Svolgimento Esercizio 1

(i) Costruzione $\mathbb{P}^1(\mathbb{R})$ e legame con retta affine reale

$$\mathbb{P}^1(\mathbb{R}) = \frac{\mathbb{R}^2 \setminus \{\underline{0}\}}{\sim}$$

$\sim$ = proporzionalità

$\underline{v}, \underline{w} \in \mathbb{R}^2 \setminus \{0\}$ $\quad \underline{v} \sim \underline{w} \Leftrightarrow$
$\exists \lambda \in \mathbb{R}^* \text{ t.c. } \underline{w} = \lambda \underline{v}$

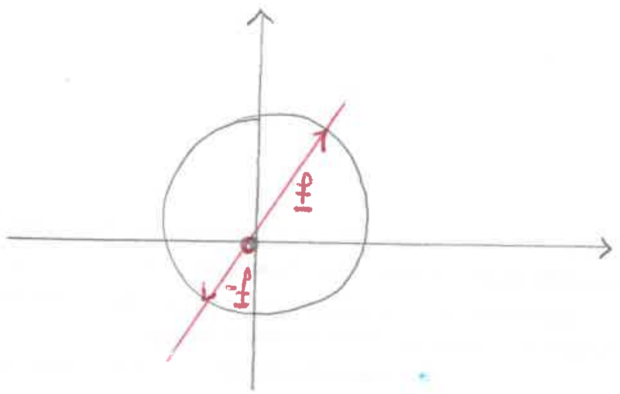
$$[\underline{v}] = \operatorname{Span}\{\underline{v}\} \setminus \{\underline{0}\} \longleftrightarrow \text{punti di } \mathbb{P}^1(\mathbb{R})$$

- Se $(\mathbb{R}^2, \mathcal{E}, \langle\,,\rangle_{st})$ considero

$$\boxed{S^1 = \{\underline{v} \in \mathbb{R}^2 \setminus \{0\} \mid \|\underline{v}\| = 1\} \subset \mathbb{E}^2_{\mathbb{R}}}$$

Perciò $\forall\, \underline{v} \in \mathbb{R}^2 \setminus \{\underline{0}\} \Rightarrow \dfrac{\underline{v}}{\|\underline{v}\|} = \underline{f}_{\underline{v}} \in S^1$

- Viceversa $\forall \underline{f} \in S^1 \Rightarrow \operatorname{Span}\{\underline{f}\} = \operatorname{Span}\{t \cdot \underline{f}\}, \forall t \in \mathbb{R} \Rightarrow [\underline{f}] = [t\underline{f}]$, $\forall t \in \mathbb{R}^*$

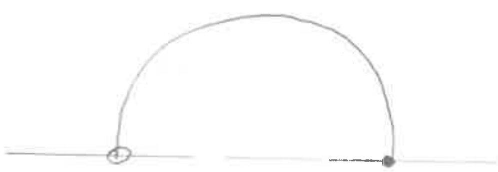


- Però $\forall\, \underline{f} \in S^1 \Rightarrow [\underline{f}] = [-\underline{f}] \Rightarrow \sim$ su $\mathbb{R}^2 \setminus \{0\}$

diventa $\sim_{ap}$ = antipodalità su S^1

Perciò

$$\mathbb{P}^1(\mathbb{R}) = \frac{\mathbb{R}^2 \setminus \{\underline{0}\}}{\sim} = \frac{S^1}{\sim_{ap}}$$



$$\frac{S^1}{\sim_{ap}} = \{ e^{i\pi\theta} \mid \theta \in [0,1)\} = \left\{ (\cos\alpha, \sin\alpha) \;\middle|\; \begin{array}{l}\alpha = \pi\cdot\theta \\ \theta \in [0,1)\end{array} \right\}$$

$\cap$

$\mathbb{E}^2_{\mathbb{R}}$

$\frac{S^1}{\sim_{ap}} = S^1_+ \setminus \{(-1,0)\}$ = calotta superiore di $S^1 \setminus \{(-1,0)\}$

$$S^1_+ \setminus \{(-1,0)\} \longleftrightarrow [0,1)$$
$$e^{i\pi\theta} \longleftrightarrow \theta$$

Modelli di $\mathbb{P}^1(\mathbb{R})$

$$\boxed{\mathbb{P}^1(\mathbb{R}) = S^1_+ \setminus \{(-1,0)\} \equiv [0,1)}$$

Ma

$$[0,1) \longleftrightarrow S^1$$
$$\theta \longrightarrow e^{2\pi i\theta} = (\cos 2\pi\theta, \sin 2\pi\theta)$$

biunivoca

$$\Rightarrow \boxed{\mathbb{P}^1(\mathbb{R}) \equiv S^1}$$

Come si collega con retta affine reale?

