Esercizio 1

$f \in End_{\mathbb{R}}(\mathbb{R}^3)$ t.c. $\boxed{f(x,y,z) = (2x, -x-3y-z, -x+4y+z)}$

$$M_{\mathcal{E},\mathcal{E}}(f) = A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & -3 & -1 \\ -1 & 4 & 1 \end{pmatrix} \in M(3,3;\mathbb{R})$$

$$P_f(t) = P_A(t) = \det(A - tI_3) \underset{\substack{\downarrow \\ \text{Laplace} \\ \text{I riga}}}{=} (2-t)\det\begin{pmatrix} -3-t & -1 \\ 4 & 1-t \end{pmatrix} =$$

$$= (2-t)\left[-(3+t)(1-t)+4\right] = (2-t)\left[-(3-3t+t-t^2)+4\right] =$$
$$= (2-t)\left[-3+3t-t+t^2+4\right] = (2-t)\left[t^2+2t+1\right] \Rightarrow$$

$$\boxed{P_f(t) = (2-t)\cdot(t+1)^2}$$

$\Rightarrow \boxed{m_a(2) = 1 \qquad m_a(-1) = 2}$

Spettro $(f) = \{-1, 2\} \subseteq \mathbb{R} \Rightarrow$ f ammette f.c. di Jordan su $\mathbb{R}$ (eventualmente diagonale)

Passo 1: separazione autovalori

* $\lambda = 2$ autovalore semplice $\Rightarrow E_2(f)$ ha dimensione 1

$\Rightarrow m_g(2) = \dim(E_2(f)) = 1$

$E_2(f) \equiv$ Autospazio generalizzato $= V_2(f)$

Infatti $(A - 2I_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -5 & -1 \\ -1 & 4 & -1 \end{pmatrix} \Rightarrow$

$$E_2(f) = \begin{cases} -x-5y-z=0 \\ -x+4y-z=0 \end{cases} \Leftrightarrow \begin{cases} -x-z = 5y \\ 9y = 0 \end{cases} \Leftrightarrow \begin{cases} x+z=0 \\ y=0 \end{cases}$$

$$E_2(f) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} = \mathrm{Span}\{\underline{a}\} \qquad \boxed{\underline{a} := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$A|_{E_2(f)} : t\underline{a} \longrightarrow 2\cdot t\underline{a}, \ \forall t \in \mathbb{R} \Rightarrow E_2(f)$ è $\overset{f\text{-stabile}}{\text{A-stabile}}$

$\downarrow$

$A(E_2(f)) = E_2(f)$

e $(A-2I_3)|_{E_2(f)} = 0_{E_f^2} \Rightarrow$ indice di nilpotenza di $(A-2I_3)|_{E_2(f)} = k = 1$

$\Rightarrow \exists!$ blocco di Jordan di ordine 1 rispetto a $\lambda = 2$

$$\boxed{J_{2,1} = (2)}$$

$\underset{\text{Ricordo notazione}}{\rightsquigarrow}$ $J_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \in M(k \times k; \mathbb{R})$

* $\underline{\lambda = -1}$

$m_a(-1) = 2 \Rightarrow \boxed{A \text{ diag.} \Leftrightarrow m_g(-1) = 2}$

Ma $E_{-1}(f) = \ker(A + I_3) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} 3x = 0 \\ -x - 2y - z = 0 \\ -x + 4y + 2z = 0 \end{cases} \right\}$

$(A + I_3) = \begin{pmatrix} 3 & 0 & 0 \\ -1 & -2 & -1 \\ -1 & 4 & 2 \end{pmatrix}$

$\Leftrightarrow \begin{cases} x = 0 \\ 2y + z = 0 \\ 4y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ 2y + z = 0 \end{cases}$

$E_{-1}(f) = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ +2 \end{pmatrix} \right\} = \text{Span}\{b\}$ dove $\boxed{b := \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}}$

$\Rightarrow \underline{m_g(-1) = \dim(E_{-1}(f)) = 1 < m_a(-1) = 2}$

$\Rightarrow$ A non diag. $\Rightarrow$ <u>f ammette f.c. Jordan non diagonale</u>

