Svolgimento Esercizio 1

(i) $M_{\mathcal{E},\mathcal{V}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \underset{\substack{\downarrow \\ \text{Laplace} \\ \text{II colonna}}}{=} -\det\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = 1 \neq 0 \Rightarrow \mathcal{V}$ base di V

$\mathcal{V}^* = \{ v_1^*, v_2^*, v_3^* \}$ è base duale di $\mathcal{V}$ se e solo se

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{se } i=j \\ 0 & \text{se } i \neq j \end{cases}$$

Scrittura v_1^*

$v_1^* = \alpha_1 e_1^* + \alpha_2 e_2^* + \alpha_3 e_3^*$ è t.c.

$$\begin{cases} 1 = v_1^*(v_1) = v_1^*(e_1+e_3) \underset{\substack{\downarrow \\ \mathcal{E}^* \text{ base duale di } \mathcal{E}}}{=} \alpha_1 + \alpha_2 \\ 0 = v_1^*(v_2) = v_1^*(e_3) = \alpha_3 \\ 0 = v_1^*(v_3) = v_1^*(e_1 - e_2) = \alpha_1 - \alpha_2 \end{cases} \Rightarrow \begin{cases} 1 = \alpha_1 + \alpha_2 \\ 0 = \alpha_3 \\ 0 = \alpha_1 - \alpha_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha_1 = 1 \\ \alpha_3 = 0 \\ \alpha_2 = 1 \end{cases} \Rightarrow \boxed{v_1^* = e_1^* + e_2^*}$$

Scrittura v_2^*

$v_2^* = \beta_1 e_1^* + \beta_2 e_2^* + \beta_3 e_3^*$

$$\begin{cases} 0 = v_2^*(v_1) = \beta_1 + \beta_3 \\ 1 = v_2^*(v_2) = \beta_3 \\ 0 = v_2^*(v_3) = \beta_1 - \beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = -\beta_3 \\ \beta_3 = 1 \\ \beta_1 = \beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = -1 \\ \beta_3 = 1 \\ \beta_2 = -1 \end{cases}$$

$$\Rightarrow \boxed{v_2^* = -e_1^* - e_2^* + e_3^*}$$

Scrittura v_3^*

$v_3^* = \gamma_1 e_1^* + \gamma_2 e_2^* + \gamma_3 e_3^*$

$$\begin{cases} 0 = v_3^*(v_1) = \gamma_1 + \gamma_3 \\ 0 = v_3^*(v_2) = \gamma_3 \\ 1 = v_3^*(v_3) = \gamma_1 - \gamma_2 \end{cases} \Rightarrow \begin{cases} \gamma_1 + \gamma_3 = 0 \\ \gamma_3 = 0 \\ \gamma_1 - \gamma_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \gamma_1 = 0 \\ \gamma_3 = 0 \\ \gamma_2 = -1 \end{cases}$$

$$\Rightarrow \boxed{v_3^* = -e_2^*}$$

(ii) $\underline{e}_i^* = \langle \underline{e}_i, - \rangle_{st}$

infatti, $\mathcal{E}_{can}$ è ortonormale $\Rightarrow$

$$\underline{e}_i^*(\underline{e}_j) = \langle \underline{e}_i, \underline{e}_j \rangle_{st} = \delta_{ij} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$\Rightarrow$

$$\underline{v}_1^* = \underline{e}_1^* + \underline{e}_2^* \in V^* \text{ è t.c.}$$

$$\underline{v}_1^*(x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3) = x + y, \ \forall \underline{v} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3 \in V$$

$\Rightarrow$ $\boxed{\underline{v}_1^* = \langle \underline{w}_1, - \rangle_{st} \text{ con } \underline{w}_1 = \underline{e}_1 + \underline{e}_2 \in V}$ infatti

$$\langle \underline{w}_1, \underline{v} \rangle_{st} = (1 \ 1 \ 0) \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x + y. \ \text{ok}$$

Analoghi $\boxed{\underline{v}_2^* = \langle -\underline{e}_1 - \underline{e}_2 + \underline{e}_3, - \rangle_{st}}$