Gli elementi di $\mathbb{P}^1(\mathbb{R})$ si identificano a classi di proporzionalità di vettori direttori di rette vettoriali in $\mathbb{R}^2$.

$[\underline{v}] = [(l, m)]$ con $(l,m) \neq (0,0) \in \mathbb{R}^2$

e $(\lambda l, \lambda m) \sim (l, m)$, $\forall \lambda \in \mathbb{R}^*$

vettori direttori di stessa retta

Perciò

$$\{\text{punti di } \mathbb{P}^1(\mathbb{R})\} \longleftrightarrow \boxed{\text{vettori di direzione rette del fascio per } O=(0,0)}$$

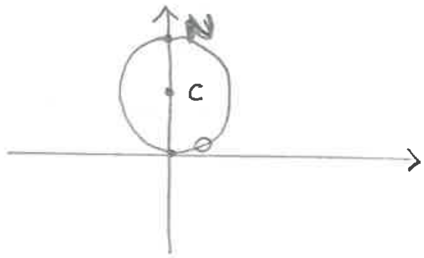
Prendo fascio di rette per O traslato in

$$N = (0, 2) \in \mathbb{E}^2(\mathbb{R})$$

Corridoro

$\mathscr{C}$ = Circonferenza di centro $C = (0,1) \in \mathbb{E}^2(\mathbb{R})$ e raggio $R = 1$: $\boxed{x^2 + (y-1)^2 = 1}$

$$\mathscr{C} \longleftrightarrow S^1$$



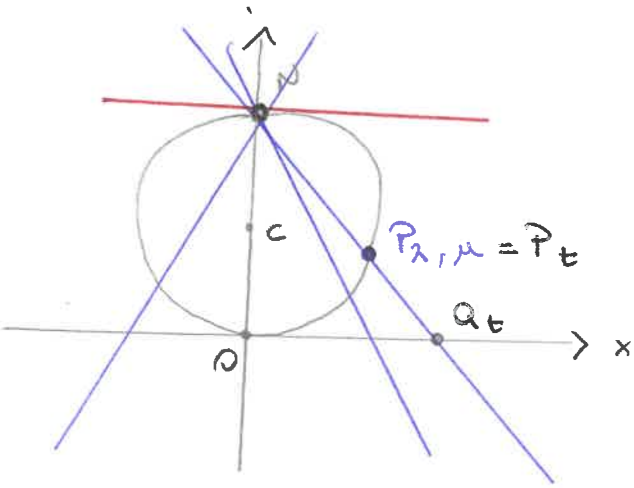
$N = (0,2)$ POLO NORD di $\mathscr{C}$

$C = (0,1)$ Centro di $\mathscr{C}$

$O = (0,0)$ POLO SUD di $\mathscr{C}$

* Fascio di rette per N

$$N = \begin{cases} x = 0 \\ y = 2 \end{cases} \Rightarrow \boxed{\lambda x + \mu(y-2) = 0} \quad \mathcal{F}_{\lambda,\mu}, \quad [\lambda,\mu] \in \mathbb{P}^1(\mathbb{R})$$



Intersezioni $\mathcal{F}_{\lambda,\mu} \cap \mathscr{C}$

$$\begin{cases} x^2 + y^2 - 2y = 0 \\ \lambda x + \mu(y-2) = 0 \end{cases}$$

$\boxed{\text{Se } \lambda = 0}$ $\Rightarrow$ $\mu \neq 0$ e

$$\begin{cases} x^2 + y^2 - 2y = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x^2 = 0 \\ y = 2 \end{cases}$$

$$\mathscr{C} \cap \mathcal{F}_{0,\mu} = \{N\}$$

come punto con molt. alg. 2

cioè $y = 2$ è tangente a $\mathscr{C}$ in N

vettore direttore è $(1,0)$

$\boxed{\text{Se } \lambda \neq 0}$

$$\begin{cases} x^2 + y^2 - 2y = 0 \\ \lambda x + \mu(y-2) = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x^2 + y^2 - 2y = 0 \\ x + \frac{\mu}{\lambda}(y-2) = 0 \end{cases} \qquad \boxed{t := \frac{\mu}{\lambda} \in \mathbb{R}}$$