Poiché $\dim E_{-1}(f) = 1 \Rightarrow$ <u>$\exists !$ blocco Jordan rispetto a</u> <u>$\lambda = -1$</u> e poiché <u>$m_a(-1) = 2$</u> $\Rightarrow \dim(V_2(f)) = 2$

$\downarrow$

<u>autospazio generalizzato</u>

$\Rightarrow$ il <u>blocco di Jordan</u> è necessariamente

$$J_{-1,2} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad (J_{\lambda,k})$$

$\Rightarrow$ <u>Forma canonica di Jordan</u> è (a meno di permutazione dei blocchi)

$$J_f = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right)$$

* <u>Cerchiamo autospazio generalizzato per $\lambda = -1$</u>

Se k = indice nilpotenza di $A + 1 I_3 |_{V_{-1}(f)}$ $\Rightarrow$ <u>$k \leq n = 3$</u>

e sicuramente $\left(A + I_3|_{V_{-1}(f)}\right)^k = \left(A + I_3|_{V_{-1}(f)}\right)^3 = 0_{V_{-1}(f)}$

Notiamo che

- $(A + I_3)^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ -9 & 0 & 0 \end{pmatrix}$
- $(A + I_3)^3 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & -2 & -1 \\ -1 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ -9 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 0 & 0 \\ -27 & 0 & 0 \end{pmatrix}$

matrici proporzionali

$\Rightarrow \boxed{V_{-1}(f) = \ker((A + I_3)^3) = \ker((A + I_3)^2) = \{x = 0\}}$

$V_1(f) = \{x \quad = 0\} = \text{Span}\{e_2, e_3\} = \text{Span}\{\underset{\substack{\downarrow \\ \text{generatore} \\ \text{di } E_1(f)}}{b}, e_3\}$

Su $V_1(f)$: $\quad A + I_3|_{V_1(f)} = \begin{pmatrix} 3 & 0 & 0 \\ -1 & -2 & -1 \\ -1 & 4 & 2 \end{pmatrix}\Big|_{V_{-1}(f)}$

cioè

$$e_2 \longrightarrow -2e_2 + 4e_3$$
$$e_3 \longrightarrow -e_2 + 2e_3 \qquad \Rightarrow \quad A + I_3|_{V_1(f)} \neq 0_{V_1(f)}$$

Invece $\left(A + I_3|_{V_1(f)}\right)^2 = 0_{V_1(f)} \quad \Rightarrow \quad$ $k = 2$ indice nilpotenza di $(A + I_3)|_{V_1(f)}$

Passo II riduzione dell'autovalore nullo

A ha autovalore $\lambda = -1 \iff A v = -v \iff (A + I_3)v = 0$

$\iff A + I_3$ ha autovalore $\lambda' = 0$ ed in effetti

$A + I_3|_{V_{-1}(f)}$ nilpotente di indice 2 $\iff A + I_3|_{V_{-1}(f)}$ ha solo autovalore nullo

Notiamo che

$$\{0\} \subsetneq \underset{\substack{\| \\ \text{Span}(b) = \text{Span}\left\{\begin{pmatrix}0\\-1\\2\end{pmatrix}\right\}}}{\overset{\substack{\dim = 1 \\ \downarrow}}{E_1(f)}} \subsetneq \overset{\substack{\dim = 2 \\ \downarrow}}{V_{-1}(f)} = \text{Span}\{b, e_3\} = \text{Span}\{e_2, e_3\}$$

Se prendo $e_3 \in V_{-1}(f) \setminus E_{-1}(f)$

$$(A + I_3)(e_3) \underset{\substack{\downarrow \\ \text{III colonna} \\ \text{di } A+I_3}}{=} \begin{pmatrix}0\\-1\\2\end{pmatrix} = b \in E_{-1}(f) \quad \text{(anzi è generatore)}$$

In altre parole, posto $f' := f + I_3|_{V_{-1}(f)}$

$$\{0\} \subset E_{-1}(f) \subsetneq V_{-1}(f)$$
$$0 \xleftarrow{f'} b \xleftarrow{f'} e_3$$

perché $\underset{\substack{\| \\ E_{-1}(f)}}{b \in \ker(f')}$, $\quad \underset{\substack{\cup\!\!\!| \\ e_3}}{\ker((f')^2)} \supsetneq \underset{\substack{\not\ni \\ e_3}}{\ker(f')}$ e $f'(e_3) \in \ker(f')$ infatti

$$\underset{\substack{\uparrow \\ e_3 \in \ker(f')^2}}{0 =} (f')^2(e_3) = f'(f'(e_3))$$

