blow-up of S along N), we know that

$$(2.5) \quad \omega_C \cong \mathcal{O}_C(\nu^*(L)(-\tilde{N}));$$

thus,

$$(2.6) \quad \nu_*(\omega_C) \cong \mathcal{J}_{N/X}(L),$$

which gives the following commutative diagram:

$$(2.7) \quad \begin{array}{ccc} C & \xrightarrow{\nu} & X = \varphi_L(S) \cap H \subseteq \mathbb{P}^{p-1} \\ & \searrow \Phi_C & \downarrow \tau_X \\ & & \Gamma \subseteq \mathbb{P}^{g-1}, \end{array}$$

where τ_X is the map induced on X by $|\mathcal{J}_{N/X}(L)|$, that is the one induced on X by the linear projection from Λ_N

$$\pi_{\Lambda_N} : \mathbb{P}^{p-1} \dashrightarrow \mathbb{P}^{g-1}.$$

Therefore, by Diagram (2.7) we can conclude. \square

The previous proposition can be rephrased as follows: C is hyperelliptic if and only if the projection from the span of the nodes of X maps X onto a smooth curve in \mathbb{P}^{g-1} lying on a number of hyperquadrics which is bigger than the expected one.

From (2.1) it follows that, when (S, H) is primitive, and H is very ample (that is, $p_a(H) \geq 3$), the first nontrivial example to consider is the following.

Example 2.8. Let $S = Q_2 \cap Q_3 \subset \mathbb{P}^4$ be a general, primitive, smooth $K3$ surface which is the complete intersection of a quadric and a cubic hypersurface. If $|H|$ denotes the hyperplane linear system of S , such that $\text{Pic}(S) = \mathbb{Z}[H]$, then $p_a(H) = 4$.

From (2.1), we want to consider $\delta = 1 = p_a(H) - 3$; by Remark 0.4, we know that $V_{|H|,1} \neq \emptyset$ on such a S .

Take $q \in S$ a general point and consider $H_q := T_q(Q_2)$ the hyperplane in \mathbb{P}^4 which is the tangent space to Q_2 at the point q . Let $Q_q := Q_2 \cap H_q$, which is a quadric cone in $H_q \cong \mathbb{P}^3$. Thus, the curve $X := H_q \cap S$ is such that $[X] \in V_{|H|,1}$ on S . Moreover $\deg(X) = 6$ and its normalization C has genus $g = 3$.

Observe that, since Q_q is a quadric cone in \mathbb{P}^3 and X has its node at the vertex q of the cone Q_q , the one-dimensional family of the ruling of Q_q cuts out on X a "generalized" \mathfrak{g}_3^1 (i.e. an element of $\overline{W}_3^1(X)$, cf. (1.3)), with the node q as a fixed point. Furthermore, the projection of X from q maps X 2 : 1 onto the base of the quadric cone in H_q . By Proposition 2.3, this determines a \mathfrak{g}_2^1 on the normalization C , which is a smooth, hyperelliptic curve of genus $g = 3$. Since this holds for a general $q \in S$, we have $\dim(V_{|H|,1}^{hyper}) \geq 2$.

Conversely, let $X = H_X \cap Q_2 \cap Q_3$, which is irreducible, 1-nodal at a point q and where $H_X \cong \mathbb{P}^3$. Thus, $T_q(S) \subset H_X$. Since $X \subset H_X \cap Q_2$, then X is contained in a quadric Q_X of $H_X \cong \mathbb{P}^3$. If we assume that X has normalization C which is a hyperelliptic curve of genus $g = 3$, we claim that Q_X is forced to be a quadric cone with vertex at the node q of X . To prove this, observe that, since $|\omega_C|$ is a special \mathfrak{g}_4^2 on C , then it is composed with the hyperelliptic involution on C . Therefore, the map $\Phi_{|\omega_C|}$ given by the canonical linear system is a $(2 : 1)$ -covering of a plane conic Γ . On the other hand, if $\nu : C \rightarrow X \subset H_X \cong \mathbb{P}^3$ denotes the normalization morphism and if π_q is the projection of X from its node to \mathbb{P}^2 , then we have $\Phi_{|\omega_C|} = \pi_q \circ \nu$. For each $x \in \Gamma$, let $L_{q,x}$ be the line in \mathbb{P}^3 connecting x with the node q of X . Then, one can easily show that X is contained in the quadric cone $\mathcal{C}_X \subset \mathbb{P}^3$ determined by $\mathcal{C}_X := \bigcup_{x \in \Gamma} L_{q,x}$ having vertex at the node q of X . Therefore, $\mathcal{C}_X = Q_X$. Thus, $\dim(V_{|H|,1}^{hyper}) = \dim(S) = 2$ whereas $\dim(V_{|H|,1}) = 3$.

Note that C is not general from the classical Brill-Noether theoretical point of view, indeed $\rho(3, 1, 2) = -1$, whereas $\rho_{sing}(4, 1, 2, 3) = 0$ and we actually have a \mathfrak{g}_2^1 on C .

In particular, this example shows the sharpness of the bound (1.11) in Corollary 1.10, whence also of Theorem 1.

Remark 2.9. Following Voisin (see [27, pp. 366-367]), Example 2.8 can also be obtained as follows. Let S be a $K3$ surface with $\text{Pic}(S) \cong \mathbb{Z}[L]$, such that $L^2 = 2p - 2 \geq 2$. The smooth members D of $|L|$ are of geometric genus p and Brill-Noether general by Lazarsfeld's theorem quoted in the introduction. In particular, they have the same gonality as a generic curve.