$\Leftrightarrow$

$$\begin{cases} x^2 + y^2 - 2y = 0 \\ x = t(2-y) \longrightarrow \text{retta } \ell_t \end{cases}$$

$$\begin{cases} t^2(2-y)^2 + y^2 - 2y = 0 \\ x = t(2-y) \end{cases} \Leftrightarrow \begin{cases} t^2(y-2)^2 + y(y-2) = 0 \\ x = t(2-y) \end{cases}$$

$$\begin{cases} (y-2)\left(t^2(y-2)+y\right)=0 \\ x = t(2-y) \end{cases} \iff \begin{cases} (y-2)\left((t^2+1)y - 2t^2\right)=0 \\ x = t(2-y) \end{cases}$$ (4)

Perciò:

- o $y = 2 \Rightarrow x = 0 \Rightarrow$ N = centro fascio è su $\mathscr{C}$
- oppure $y = \dfrac{2t^2}{t^2+1} \Rightarrow x = t\left(2 - \dfrac{2t^2}{t^2+1}\right) =$

$$= t\left(\frac{\cancel{2t^2}+2-\cancel{2t^2}}{t^2+1}\right)$$

$$= \left(\frac{2t}{t^2+1}\right)$$

$$P_{\lambda,\mu} = P_t = \begin{pmatrix} \dfrac{2t}{t^2+1} \\ \dfrac{2t^2}{t^2+1} \end{pmatrix} \in \mathscr{C} \subset \mathbb{E}^2_{\mathbb{R}}$$

$t = \dfrac{\mu}{\lambda} \in \mathbb{R}$
parametro

Notare che

$0 = x - t(2-y) = x + ty - 2t \Rightarrow (1, t)$ = vettore normale retta a ℓ_t

$\Rightarrow (t, -1)$ vettore direttore di ℓ_t

Intersezione $\mathcal{F}_{\lambda,\mu} \cap$ asse x

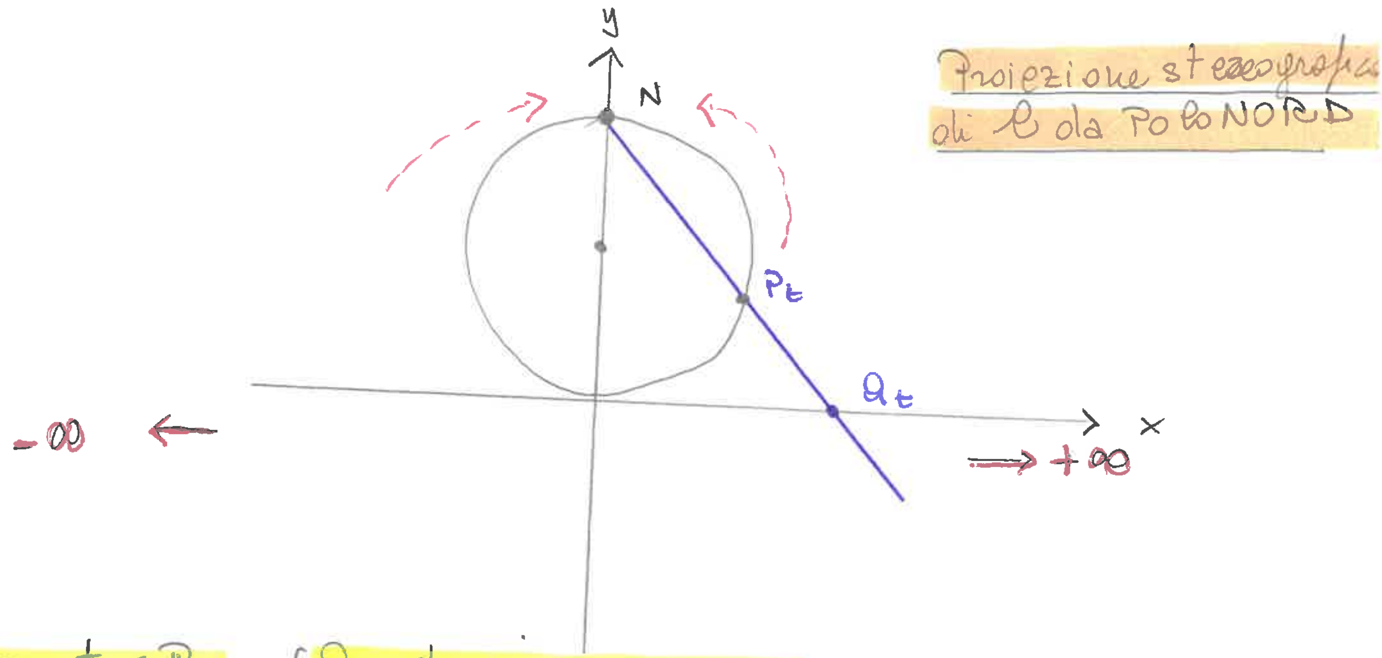
$$\begin{cases} \lambda x + \mu(y-2) = 0 \\ y = 0 \end{cases} \iff \begin{cases} \lambda x - 2\mu = 0 \\ y = 0 \end{cases}$$