$\Rightarrow \mathcal{B} = \{ \underline{b}, \underline{e}_3 \}$ base ciclica di stringhe per $V_{-1}(f)$ (4)

Infatti

$$f'(\underline{e}_3) = \underline{b} = 1\underline{b} + 0\,\underline{e}_3$$
$$f'(\underline{b}) = (f')^2(\underline{e}_3) = \underline{0}$$

$$\Rightarrow \quad M_{\mathcal{B},\mathcal{B}}(f') = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Ma $\quad f' = (f + I_3)|_{V_{-1}(f)} \Rightarrow$

$$f'(\underline{e}_3) = (f + I_3)(\underline{e}_3) = \underline{b} \quad \Leftrightarrow \quad f(\underline{e}_3) = \underline{b} - \underline{e}_3$$
$$f'(\underline{b}) = (f + I_3)(\underline{b}) = \underline{0} \quad \Leftrightarrow \quad f(\underline{b}) = -\underline{b}$$

$$\Rightarrow \quad M_{\mathcal{B},\mathcal{B}}\left(f|_{V_{-1}(f)} \right) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

che è come fare $M_{\mathcal{B},\mathcal{B}}(f') - M_{\mathcal{B},\mathcal{B}}(I_3|_{V_{-1}(f)}) =$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Pertanto in base

$$\boxed{\mathcal{J} = \{ \underline{a}, \underline{b}, \underline{e}_3 \}}$$ per $\mathbb{R}^3$ si ha

$$M_{\mathcal{J},\mathcal{J}}(f) = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right) \Rightarrow \mathcal{J} = \text{base di Jordan per } f$$

$$\Rightarrow \boxed{J_f = \left(\begin{array}{c|c} J_{2,1} & 0 \quad 0 \\ \hline 0 & \\ 0 & J_{-1,2} \end{array}\right)}$$

* Se

$$M := M_{\mathcal{E},\mathcal{J}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$
$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \underline{a} & \underline{b} & \underline{e}_3 \end{array}$$

$$\begin{tikzcd}
(\mathbb{R}^3, \mathcal{E}) \arrow[rr, "f", "M_{\mathcal{E},\mathcal{E}}(f) = A"'] \arrow[d, leftrightarrow, "Id"] & & (\mathbb{R}^3, \mathcal{E}) \arrow[d, leftrightarrow, "Id"] \\
(\mathbb{R}^3, \mathcal{J}) \arrow[rr, "f", "M_{\mathcal{J},\mathcal{J}}(f) = J_f"'] & & (\mathbb{R}^3, \mathcal{J})
\end{tikzcd}$$

$$J_f = M_{\mathcal{J},\mathcal{J}}(f) = M_{\mathcal{J},\mathcal{J}}(Id_{\mathbb{R}^3} \circ f \circ Id_{\mathbb{R}^3}) = M_{\mathcal{J},\mathcal{E}}(Id_{\mathbb{R}^3}) \circ M_{\mathcal{E},\mathcal{E}}(f) \circ M_{\mathcal{E},\mathcal{J}}(Id_{\mathbb{R}^3})$$

$$= M_{\mathcal{J},\mathcal{E}} \circ A \circ M_{\mathcal{E},\mathcal{J}} = M_{\mathcal{E},\mathcal{J}}^{-1} \circ A \circ M_{\mathcal{E},\mathcal{J}} = M^{-1} \circ A \circ M$$

La similitudine tra J_f e A si ha con $M = M_{\mathcal{E},\mathcal{J}}$

$$\boxed{J_f = M^{-1} \circ A \circ M}$$

Esercizio 2

$$A = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -3 & 2 & -1 \\ 0 & -3 & 2 & -1 \end{pmatrix} \in M(4,4;\mathbb{R})$$

$$P_A(t) \underset{\downarrow}{=} (t^4 - 4t^3 + 6t^2 - 4t + 1)$$

Laplace I riga

Ricordo $\binom{4}{3} = 4 = \binom{4}{1}$, $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6 \implies$

$$\boxed{P_A(t) = (t-1)^4}$$

Se $f = L_A$

Spettro$(f) = \{1\} \subset \mathbb{R} \implies$ f ammette f.c. Jordan su $\mathbb{R}$

Quante possibilità a priori (a meno di permutazioni)?