Therefore, if $p = 2k$, $k \geq 2$, a general member D is $(k + 1)$ -gonal. Let \mathcal{A} be the \mathfrak{g}_{k+1}^1 on D . Following [22], there exists a rank-two vector bundle $E := E(D, \mathcal{A})$ on S , such that

$$c_1(E) = L, \quad c_2(E) = k + 1 \quad \text{and} \quad h^0(S, E) = k + 2.$$

The zero-locus of a general section of E is a general member of a \mathfrak{g}_{k+1}^1 on some D . If x_1, \dots, x_δ are general points on S such that $2\delta \leq k$, the vector space

$$H := H^0(S, \mathcal{J}_{\{x_1, \dots, x_\delta\}} \otimes E)$$

has dimension at least two. For α and β general in H , the curve $X := V(\alpha \wedge \beta)$, where $\alpha \wedge \beta \in H^0(S, L)$, is irreducible, with nodes exactly at the points x_i 's, $1 \leq i \leq \delta$. On

the other hand, the given two sections generate a rank-one subsheaf of the restriction $E|_X$. If $\nu : C \rightarrow X$ is the normalization, this rank-one subsheaf induces a line sub-bundle $\mathcal{L}_{\mathcal{A}} \subset \nu^*(E)$ with two sections and whose moving part on C is of degree $k + 1 - \delta$. Hence C is $(k + 1 - \delta)$ -gonal.

If we want C to be hyperelliptic, we set $\delta = k - 1$. Since $2\delta \leq k$, then $k \leq 2$. Therefore, the only case one obtains using this procedure, is the one discussed in Example 2.8, with $k = 2$, $p = 4$ and $\delta = 1$.

The second example we want to consider is the following.

Example 2.10. Let $p = 5$ and let S be a general, primitive complete intersection surface of type $(2, 2, 2)$ in \mathbb{P}^5 . We consider the case $\delta = 2$ which, by (2.1), is the only possible, nontrivial example to construct. By what recalled in Remark 0.4, $V_{|H|,2} \neq \emptyset$. Let $[X] \in V_{|H|,2}$ and let $N = \text{Sing}(X)$; let $\Lambda_N := l$ be the line spanned by the two nodes of X . Since C , the normalization of X , has genus $g = 3$ then, by using Proposition 2.3, we first show the following:

Step 1: $\tau_1(X) = \Gamma \subset \mathbb{P}^2$ is a conic $\Leftrightarrow X \subset H = \mathbb{P}^4$ lies on a quadric cone, which is a rank-3 quadric in \mathbb{P}^4 with vertex the line l and base the conic $\Gamma \Leftrightarrow X$ is in the join variety of Γ and l in \mathbb{P}^4 .

Proof. (\Rightarrow) If $U \in |\mathcal{J}_{l/H}(1)|$, then $\#U \cap X = 8$ and 4 intersections are absorbed by the two nodes. Let \mathbb{P}_{Γ}^2 be the plane containing Γ ; since $U \cap \mathbb{P}_{\Gamma}^2$ is a line, then $U \cap \Gamma$ are two points $\gamma_1, \gamma_2 \in \mathbb{P}_{\Gamma}^2$. By assumption, there exist points $n_i, q_i \in X$, $1 \leq i \leq 2$, such that

$$\tau_X(n_i) = \tau_X(q_i) = \gamma_i, \quad 1 \leq i \leq 2;$$

furthermore, the configuration of points $\{n_i, q_i, \gamma_i\}$ lies on the same plane $\pi_i \subset U$ through l , $1 \leq i \leq 2$. Therefore, the configuration of points $\{n_1, q_1, \gamma_1, n_2, q_2, \gamma_2\}$ lies on the singular quadric $\Sigma_l = \pi_1 \cup \pi_2 \subset U$. If we take the pencil of \mathbb{P}^3 's in H , passing through l and cutting 2 points on Γ , we get that X is contained in the cone over Γ in \mathbb{P}^4 , with vertex l .

(\Leftarrow) Let \mathcal{Q}_l be a rank-3 quadric in \mathbb{P}^4 and assume that $X \subset \mathcal{Q}_l$. Let $U \in |\mathcal{J}_{l/H}(1)|$; since $U \cap X \subset U \cap \mathcal{Q}_l = \Sigma_l = \pi_1 \cup \pi_2$ and since $\#U \cap X = 8$, by symmetry we have two smooth points of X on each π_i , $1 \leq i \leq 2$. By definition of the maps τ_X and π_l , we conclude. \square

Now, we want to lift-up to S the geometric information proved in Step 1. Precisely, we show:

Step 2: $X \subset H = \mathbb{P}^4$ lies on a rank-3 quadric in H , with the two nodes on the linear vertex l of the quadric \Leftrightarrow the surface $S = \bigcap_{i=1}^3 Q_i$ lies on a rank-5 quadric in \mathbb{P}^5 having l as a generatrix, which is a 2-secant of S .

Proof. (\Leftarrow) After a possible change of basis, one can assume that Q_1 is a rank-5 quadric in \mathbb{P}^5 , which is a cone of vertex a point v and base a smooth quadric \mathcal{Q} in \mathbb{P}^4 . If we project from v to \mathbb{P}^4 , then Q_1 projects onto \mathcal{Q} and S onto a birationally equivalent surface $F \subset \mathcal{Q}$, having a double curve in its singular locus. This means that, through v , we have a pencil of generatrices of Q_1 which are 2-secants of S . Let l be one of such generatrices. Let $H \in |\mathcal{J}_{l/\mathbb{P}^5}(1)|$ which is tangent to Q_1 along l . Then $H \cap S = X$ is a curve with two nodes at the points $S \cap l$. Since $S \subset Q_1$, then $X \subset Q_1 \cap H := \mathcal{Q}_l$, which is a rank 3 quadric in \mathbb{P}^4 having l as the linear vertex.