$\boxed{\text{Se } \lambda \neq 0}$ $\begin{cases} x = 2\dfrac{\mu}{\lambda} = 2t \\ y = 0 \end{cases} \Rightarrow Q_t = (2t, 0)$

$\boxed{\text{Se } \lambda = 0}$ $\begin{cases} -2\mu = 0 \\ y = 0 \end{cases}$ $\mu = 0$ ✗ no retta fascio

$\Rightarrow \nexists$ soluzione

Ma $\lambda = 0 \iff \mathcal{F}_{0,\mu}: y - 2 = 0 \iff$ retta // asse x



Per $t \in \mathbb{R}$ $\Rightarrow \begin{cases} Q_t \text{ descrive asse } x \\ P_t \text{ descrive } \mathscr{C} \setminus \{N\} \end{cases}$

$(P_t) \to N \leftarrow (-P_t)$

$Q_t = (2t, 0) \to \infty$ se $t \to +\infty$

$-Q_t = (-2t, 0) \to -\infty$ se $t \to -\infty$

$\boxed{\text{N su } \mathscr{C} \text{ completa la retta affine } \Rightarrow \mathbb{P}^1(\mathbb{R}) \text{ completa } \mathbb{R} \text{ con elemento ! } P_\infty}$

$\Rightarrow$ $\mathbb{P}^1(\mathbb{R}) = \{\text{retta asse } x\} \cup \{P_\infty\} = \mathbb{R}_t \cup \{P_\infty\}$

$\mathbb{R}_t : \longrightarrow t = \frac{\mu}{\lambda}$ se $\lambda \neq 0$ $\Rightarrow$ $[\lambda, \mu] \in \mathbb{P}^1 \setminus [0, 1]$

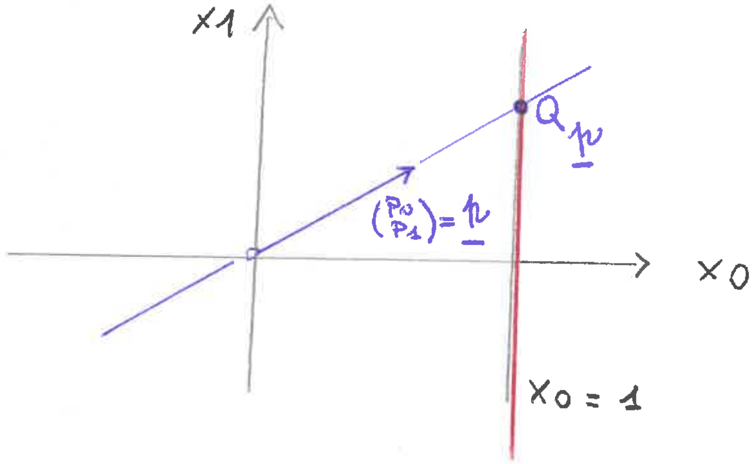
P_∞ : $\lambda = 0 \Rightarrow \mu = 1$ $\Rightarrow$ $[0, 1] \in \mathbb{P}^1$

Pertanto dentro $\mathbb{P}^1$ trovo

$$\boxed{A_0 = \{[\lambda, \mu] \in \mathbb{P}^1(\mathbb{R}) \mid \lambda \neq 0\} = \{[1, \tfrac{\mu}{\lambda}] \in \mathbb{P}^1(\mathbb{R}) \mid \lambda \neq 0\} = \mathbb{R}^1_t = \mathbb{A}^1_t}$$ retta affine