1! blocco

$$J_{1,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 blocchi

$$\left(\begin{array}{c|ccc} J_{1,1} & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & J_{1,3} & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{J_{1,2}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{J_{1,2}} \\ 0 & 0 & & \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

3 blocchi

$$\left(\begin{array}{c|c|cc} J_{1,1} & 0 & 0 & 0 \\ \hline 0 & J_{1,1} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{J_{1,2}} \\ 0 & 0 & & \end{array}\right) = \left(\begin{array}{c|c|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

4 blocchi

$$\left(\begin{array}{c|c|c|c} J_{1,1} & 0 & 0 & 0 \\ \hline 0 & J_{1,1} & 0 & 0 \\ \hline 0 & 0 & J_{1,1} & 0 \\ \hline 0 & 0 & 0 & J_{1,1} \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \text{diagonale}$$

Escludiamo subito alcuni casi

Notiamo che

$$\mathbb{R}^4 = \ker\left((L_A - I_4)^4\right) = V_1(L_A)$$ è autospazio generalizzato perché ho solo $\lambda = 1$

con $\boxed{m_a(1) = 4 = \dim(V_1(L_A))}$

$m_g(1) = ?$

Considero

$$\boxed{B := A - 1 I_4 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -3 & 1 & -1 \\ 0 & -3 & 2 & -2 \end{pmatrix}}$$

$\Rightarrow \ rg(B) \geq 2$ perché $\det(\square) = -3 \neq 0$

Però

$$B_{(\text{III col})} = -B_{(\text{IV col})} \Rightarrow rg(B) \leq 3$$

Se orlo $\square$ in tutti i modi possibili $\rightarrow$ det $= 0$ sempre

$$\Rightarrow \boxed{rg(B) = 2}$$

$$\Rightarrow \boxed{\dim(\ker(B)) = 4 - rg(B) = 2}$$

Ma $\ker(B) = E_1(L_A) \Rightarrow$

$$\boxed{m_g(1) = 2 < m_a(1) = 4}$$

$\Rightarrow$ A non diagonalizzabile e A ammette 2 blocchi di Jordan

(*)
$$\left(\begin{array}{c|c} J_{1,1} & 0\ 0\ 0 \\ \hline \begin{matrix}0\\0\\0\end{matrix} & J_{1,3} \end{array}\right)$$

(**)
$$\left(\begin{array}{c|c} J_{1,2} & \begin{matrix}0 & 0\\0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\0 & 0\end{matrix} & J_{1,2} \end{array}\right)$$

* Notiamo che
$$B^2 = B \circ B = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -3 & 1 & -1 \\ 0 & -3 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -3 & 1 & -1 \\ 0 & -3 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & & & \\ & & & \end{pmatrix} \neq O$$

$\Rightarrow$ $k = 2$ NON È indice nilpotenza di B

Invece $k = 3$ è indice nilpotenza di B i.e. $B^3 = O$

$\Rightarrow$ Se $f = L_A$ e $f' := L_A - I_4 = L_B$

$$\{0\} \subsetneq \underset{\ker(f')}{\underset{\|}{E_1(f)}} \subsetneq \ker((f')^2) \subsetneq \ker((f')^3) = \underset{\mathbb{R}^4}{\underset{\|}{\ker(0)}}$$

$$\underline{0} \leftarrow (f')^2(\underline{v}) \leftarrow f'(\underline{v}) \leftarrow \underline{v}$$

$\Rightarrow$ f' deve avere un blocco di ordine 3 $\Rightarrow$ f deve avere un blocco di ordine 3 $\Rightarrow$ escludo la II possibilità (**)

$$\boxed{\Rightarrow J_f = \left(\begin{array}{c|ccc} J_{1,1} & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & J_{1,3} & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)}$$