(\Rightarrow) Let \mathcal{Q}_l be a rank 3 quadric in \mathbb{P}^4 vertex a line l and assume that $X \subset \mathcal{Q}_l$. By assumption $\text{Pic}(S) = \mathbb{Z}[H]$, therefore l is not contained in S . Since $X \sim H$ on S , the hyperplane H must be tangent to S only at the two nodes of X , which are $l \cap S$. Since $\mathcal{Q}_l \subset \mathbb{P}^4$ contains both $X = S \cap H$ and $l \subset H$ and since S is non-degenerate, irreducible and linearly normal, then \mathcal{Q}_l extends to a unique, irreducible quadric Q_l in \mathbb{P}^5 containing S and obviously l (see e.g. [25, Lemma 7.9], whose arguments easily extend to the case of an irreducible, nodal hyperplane section of S).

Now, $l = \text{Sing}(\mathcal{Q}_l) = \text{Sing}(Q_l \cap H)$; therefore, by standard facts on hyperquadrics (see e.g. [14], page 283), we have that either

- (i) Q_l is a rank 3 quadric in \mathbb{P}^5 , with vertex a plane Π_l containing l , or
- (ii) Q_l is a rank 4 quadric in \mathbb{P}^5 with vertex l , or
- (iii) l is not contained in $\text{Sing}(Q_l)$ and H is a tangent hyperplane to Q_l along l .

We claim that only case (iii) can occur. Indeed, in case (i), since $S \subset Q_l$ is of type $(2, 2, 2)$ in \mathbb{P}^5 , then other two linearly independent quadrics would intersect Π_l , so $S \cap \Pi_l \neq \emptyset$, which would give that S is singular, a contradiction. In case (ii), since $S \cap l = \text{Sing}(X)$, then S would be singular too. Therefore we must be in case (iii).

Now, in case (iii), Q_l must be necessarily a rank 5 quadric with vertex a point and l a generatrix. Indeed, if Q_l were smooth, then the Gaussian map would give an isomorphism between Q_l and its dual hypersurface, which is still a hyperquadric, so the points of l could not have the same tangent hyperplane. \square

To sum up, in order to construct examples of 2-nodal curves which are hyperplane sections on a general, primitive, complete intersection surface S of type $(2, 2, 2)$ in \mathbb{P}^5 , it is sufficient to consider the quadric cones, with vertex a point, containing S and hyperplanes which are tangent to these cones along generatrices which are 2-secants of S . Observe that such a S is contained in a two-dimensional family of hyperquadrics in \mathbb{P}^5 . Among these hyperquadrics, we have a one-dimensional family of quadric cones having a zero-dimensional vertex. Each such a cone has a one-dimensional family of 2-secants of S through its vertex. Therefore, by recalling notation as in Definition 0.5, we have, a priori, $\dim(V_{|H|,2}^{hyper}) \geq 2$. Furthermore,

$$V_{|H|,2}^{hyper} \subseteq V_{|H|,2}$$

and, by Remark 0.4, $\dim(V_{|H|,2}) = 3$.

We claim that the above inclusion is proper. Indeed, let $q_1 \neq q_2 \in S$ be such that the general element $[X] \in V_{|H|,2}$ has $\text{Sing}(X) = \{q_1, q_2\}$. Therefore, there exists a hyperplane $H_X \subset \mathbb{P}^5$ s.t.

$$T_{q_1}(S), T_{q_2}(S) \subset H_X \text{ and } X = H_X \cap S.$$

Since H_X intersects each plane of \mathbb{P}^5 along (at least) a line, in order to contain $T_{q_i}(S)$, $1 \leq i \leq 2$, we have to impose two independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^5}(1)|$.

On the other hand, S is a complete intersection of quadrics in \mathbb{P}^5 , i.e. $S = V(F_1, F_2, F_3)$, with $\deg(F_i) = 2$. Therefore, to impose to a general hyperplane $\sum_{i=0}^5 a_i x_i = 0$ to be tangent at q_1 and at q_2 to one of the three quadrics $V(F_i)$ gives more than two independent conditions on the coefficients $\{a_0, \dots, a_5\}$.

In conclusion, we have $\dim(V_{|H|,2}^{hyper}) = 2$.

This shows that all the elements of $V_{|H|,2}^{hyper}$ on S are obtained from the above construction; furthermore, this gives examples of 2-hyperelliptic, nodal curves in the range $\rho_{sing} > 0$. Indeed, in this case we have $g = 3$, so $\rho_{sing}(5, 1, 2, 3) = \rho(3, 1, 2) + 2 = 1$.

Remark 2.11. In both examples presented above, $V_{|L|,\delta}^{hyper}$ is nonempty and two-dimensional. We will return in a forthcoming work to the study of the nonemptiness and the dimension of $V_{|L|,\delta}^{hyper}(S)$, for $\rho_{sing}(p_a(L), 1, 2, p_a(L) - \delta) \geq 0$ and (S, L) a general, primitive $K3$.

Note also that, in Examples 2.8 and 2.10, $V_{|L|,\delta}^{hyper}$ is of codimension one in $V_{|L|,\delta}$, for $\delta = 1, 2$, respectively. This shows that the property of having hyperelliptic normalizations is not constant among all the nodal curves in a Severi variety, contrary to the case of *smooth* curves (cf. the theorems of Saint-Donat [25] and Green and Lazarsfeld [12] stated in the introduction). Moreover, the “unexpected” property of having hyperelliptic normalization is not induced by a line bundle on the surface. This indicates that the Brill-Noether theory for singular curves on $K3$ surfaces appears to be more subtle than the one for smooth curves.

3. NEUTRAL AND NON-NEUTRAL NODES

In this section we briefly make some comments on the behavior of the nodes of a δ -hyperelliptic curve under the hyperelliptic involution, first in the general case, and then when the curve lies on a $K3$ surface.