Questa retta affine è compatibile con costruzione formule di $\mathbb{P}^1(\mathbb{R})$ (costruzione dovuta a Desargues) (6)

$\mathbb{R}^2 \equiv \mathbb{A}^2(\mathbb{R})$ con $RA(O, \mathcal{E})$ e coord. (x_0, x_1)



$r : x_0 = 1$ retta // asse x_1 $\Rightarrow$ schermo affine per $\mathbb{P}^1$

$$\boxed{\begin{array}{ccc} \mathbb{R}^2 \setminus \{\underline{0}\} = \mathbb{A}^2(\mathbb{R}) \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}^1(\mathbb{R}) \\ (x_0, x_1) & \longrightarrow & [x_0, x_1] \end{array}}$$

$\forall\ [p_0, p_1] \in \mathbb{P}^1(\mathbb{R}) \Rightarrow$

$$\left\{ \begin{pmatrix} \lambda p_0 \\ \lambda p_1 \end{pmatrix} \middle| \ \lambda \in \mathbb{R}^* \right\} = \mathrm{Span}\Big\{ \underbrace{\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}}_{\underline{v}} \Big\} \setminus \{\underline{0}\} = \pi^{-1}([p_0, p_1])$$

Se $[0, 1] \in \mathbb{P}^1(\mathbb{R})$

$$\mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \setminus \{\underline{0}\} = \pi^{-1}([0,1]) = \{\text{asse } x_1\} \setminus \{0\}$$
$$= \{x_0 = 0\}$$

$\Rightarrow \forall\ [p_0, p_1] \in \mathbb{P}^1(\mathbb{R}) \setminus \{[0, 1]\}$

$$\underline{v} = \begin{pmatrix} \lambda p_0 \\ \lambda p_1 \end{pmatrix} \Rightarrow \begin{cases} x_0 = \lambda p_0 \\ x_1 = \lambda p_1 \end{cases}$$

Interseco con schermo affine $x_0 = 1$

$[p_0, p_1] \neq [0, 1] \Rightarrow p_0 \neq 0$

$$\begin{cases} x_0 = \lambda p_0 \\ x_1 = \lambda p_1 \\ x_0 = 1 \end{cases} \Rightarrow \begin{cases} \lambda p_0 = 1 \\ x_1 = \lambda p_1 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{1}{p_0} \\ x_1 = \frac{p_1}{p_0} \\ x_0 = 1 \end{cases}$$

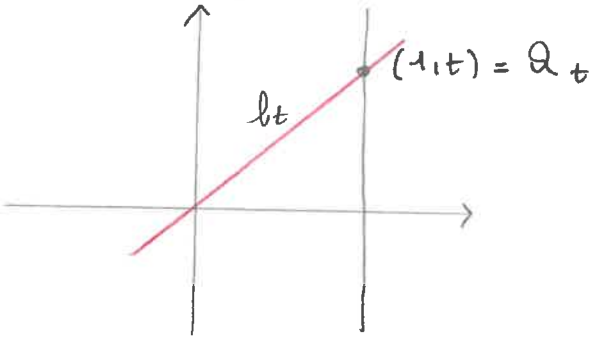
$\Rightarrow Q_{\underline{v}} = \left(1, \frac{p_1}{p_0}\right)$ e $[p_0, p_1] = \left[1, \frac{p_1}{p_0}\right] \in \mathbb{P}^1(\mathbb{R})$

Perciò

$$\boxed{\begin{array}{ccccc} [p_0, p_1] & \longleftrightarrow & \text{retta Span}\{\underline{p}\} & \longleftrightarrow & \text{Punto} \\ \cap & & \text{in } \mathbb{A}^2(\mathbb{R}) & & Q_{\underline{p}} = \left(1, \frac{p_1}{p_0}\right) \\ \mathbb{P}^1(\mathbb{R}) \setminus \{[0,1]\} & & \text{non parallela} & & \text{su schermo} \\ & & \text{a schermo} & & \text{affine} \\ & & x_0 = 1 & & x_0 = 1 \end{array}}$$

Viceversa

$\forall\ Q_t = (1, t) \in \{x_0 = 1\} \Rightarrow \exists!$ retta vettoriale in $\mathbb{A}^2(\mathbb{R})$ che congiunge $O = (0,0)$ con $Q_t = (1,t)$