Prendo

$$\ker(f') = \{B \cdot \underline{x} = \underline{0}\} = \begin{cases} x_1 - x_3 + x_4 = 0 \\ x_1 - x_3 + x_4 = 0 \\ x_1 - 3x_2 + x_3 - x_4 = 0 \\ -3x_2 + 2x_3 - 2x_4 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 - x_3 + x_4 = 0 \\ 3x_2 - 2x_3 + 2x_4 = 0 \end{cases} \rightarrow \begin{array}{l} x_4 = t \\ x_3 = s \\ x_2 = \frac{2}{3}s - \frac{2}{3}t \\ x_1 = s - t \end{array}$$

$$\boxed{\ker(f') = E_1(f) = \mathrm{Span}\left\{ \underline{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -3 \end{pmatrix} \right\}}$$

Noto che $\boxed{B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 \end{pmatrix}}$

$$\Rightarrow \boxed{\ker((f')^2) = \ker(B^2) = \{x_1 - x_3 + x_4 = 0\}}$$

iperpiano intermedio

Visto che

$$\{\underline{0}\} \subsetneq \underset{\ker(B)}{\underset{\parallel}{\mathrm{Span}\{v_1, v_2\}}} \subsetneq \underset{\ker(B^2)}{\underset{\parallel}{\{x_1 - x_3 + x_4 = 0\}}} \subsetneq \underset{\ker(B^3)}{\underset{\parallel}{\mathbb{R}^4}}$$

Noto che e.g. $\underline{e}_3 \in \mathbb{R}^4 \setminus \ker(B^2) = \ker(B^3) \setminus \ker(B^2)$
↳ non soddisfa eq. cartesiana di iperpiano

$\Rightarrow$

$$\underline{0} \xleftarrow{B=f'} B^2(\underline{e}_3) \xleftarrow{B=f'} B(\underline{e}_3) \xleftarrow{B=f'} \underline{e}_3$$

Sistema ciclico o stringa per $\mathbb{R}^4$

$$\underline{e}_3 \rightarrow B(\underline{e}_3) = \text{III colonna di } B = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 2 \end{pmatrix} \sim$$

$$\longrightarrow B^2(\underline{e}_3) = \text{III colonna di } B^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} v_1 - \frac{1}{3} v_2 \in \underset{E_1(f)}{\underset{\parallel}{\ker(B)}}$$

* Pertanto prendiamo ad esempio

$$J = \left\{ \underline{j}_1 = v_1,\ \underline{j}_2 = B^2(\underline{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ \underline{j}_3 = B(\underline{e}_3) = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 2 \end{pmatrix},\ \underline{j}_4 = \underline{e}_3 \right\}$$

In base J

$B(\underline{j}_1) = B(v_1) = \underline{0}$, $\quad v_1 \in \ker(B) = E_1(L_A)$

$B(\underline{j}_2) = B(B^2(\underline{e}_3)) = B^3(\underline{e}_3) = O_{\mathbb{R}^4}(\underline{e}_3) = \underline{0}$

$B(\underline{j}_3) = B(B(\underline{e}_3)) = B^2(\underline{e}_3) = \underline{j}_2$

$B(\underline{j}_4) = B(\underline{e}_3) = \underline{j}_3$

$\Rightarrow$

$$\underset{f' = L_B}{\underset{\downarrow}{M_{J,J}(f')}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{B = A - I_4 \\ f' = f - I_{\mathbb{R}^4}}}{\Longmapsto} \underset{f = L_A}{\underset{\downarrow}{M_{J,J}(f)}} = M_{J,J}(f') + I_4$$

$$M_{\mathcal{J},\mathcal{J}}(f) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) = J_f$$

Pertanto la matrice che fornisce la similitudine è

$$M := M_{\mathcal{E},\mathcal{J}} = \begin{array}{c} \begin{pmatrix} 3 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \\ \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \underline{v}_1 = \underline{u}_1 & B^2(\underline{e}_3) = \underline{u}_2 & B(\underline{e}_3) = \underline{u}_3 & \underline{e}_3 \end{array} \end{array}$$

i.e. $\boxed{M^{-1} \circ A \circ M = J_f}$