Let X be a nodal curve (not necessarily lying on a $K3$ surface) and $N := \text{Sing}(X)$ be its scheme of nodes. If \mathcal{A} is a rank-one, torsion-free sheaf on X then, by the hypotheses, one has:

$$\text{Sing}(\mathcal{A}) := \{q \in X \mid \mathcal{A}_q \text{ is not locally free}\} \subseteq N \text{ and } \mathcal{A}_q \cong \mathfrak{m}_q \text{ if } q \in \text{Sing}(\mathcal{A}),$$

where $\mathfrak{m}_q \subset \mathcal{O}_{X,q}$ is the maximal ideal (cf. e.g. [13] or [26], pp. 163-165).

From now on in this section, unless otherwise stated, we shall focus on the following situation:

Notation 3.1. Assume that:

- (i) X is an irreducible curve of arithmetic genus $p = p_a(X)$ with δ nodes as the only singularities;
- (ii) $N = \text{Sing}(X)$ is the (reduced) scheme of nodes of X ;
- (iii) for each subscheme $Z \subseteq N$, denote by $\nu_Z : X_Z \rightarrow X$ the partial normalization of X along Z . In particular, when $Z = N$, then $X_N = C$ and $\nu_N = \nu : C \rightarrow X$ is the (total) normalization of X ; in this case the smooth curve C has (geometric) genus $g = p - \delta$, which is assumed to be $g \geq 2$;
- (iv) there exists a line bundle Δ_C on C such that $|\Delta_C| = \mathfrak{g}_2^1$, i.e. C is hyperelliptic. From the assumption $g \geq 2$, Δ_C is unique.

Note that the sheaf $\nu_*(\Delta_C)$ on X has the following properties:

- $\nu_*(\Delta_C)$ is torsion-free of rank one,
- $\deg(\nu_*(\Delta_C)) = 2 + \delta$,
- $\text{Sing}(\nu_*(\Delta_C)) = N$, and
- $h^0(X, \nu_*(\Delta_C)) = h^0(C, \Delta_C) = 2$.

We now recall the classical notion of *neutral* (resp. *non-neutral*) pairs with respect to the \mathfrak{g}_2^1 on C .

Definition 3.2. Let $n \in N$ and $\nu^{-1}(n) = \{n', n''\} \subset C$. The pair (n', n'') is called a neutral pair if $n' + n'' \in |\Delta_C|$ (otherwise the pair is called non-neutral). For brevity, the node $n \in X$ will be called a neutral node (non-neutral node, respectively).

We extend the notion of neutral (non-neutral, resp.) nodes to linear series composed with the hyperelliptic involution of C . The typical situation is given by the canonical linear system $|\omega_C| = \mathfrak{g}_{2g-2}^{g-1}$. Since C is by assumption hyperelliptic, with $g \geq 2$, then $|\omega_C| = (g-1)|\Delta_C|$.

If $|L_C| = \mathfrak{g}_{2r}^r = \mathfrak{g}_2^1 \circ \mathfrak{g}_2^1 = r|\Delta_C|$, then the associated morphism

$$(3.3) \quad C \xrightarrow{\Psi} \Gamma_r \subset \mathbb{P}^r,$$

which is 2 : 1 onto a rational normal curve $\Gamma_r \subset \mathbb{P}^r$, is the composition of the 2 : 1 map given by $|\Delta_C|$, and of the r -tuple Veronese embedding. Hence, it makes sense to give the following:

Definition 3.4. Let $n \in N$ be a node of X and let (n', n'') be the corresponding pair on C . Then (n', n'') is a neutral pair (equiv., n is a neutral node) for $|L_C|$ if and only if it is a neutral pair (equiv., a neutral node) for $|\Delta_C|$.

Proposition 3.5. Let $n \in N$ and let $|L_C| = \mathfrak{g}_{2r}^r$, with $r \leq g-1$, be a base point free linear system on C which is composed with $|\Delta_C|$. Then $\nu_*(L_C)$, as a rank-one torsion-free sheaf on X , is not globally generated at the neutral nodes, and globally generated elsewhere.

Proof. First of all, observe that as a straightforward consequence of Definition 3.5, $\nu_*(L_C)$ is not g.g. at a point $p \in X$ if and only if $\nu_*(\Delta_C)$ is. Therefore, since ν is an isomorphism outside N , it is sufficient to study the generation of $\nu_*(\Delta_C)$ at the nodes of X . Using Notation 3.1, let $A \subseteq N$ be the (possibly empty) subset of neutral nodes and $Z \subseteq N$ be the (possibly empty) subset of non-neutral nodes of X . Consider the following commutative diagram:

$$(3.6) \quad \begin{array}{ccc} C & \xrightarrow{\tau} & X_Z \\ & \searrow \nu & \downarrow \varphi \\ & & X, \end{array}$$

where φ normalizes the nodes in Z (so X_Z has still $a := \text{Card}(A)$ nodes), and $\tau = \nu_A$ normalizes the residual a nodes. Denote by $f := f_{\Delta_C} : C \rightarrow \mathbb{P}^1$ the (2 : 1)-morphism induced by $|\Delta_C|$.

Since the map τ identifies a neutral pairs, we have that the morphism f induces on X_Z a (2 : 1)-morphism $h : X_Z \rightarrow \mathbb{P}^1$, which gives the commutative diagram

$$(3.7) \quad \begin{array}{ccc} C & \xrightarrow{\tau} & X_Z \\ & \searrow f & \downarrow h \\ & & \mathbb{P}^1, \end{array}$$

i.e. Δ_C induces a line bundle $R \in \text{Pic}^2(X_Z)$ such that:

- (i) R is globally generated on X_Z (since h is a morphism), and
- (ii) $\tau^*(R) = \Delta_C$ (since the \mathfrak{g}_2^1 on C is unique).

From (ii), we have that $\tau_*(\Delta_C) = R \otimes \tau_*\mathcal{O}_C$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{X_Z} \rightarrow \tau_*\mathcal{O}_C \rightarrow \mathcal{O}_A \rightarrow 0$$

tensored by R , we see that the cokernel of the evaluation morphism

$$H^0(\tau_*(\Delta_C)) \otimes \mathcal{O}_{X_Z} \cong H^0(R) \otimes \mathcal{O}_{X_Z} \rightarrow \tau_*(\Delta_C)$$

is exactly \mathcal{O}_A . So $\nu_*(\Delta_C)$ is not generated at the neutral nodes.