$\underline{v}_{\ell_t} = \overrightarrow{OQ_t} = (1, t)$ e $[1, t] \in \mathbb{P}^1(\mathbb{R}) \setminus \{[0,1]\}$

Perciò

$$\boxed{\begin{array}{ccccc} \mathbb{P}^1(\mathbb{R}) \setminus \{[0,1]\} & \longleftrightarrow & \text{Schermo affine} & \longleftrightarrow & \text{retta affine} \\ & & x_0 = 1 & & \mathbb{A}^1_t \text{ di} \\ & & \begin{pmatrix} 1 \\ t \end{pmatrix},\ t \in \mathbb{R} & & \text{coordinata } t \\ & & & & =: A_0 \end{array}}$$

Estendiamo corrisp. biunivoca

$$\mathbb{P}^1(\mathbb{R}) \longleftrightarrow \{\text{Schermo affine } x_0 = 1\} \cup \{\infty\} \leftrightarrow A_0 \cup \{P_\infty\}$$

$$[0, 1] \longleftrightarrow \{\infty\} \longleftrightarrow \{P_\infty\}$$

$\{\infty\}$
$\downarrow$
$\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

retta $x_0 = 0$
$\parallel$ $x_0 = 1$ non
taglia punti
su schermo
affine

$|t| \to +\infty$ trovo sempre $[0, 1]$

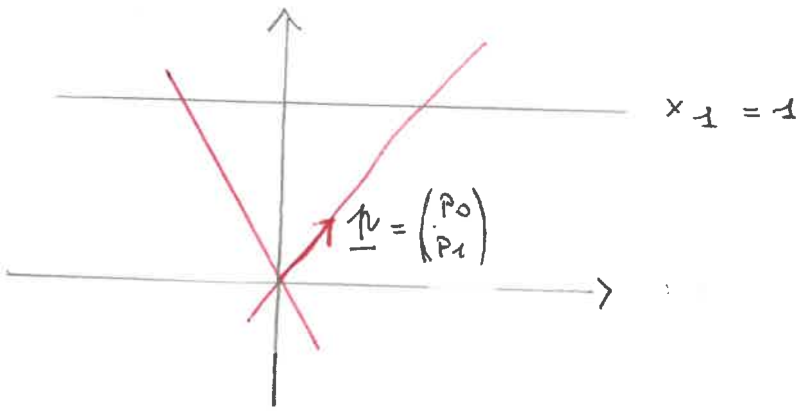
Perciò

$$\mathbb{P}^1(\mathbb{R}) = \underset{\substack{\downarrow \\ \text{retta affine}}}{A_0} \cup \underset{\substack{\downarrow \\ \text{punto all'infinito} \\ \text{per retta affine } A_0}}{\{P_\infty\}}$$

* Ma lo schermo affine $x_0 = 1$ NON È PRIVILEGIATO

Se prendo

$$x_1 = 1$$



$$\mathbb{P}^1(\mathbb{R}) \setminus \{[1,0]\} \longleftrightarrow \left\{ \begin{array}{l} \text{schermo affine} \\ x_1 = 1 \end{array} \right\} \longleftrightarrow \begin{array}{l} \text{retta } \mathbb{A}^1_s \text{ con} \\ s = \begin{pmatrix} \frac{p_0}{p_1} \end{pmatrix} \\ A_1 \end{array}$$

A_1 è retta affine con coordinata $s = \frac{p_0}{p_1}$

Su $A_0 \cap A_1$ $\quad \boxed{s = \frac{1}{t}}$

Inoltre, poiché $[p_0, p_1] \in \mathbb{P}^1(\mathbb{R}) \Longleftrightarrow$ o $p_0 \neq 0$ o $p_1 \neq 0$

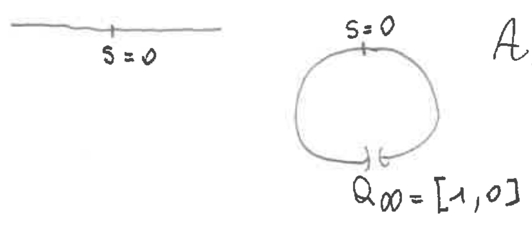
$$\Longrightarrow \boxed{\mathbb{P}^1(\mathbb{R}) = A_0 \cup A_1}$$