On the other hand, if $\varphi_*(R)$ were not globally generated as a rank-one torsion-free sheaf on X , then consider $\mathcal{L} := \text{im}(ev)$, where $ev : H^0(\varphi_*(R)) \otimes \mathcal{O}_X \rightarrow \varphi_*(R)$ is the (non-surjective) evaluation map. It is easy to show that $\mathcal{L} \in \text{Pic}(X)$ and $\varphi^*(\mathcal{L}) = R$. The latter is impossible, since $\varphi : X_Z \rightarrow X$ identifies the non-neutral pairs, so $\varphi_*(R)$ is globally generated.

From Diagram (3.7), it follows that $\nu_*(\Delta_C)$, as rank-one torsion-free sheaf on X , is globally generated at the non-neutral nodes of X . This concludes the proof. \square

Remark 3.8. We note that from the last proposition and the discussion above, it is clear that the non-neutral nodes N_0 of X can be characterized by any of the three following equivalent descriptions:

- (i) N_0 is the set of the nodes whose two preimages on the normalization of X are not conjugated by the hyperelliptic involution.
- (ii) N_0 is the minimal set of nodes such that the curve obtained by desingularizing these nodes possesses a \mathfrak{g}_2^1 , i.e. it admits a $(2 : 1)$ -morphism to \mathbb{P}^1 .
- (iii) N_0 is the set of nodes at which $\nu_*(\omega_C)$ is globally generated.

If we furthermore assume that X lies on a $K3$ surface and $X \in |L|$ for $L \in \text{Pic}(S)$, then by (2.6) and by the short exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{J}_{N/S} \otimes L \rightarrow \mathcal{J}_{N/X} \otimes L \rightarrow 0,$$

the conditions (i)-(iii) are equivalent to:

- (iv) N_0 is the set of nodes at which $\mathcal{J}_{N/S} \otimes L$ is globally generated as a sheaf on S .

If $|L|$ is hyperelliptic on a $K3$ surface S , then φ_L maps the surface S generically $2 : 1$ onto a rational ruled surface or onto the Veronese surface in \mathbb{P}^5 , by Saint-Donat [25], so that any irreducible member of $|L|$ is mapped $2 : 1$ onto a smooth rational curve. Hence, if $V_{|L|,\delta}$ is not empty, any node of a curve in it must be neutral. To see such an example, it is sufficient to consider S as the general degree-2 $K3$ and take X as the pull-back of a line tangent to the branched plane sextic curve.

The next two examples show that the nodes in Examples 2.8 and 2.10 above are all non-neutral. This will also follow from the results in the next section (more precisely from Theorem 2). However, since this can be seen geometrically, we find it instructive to include these examples at this point.

Example 3.9. Let $S \subset \mathbb{P}^4$ be a general complete intersection of type $(2, 3)$ and let H be the hyperplane section of S . From Example 2.8, we know that $V_{|H|,1}^{hyper} \neq \emptyset$. Let $[X] \in V_{|H|,1}^{hyper}$ and let $\{p\} = \text{Sing}(X)$. Let $\nu : C \rightarrow X$ be the normalization

morphism and let $\nu^{-1}(p) = \{p', p''\}$. Suppose that p is a neutral node of X , i.e. $|\mathcal{O}_C(p' + p'')| = |\Delta_C| = \mathfrak{g}_2^1$ on C . Thus,

$$(3.10) \quad h^0(C, \omega_C \otimes \mathcal{O}_C(-p' - p'')) = h^0(C, \omega_C \otimes \mathcal{O}_C(-p')) = 2.$$

On the other hand, by (2.5), formula (3.10) becomes

$$(3.11) \quad h^0(C, \nu^*(H) - 2p' - 2p'') = h^0(C, \nu^*(H) - 2p' - p'') = 2,$$

which is a contradiction. Indeed, the geometric meaning of (3.11) is that there should exist a pencil of planes in $\mathbb{P}^3 = H$ such that each plane of the pencil passes through p and, since it is tangent at p to one of the two branches of X around p , it must be tangent also to the other branch of X through p . This cannot occur since only one such plane does exist; thus p is a non-neutral node for X .

Example 3.12. From Example 2.10, let $S \subset \mathbb{P}^5$ be a general, primitive $K3$ surface of type $(2, 2, 2)$ and let $[X] \in V_{|H|, 2}^{hyper}(S)$. Let $\text{Sing}(X) = \{n, q\}$. We want to show that both n and q are non-neutral nodes for X .

By contradiction, two cases must be considered. Let $\nu : C \rightarrow X$ be the normalization. Then:

Case 1: Suppose n neutral and q non-neutral. Let $H_X \subset \mathbb{P}^5$ be the hyperplane such that $X = S \cap H_X$.

If $\nu^{-1}(n) = \{n', n''\}$ and $\nu^{-1}(q) = \{q', q''\}$, then $|\mathcal{O}_C(n' + n'')| = |\Delta_C| = \mathfrak{g}_2^1$ on C . Thus, as in Example 3.9, we would have:

$$(3.13) \quad h^0(\nu^*(H) - 2n' - 2n'' - q' - q'') = h^0(\nu^*(H) - 2n' - n'' - q' - q'') = 2,$$

which is a contradiction. Indeed, as above, (3.13) would imply there exist a pencil of hyperplanes in $\mathbb{P}^4 = H_X$ such that each hyperplane of the pencil passes through q and, since it is tangent at n to one of the two branches of X through n , it must be tangent also to the other branch of X through n ; in fact only one such hyperplane actually exists in $\mathbb{P}^4 = H_X$.

Case 2: Suppose both n and q neutral. One can conclude as above.