A_0 e A_1 carte affini di $\mathbb{P}^1(\mathbb{R})$

Nel modello di S^1

A_0

A_1



(ii) $\mathbb{P}^1(\mathbb{C})$

Le costruzioni di schermi affini e carte affini sono identiche ($\Rightarrow$ indip. da $\mathbb{K}$)

$$\mathbb{P}^1(\mathbb{C}) = A_0 \cup \{[0,1]\} = A_1 \cup \{[1,0]\} = A_0 \cup A_1$$

Modello geometrico cambia

Visto che $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} = \mathbb{A}^1_{\mathbb{C}} \cup \{P_\infty\}$

$\mathbb{A}^1_{\mathbb{C}} = \mathbb{C} =$ piano di Argand Gauss

$$\underline{w} = x + iy \in \mathbb{C}$$

Mi metto in $\mathbb{E}^3_{\mathbb{R}}$, $RC(O, \mathcal{E})$, coord (x, y, z)

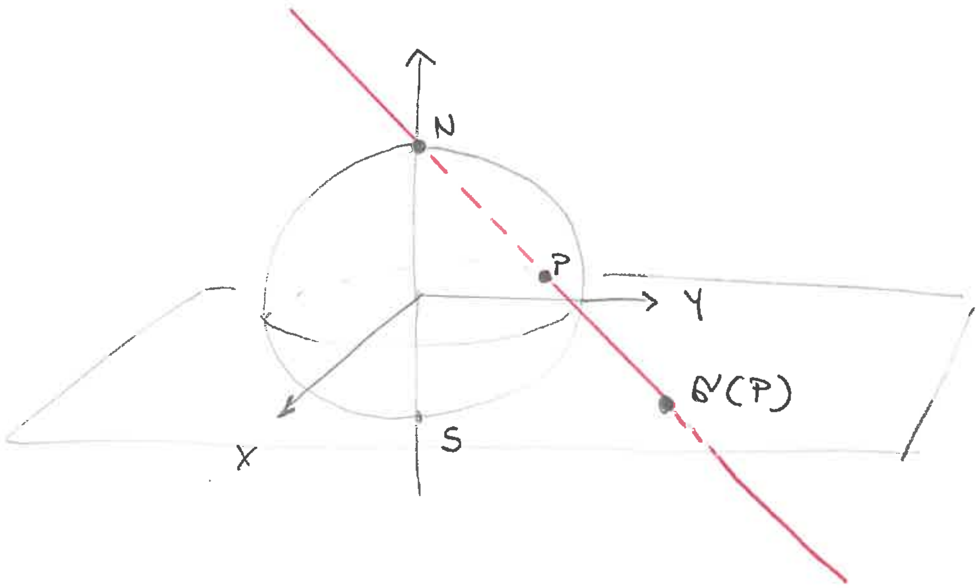
e identifico piano Argand-Gauss con $\pi : \{z = 0\}$

$$\mathbb{S}^2 = \{P \in \mathbb{E}^3(\mathbb{R}) \mid d(O,P) = 1\} = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 = 1\right\}$$

$$x^2+y^2+z^2 = 1 \iff |\underline{w}|^2 + z^2 = 1$$

$N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$ polo NORD di $\mathbb{S}^2$

$S = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} =$ polo SUD di $\mathbb{S}^2$



$\forall\ P = (P_1, P_2, P_3) \in S^2 \setminus \{N\}$ $\exists!$ retta per N e P

$l_{N,P}$ non è parallela a π $\Rightarrow$

$l_{N,P} \cap \pi = \sigma(P)$ $\qquad \sigma : S^2 \setminus \{N\} \longrightarrow \pi$

$\qquad \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \longrightarrow \sigma\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$

$l_{N,P} : \begin{cases} x = P_1 t \\ y = P_2 t \\ z = (P_3 - 1)t + 1 \end{cases}$ perché ha vettore direttore $\overrightarrow{NP}$ e passa per N

$\Rightarrow \quad \sigma\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{cases} \pi \\ l_{N,P} \end{cases} \Leftrightarrow \left(\dfrac{P_1}{1-P_3}, \dfrac{P_2}{1-P_3}, 0 \right) \in \pi$

* Viceversa, $\forall Q = (u, v, 0) \in \pi$ $\exists!$ retta $l_{N,Q}$ $\quad \overrightarrow{NQ} = (u, v, -1)$

$l_{N,Q} \begin{cases} x = tu \\ y = tv \\ z = 1 - t \end{cases} \Rightarrow \quad l_{N,Q} \cap S^2 = \left(\dfrac{2u}{u^2+v^2+1}, \dfrac{2v}{u^2+v^2+1}, \dfrac{u^2+v^2-1}{u^2+v^2+1} \right)$

$\Rightarrow$ definisce $\sigma^{-1} : \pi \longrightarrow S^2 \setminus \{N\}$

$\Rightarrow$ σ consente di rappresentare $S^2 \setminus \{N\}$ su π

proiezione stereografica di $S^2 \setminus \{N\}$ su piano

se aggiungiamo $\{N\}$ $\Rightarrow$ interpreto come P_∞ per π

perché se Q si allontana da O su π (i.e. $\|\overrightarrow{OQ}\| \to +\infty$)

$\sigma^{-1}(Q)$ tende a N

Perciò $\quad \pi \longleftrightarrow \mathbb{C}$

$\qquad \updownarrow$

$\qquad S^2 \setminus \{N\}$

$\Rightarrow \quad \boxed{S^2 = \mathbb{C} \cup \{P_\infty\} = \mathbb{P}^1(\mathbb{C})}$

$\mathbb{P}^1(\mathbb{C})$ Sfera di Riemann

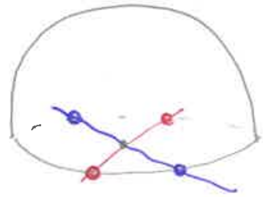
(iii) $\mathbb{P}^n(\mathbb{R})$ modello

$$\mathbb{E}^{n+1}_{\mathbb{R}} \supset S^n = \{ P \in \mathbb{E}^{n+1}_{\mathbb{R}} \mid d(0,P) = 1 \}$$

$\sim$ su S^n equivalenza antipodale $\sim_{ap}$

$$[\underline{x}] = [-\underline{x}] \quad \text{su } S^n/\sim_{ap}$$

$$S^n_+ = \{ x_n \geq 0 \} \cap S^n$$



Su $S^{n-1} \subset \{x_n = 0\}$ equatoriale identifico anche essi in modo antipodale

(iv) $\mathbb{P}^2(\mathbb{C})$ modello

$$\mathbb{P}^2(\mathbb{C}) = \mathbb{P}(\mathbb{C}^3)$$

$$\mathbb{C}^3 \longleftrightarrow \mathbb{R}^6 \longleftrightarrow \mathbb{E}^6_{\mathbb{R}}$$

$$\underline{z} = (z_0, z_1, z_2) = (x_0 + i y_0, x_1 + i y_1, x_2 + i y_2) \qquad (x_0, y_0, x_1, y_1, x_2, y_2)$$

In $\mathbb{E}^6_{\mathbb{R}}$

$$\mathbb{E}^6_{\mathbb{R}} \supset S^5 = \left\{ \sum_{i=0}^{3} x_i^2 + y_i^2 = 1 \right\} = \left\{ \underline{z} \in \mathbb{C}^3 \mid |z_0|^2 + |z_1|^2 + |z_2|^2 = 1 \right\} \subset \mathbb{C}^3$$

Se $\underline{v} = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \in \mathbb{C}^3 \setminus \{\underline{0}\} \Rightarrow \mathrm{Span}(\underline{v}) = \begin{cases} z_0 = t\lambda \\ z_1 = t\mu \\ z_2 = t\nu \end{cases}, \; t \in \mathbb{C}^*$, $\quad \mathrm{Span}(\underline{v}) \cap \mathbb{C}^3 \setminus \{\underline{0}\}$

$\Rightarrow$

$$S^5 \cap \mathrm{Span}\{\underline{v}\} = \{ |t|^2 (|\lambda|^2 + |\mu|^2 + |\nu|^2) = 1 \} \Leftrightarrow$$

$$|t|^2 = \frac{1}{|\lambda|^2 + |\mu|^2 + |\nu|^2}$$ è una S^1 di punti in $\mathbb{E}^6_{\mathbb{R}}$

infatti $S^5 \to 1$ equazione reale $+$

$P \in \mathrm{Span}(\underline{v}) = \left\{ \mathrm{rg} \begin{pmatrix} z_0 & z_1 & z_2 \\ \lambda & \mu & \nu \end{pmatrix} = 1 \right\}$ 2 eq. complesse $\Downarrow$ 4 eq. reali in $\mathbb{E}^6_{\mathbb{R}}$