4. δ -HYPERELLIPTIC NODAL CURVES: A CLASSIFICATION RESULT FOR $\rho_{sing} < 0$.

As already mentioned in the introduction, here we want to classify linear systems $|L|$ on a $K3$ surface S with *arbitrary* Picard group, such that, as in Definition 0.5, $V_{|L|, \delta}^{hyper} \neq \emptyset$ and $\rho_{sing} < 0$. This can be viewed as a natural extension to the nodal case of Saint-Donat's result.

In the particular case of S with $\text{Pic}(S) = \mathbb{Z}[L]$ and $p = p_a(L) \geq 3$, we find again Corollary 1.10 via a different approach.

The techniques used here are related to a Bogomolov-Reider type approach for separation of points by the linear system $|L|$ on S .

By recalling Notation 3.1, we first prove the following more general result, which translates the property of having k -gonal normalization into a failure of separation of zero-dimensional schemes by $|L|$.

Theorem 4.1. *Let S be a $K3$ surface and L be a globally generated line bundle on S of sectional genus $p = p_a(L) \geq 3$. Let δ be an integer such that $\delta \leq p - 2$. Let $[X] \in V_{|L|, \delta}^{hyper}$ and let $N_0 \subseteq \text{Sing}(X) := N$ be a subset of the scheme of the δ nodes of X . Let X_{N_0} be the curve obtained by desingularizing X at the nodes in N_0 .*

Then X_{N_0} possesses a base point free, complete \mathfrak{g}_k^1 , for $k \geq 2$, if and only if there exist k points $q_1, \dots, q_k \in X \setminus N$ such that the zero-dimensional subscheme $N' := N_0 \cup \{q_1, \dots, q_k\} \subseteq X \subseteq S$ does not impose independent conditions on the linear system $|L|$, but for any proper subset $Z \subsetneq \{q_1, \dots, q_k\}$, the subscheme $N_0 \cup Z$ imposes independent conditions on $|L|$; more precisely

$$(4.2) \quad h^1(\mathcal{J}_{N'/S} \otimes L) = 1 \text{ and } h^1(\mathcal{J}_{(N_0 \cup Z)/S} \otimes L) = 0 \text{ for any } Z \subsetneq \{q_1, \dots, q_k\}$$

Moreover, this property holds for every k -uple of points $q_1, \dots, q_k \in X \setminus N$ (not necessarily distinct) such that their inverse images on X_{N_0} are part of the \mathfrak{g}_k^1 .

Proof. Denote by $\mu_{N_0} : S_{N_0} \rightarrow S$ the blow-up of S along N_0 , which induces the morphism $\nu_{N_0} : X_{N_0} \rightarrow X$, which normalizes the nodes in N_0 . Let $B := \sum_{i=1}^{\delta} E_i$ be the total μ_{N_0} -exceptional divisor and let $\tilde{N}_0 = \text{Supp}(\mathcal{O}_{X_{N_0}}(B))$ be the scheme of $2\delta_0$ points of X_{N_0} , the pre-images of the $\delta_0 \leq \delta$ nodes in N_0 . As in (2.5), $\omega_{X_{N_0}} \cong \mathcal{O}_{X_{N_0}}(\nu^*L) \otimes \mathcal{O}_{X_{N_0}}(-\tilde{N}_0)$, where $\omega_{X_{N_0}}$ is the dualizing sheaf of X_{N_0} . This gives

$$(4.3) \quad H^0(X_{N_0}, \omega_{X_{N_0}}) \cong \frac{H^0(S, \mathcal{J}_{N_0/S}(L))}{H^0(\mathcal{O}_S)}.$$

Let $q'_1, \dots, q'_k \in X_{N_0} - \tilde{N}_0 - \text{Sing}(X_{N_0})$ be k (not necessarily distinct) points and $q_1, \dots, q_k \in X \setminus N$ their images via ν . Consider π the further blow-up of S_{N_0} along $\{q'_1, \dots, q'_k\}$. Therefore, $\mu_{N'} := \pi \circ \mu_{N_0}$ is the total blow-up of S along $N' := N_0 \cup \{q_1, \dots, q_k\}$. Denote by E_{q_1}, \dots, E_{q_k} the $\mu_{N'}$ -exceptional divisors over q_1, \dots, q_k , respectively.

Denote by \overline{X}_{N_0} the proper transform of X in $S_{N'}$. Then $X_{N_0} \cong \overline{X}_{N_0}$; let q''_j be the point on \overline{X}_{N_0} corresponding to q'_j on X_{N_0} , $1 \leq j \leq k$. Consider the standard exact sequences

$$(4.4) \quad 0 \rightarrow \mathcal{O}_{S_{N'}}(B) \rightarrow \mathcal{O}_{S_{N'}}(\mu_{N'}^*(L) - B - E_{q_1} - \dots - E_{q_k}) \rightarrow \omega_{\overline{X}_{N_0}}(-q''_1 - \dots - q''_k) \rightarrow 0$$

and

$$(4.5) \quad 0 \rightarrow \mathcal{O}_{S_{N'}}(B + E_{q_1} + \dots + E_{q_k}) \rightarrow \mathcal{O}_{S_{N'}}(\mu_{N'}^*(L) - B) \rightarrow \omega_{\overline{X}_{N_0}} \rightarrow 0$$

on $S_{N'}$. By the Leray isomorphism, Fujita's Lemma (cf. [19]) and by the fact that S is $K3$, we get that $h^1(\mathcal{O}_{S_{N'}}(B)) = h^2(\mathcal{O}_{S_{N'}}(\mu_{N'}^*(L) - B - E_{q_1} - \dots - E_{q_k})) = 0$ and $h^2(\mathcal{O}_{S_{N'}}(B)) = 1$. Hence, by the Leray isomorphism again, we have

$$(4.6) \quad H^0(\omega_{\overline{X}_{N_0}}) \cong \frac{H^0(S, \mathcal{J}_{N_0/S}(L))}{H^0(\mathcal{O}_S)}, \quad H^0(\omega_{\overline{X}_{N_0}}(-q''_1 - \dots - q''_k)) \cong \frac{H^0(S, \mathcal{J}_{N'/S}(L))}{H^0(\mathcal{O}_S)},$$

and

$$(4.7) \quad h^1(\omega_{\overline{X}_{N_0}}(-q''_1 - \dots - q''_k)) = h^1(S, \mathcal{J}_{N'/S}(L)) - 1.$$

From Serre duality and (4.7) we see that $|\mathcal{O}_{\overline{X}_{N_0}}(q'_1 + \dots + q'_k)|$ is a \mathfrak{g}_k^1 if and only if $h^1(\mathcal{J}_{N'/S}(L)) = 1$ or, equivalently, from (4.6) if and only if N' imposes only $\delta_0 + k - 1$ conditions on $|L|$.

At the same time, the \mathfrak{g}_k^1 is base point free if and only if, for any proper subset $Z \subsetneq \{q_1, \dots, q_k\}$, denoting the corresponding subscheme on \overline{X}_{N_0} by Z'' , we have $h^0(\mathcal{O}_{\overline{X}_{N_0}}(Z'')) = 1$, which is equivalent to saying that $h^1(\mathcal{J}_{(N_0 \cup Z)/S}(L)) = 0$, or that $N_0 \cup Z$ imposes independent conditions on $|L|$, as $h^1(L) = 0$ since L is big and nef.

To conclude, we have left to show that if N' does not impose independent conditions on $|L|$, but $N_0 \cup Z$ does, for any proper subset $Z \subsetneq \{q_1, \dots, q_k\}$, then $h^1(\mathcal{J}_{N'/S}(L)) = 1$.

Assume, to get a contradiction, that $h^1(\mathcal{J}_{N'/S}(L)) > 1$. Then, as in (4.7), we would have that $h^0(\mathcal{O}_{\overline{X}_{N_0}}(q_1'' + \dots + q_k'')) > 2$. Therefore, we can find a proper subset $Z \subsetneq \{q_1, \dots, q_k\}$, such that, denoting the corresponding subscheme on \overline{X}_{N_0} by Z'' , we have $h^0(\overline{X}_{N_0}(Z'')) \geq 2$. Hence, arguing as above, $N_0 \cup Z$ does not impose independent conditions on $|L|$, a contradiction. \square

Using Theorem 4.1 and the Reider-like results on higher order embeddings of $K3$ surfaces in [20], we are now ready to prove Theorem 2, stated in the introduction, which classifies linear systems $|L|$ on S for which $V_{|L|, \delta}^{hyper} \neq \emptyset$, with $\rho_{sing} < 0$. We remark that, as already mentioned in the introduction, a similar statement can also be obtained for curves with normalizations possessing \mathfrak{g}_k^1 's for $k > 2$, using Theorem 4.1 and the corresponding notion of neutral and non-neutral nodes for \mathfrak{g}_k^1 's. The principle is the same, but the classification becomes longer.

Proof of Theorem 2. Assume that $V_{|L|, \delta}^{hyper} \neq \emptyset$ and let $[X]$ be a point of this scheme. Let $N_0 \subseteq N := \text{Sing}(X)$ be the set of non-neutral nodes of X , which consists of $\delta_0 \leq \delta$ points, by assumption. Then, by Theorem 4.1 and Remark 3.8, there exist two points $p, q \in X \setminus N$ such that, if $N' = N_0 \cup \{p, q\}$, then $|L|$ does not separate N' .

Let $Z \subseteq N'$ be minimal, so that for each $Z' \subsetneq Z$, $|L|$ separates Z' but it does not separate Z . We claim that $Z = N'$.

Indeed, if $p \notin Z$ then in particular $|L|$ does not separate $N_0 \cup \{q\}$ and by (4.3) it follows that the inverse image of q is a base point of $|\omega_{X_{N_0}}|$, a contradiction. By symmetry we therefore have that both $p, q \in Z$. If $Z = N_1 \cup \{p, q\}$ for some $N_1 \subsetneq N_0$, then already a partial normalization of $\delta_1 := \text{length } N_1$ of the nodes of X would admit a \mathfrak{g}_2^1 , contradicting property (ii) in Remark 3.8. Hence $Z = N'$.

We have that $\text{length}(N') = \delta_0 + 2$. Assume now that

$$\delta_0 \leq \frac{p_a(L) - 3}{2},$$

or equivalently

$$(4.8) \quad L^2 \geq 4(\text{length}(N') - 1).$$

By [20, Theorem 1.1], the condition (4.8) implies that there exists an effective divisor D on S passing through N' and satisfying the numerical conditions

$$(4.9) \quad 2D^2 \stackrel{(i)}{\leq} L.D \leq D^2 + \delta_0 + 2 \stackrel{(ii)}{\leq} 2\delta_0 + 4$$

with equality in (i) if and only if $L \sim 2D$ and $L^2 \leq 4\delta_0 + 8$,
and equality in (ii) if and only if $L \sim 2D$ and $L^2 = 4\delta_0 + 8$.

Setting $D' := L - D$ one easily finds that (4.9) implies

$$(4.10) \quad D'.D \leq \delta_0 + 2.$$

Furthermore, by [20, Lemma 3.6(v)] (or the proof of [18, Prop. 1.13]), we have $L.(L - 2D) \geq 0$, whence

$$(4.11) \quad D'^2 \geq D^2.$$

By (4.8) and (4.9) we have

$$L^2 \geq 4\delta_0 + 4 > 2\delta_0 + 4 \geq D.L,$$

whence $D \not\cong X$, by the nefness of L . Therefore, since $D \supset N'$ we have by Bezout's Theorem

$$(4.12) \quad D.X = D.L \geq 2\delta_0 + 2.$$

This gives

$$(4.13) \quad D^2 = D.L - D.D' \geq 2\delta_0 + 2 - \delta_0 - 2 = \delta_0 > 0.$$

The Hodge index theorem and (4.10) yields

$$(4.14) \quad (D^2)^2 \leq D^2 D'^2 \leq (D.D')^2 \leq (\delta_0 + 2)^2,$$

which gives $\delta_0 \leq D^2 \leq \delta_0 + 2$ (note that, D^2 is even, since the intersection form on a $K3$ surface is even).

If $D^2 = \delta_0 + 2$ we get from (4.9) that $L \sim 2D$, so that $L^2 = 4\delta_0 + 8$, i.e. $p_a(L) = 2\delta_0 + 5$, which is the second case in (ii) of the theorem.

If $D^2 = \delta_0 + 1$ we get from (4.9) that either $L.D = 2\delta_0 + 2$ with $L \sim 2D$, or $L.D = 2\delta_0 + 3$.

In the first case we get $L^2 = 4\delta_0 + 4$, i.e. $p_a(L) = 2\delta_0 + 3$, which is the first case in (ii). Moreover, since L is nonhyperelliptic by assumption, we must have $D^2 \geq 4$, whence $\delta_0 \geq 3$, by Saint-Donat's theorem quoted in the introduction.

In the second we get $D.D' = D.L - D^2 = \delta_0 + 2$. From (4.11) and (4.14) and the fact that δ_0 is odd we get that $D'^2 = \delta_0 + 1$ or $\delta_0 + 3$, yielding $L^2 = D^2 + D'^2 + 2D.D' = 4\delta_0 + 6$ and $4\delta_0 + 8$ respectively. In the first subcase we get from the results in [20] (see the precise statement in [18, Prop. 1.13]) that $L \sim 2D + \Gamma$ for a smooth rational curve Γ such that $\Gamma.D = 1$ and $\Gamma \cap N' \neq \emptyset$. But since $\Gamma.L = 0$ we have $\Gamma \cap X = \emptyset$, a contradiction. In the second subcase we compute $(L - 2D)^2 = 0$ and $(L - 2D).L = 2$ so that $|L|$ is a hyperelliptic system by Saint-Donat's theorem quoted in the introduction, a contradiction.

Finally, if $D^2 = \delta_0$ then δ_0 is even and we get from (4.9) and (4.12) that $L.D = 2\delta_0 + 2$. Then (4.14) yields $D'^2 = \delta_0, \delta_0 + 2$ or $\delta_0 + 4$ or $(\delta_0, D'^2) = (2, 8)$. The last case is case (i) in the theorem and one easily sees that the first three cases yield the three cases in (iii).

We have therefore proved that if $\delta_0 \leq \frac{p_a(L)-3}{2}$ we must be in one of the three cases (i)-(iii) of Theorem 2. In particular we have proved that either $\delta_0 \geq (p_a(L) - 5)/2$ or $(\delta_0, p_a(L)) = (2, 10)$, as stated in the theorem. \square

Now Corollary 3 in the introduction immediately follows.

Proof of Corollary 4. If $\delta \leq \frac{p_a(nL)-3}{2} = \frac{1}{4}n^2L^2 - 1$, then $\delta_0 \leq \frac{p_a(nL)-3}{2}$, where δ_0 is the number of non-neutral nodes of $X \in V_{|nL|, \delta}^{hyper}$. By Theorem 2 and the fact that $\text{Pic}(S) \cong \mathbb{Z}[L]$, we must be in one of the cases (i) or (ii) of Theorem 2.

In case (i) we must have $n = 3, L^2 = 2$ and $2 = \delta_0 \leq \delta \leq 3$, whence $\delta = 2$ or 3 . This yields case (a) in the corollary.

In case (ii), we have $nL \sim 2D$, with $D \sim kL$ for some $k \geq 1$, whence $n = 2k$. Moreover $D^2 = \delta_0 + 1$ or $\delta_0 + 2$.

If $D^2 = \delta_0 + 1$ we find $\delta_0 = \frac{1}{4}n^2L^2 - 1$, so that $\delta = \delta_0$. It follows that $k = \sqrt{\frac{\delta+1}{L^2}}$ and $n = 2\sqrt{\frac{\delta+1}{L^2}}$, which is case (b) in the corollary.

If $D^2 = \delta_0 + 2$ we find $\delta_0 = \frac{1}{4}n^2L^2 - 2$, so that $\delta_0 = \delta - 1$ or δ . In the first case we end up in (b) as above, and in the second we get $k = \sqrt{\frac{\delta+2}{L^2}}$ and $n = 2\sqrt{\frac{\delta+2}{L^2}}$, which is case (c) in the corollary. \square

We remark that for what concerns the case $\delta = 1$ we have a complete picture.

Corollary 4.15. *Let S be a K3 surface and L be a globally generated, nonhyperelliptic line bundle on S of sectional genus $p = p_a(L) \geq 3$. Then:*

- (i) $V_{|L|,1}^{hyper} = \emptyset$, if $p \geq 5$;
- (ii) if, furthermore, S is a general, primitive K3 such that $\text{Pic}(S) = \mathbb{Z}[L]$, then

$$\dim(V_{|L|,1}^{hyper}) = 2, \text{ for } p = 3, 4.$$

Proof. Statement (i) is an immediate consequence of Theorem 2. Statement (ii) follows from Example 2.8 when $p = 4$. For $p = 3$, just take S as general in \mathbb{P}^3 embedded by $|L|$ and, through the general point $p \in S$, take a hyperplane section which is tangent to S at p . This yields an irreducible curve on S with one node, that is a curve of geometric genus two, which is automatically hyperelliptic. Therefore $\dim(V_{|L|,1}^{hyper}) = \dim(V_{|L|,1}) = 2$ in this case as well. \square

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