$\boxed{\underline{v}_3^* = \langle -\underline{e}_2, - \rangle_{st}}$

(iii) Visto che

$$\underline{v}_1^* = \langle \underline{w}_1, - \rangle_{st} \Rightarrow$$

$$\mathrm{Ann}_V(\underline{v}_1^*) = \{\underline{v} \in \mathbb{R}^3 \mid \underline{v}_1^*(\underline{v}) = 0\} \subset V = \mathbb{R}^3, \ \underline{v} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$$

$$\Leftrightarrow \langle \underline{w}_1, \underline{v} \rangle_{st} = 0 \Leftrightarrow x + y = 0$$

cioè $\boxed{\mathrm{Ann}_V(\underline{v}_1^*) = (\mathrm{Span}\{\underline{w}_1\})^{\perp_{\langle,\rangle_{st}}} \subset V}$

che in effetti coincide con $\mathrm{Ker}(\underline{v}_1^*) =$

$= \mathrm{Ker}(\underline{e}_1^* + \underline{e}_2^*)$

(iv) $\dim(\mathrm{Ann}_{V^*}(W)) = \dim(V) - \dim(W) = 1$

Infatti

$W = \mathrm{Span}\{\underline{v}_1, \underline{v}_2\}$

$\mathcal{V}^* = \{\underline{v}_1^*, \underline{v}_2^*\}$ è base duale di $\{\underline{v}_1, \underline{v}_2\}$

$\forall\ f \in V^*\ \exists$ t.c. $f = \alpha_1 \underline{v}_1^* + \alpha_2 \underline{v}_2^* + \alpha_3 \underline{v}_3^* \in V^*$ perché $\mathcal{V}^*$ base

Allora

$$f \in \mathrm{Ann}_{V^*}(W) \iff f|_W \equiv 0 \iff \begin{cases} f(\underline{v}_1) = 0 \\ f(\underline{v}_2) = 0 \end{cases}$$

cioè $W \subseteq \mathrm{Ker}(f)$

$$\iff \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases} \iff \mathrm{Ann}_{V^*}(W) = \mathrm{Span}\{\underline{v}_3^*\}$$

$\Rightarrow$ $\underline{v}_3^*$ è base di $\mathrm{Ann}_{V^*}(W)$ $\boxed{\underline{v}_3^* = -\underline{e}_2^*}$

Allora $\underline{e}_2^* = y$ (funzionale lineare coordinata) $\Rightarrow$ infatti $\boxed{y = 0}$ è proprio l'equazione cartesiana di $W = \mathrm{Span}\{\underline{e}_1 + \underline{e}_3, \underline{e}_3\} = \mathrm{Span}\{\underline{e}_1, \underline{e}_3\}$

Es2

$V = \mathbb{R}[x]_{\leq 4}$

$U = \{ p(x) \in V \mid p(1) = p(2) = 0 \}$

(i) U ssp. vett.? se sì dim(U) ed una base

(ii) V/U dim. ed una base

(iii) Determinare $k \in \mathbb{N} \setminus \{0\}$ t.c. $V/U \cong \mathbb{R}^k$ ed esplicitare un isomorfismo esplicito $\varphi \in \mathrm{Isom}_{\mathbb{R}}(V/U, \mathbb{R}^k)$

(iv) $x \in V$, $x^3 \in V$ è vero che $\pi_U(x) = \pi_U(x^3)$? Possono dare un sistema $\{[x], [x^3]\}$ dipendente?

(v) $\{[x], [x^2+2]\}$ può essere base di V/U?

Svolgimento

Ricordo che $\dim_{\mathbb{R}}(V) = 5$, infatti base canonica $= \{1, x, x^2, x^3, x^4\}$

(i) Per Ruffini $p(x) \in U \iff$ in $\mathbb{R}[x]$ si ha che

$$(x-1)(x-2) \mid p(x)$$

Poiché $\deg(p(x)) \leq 4 \Rightarrow$

$p(x) \in U \iff p(x) = (x-1)(x-2)(a + bx + cx^2)$

Inoltre U è ssp. perché $\forall\, \lambda, \mu \in \mathbb{R}$ e $\forall\, p(x), q(x) \in U$

$f(x) := \lambda p(x) + \mu q(x) \in V$ è t.c. $\begin{cases} f(1) = \lambda p(1) + \mu q(1) = 0 \\ f(2) = \lambda p(2) + \mu q(2) = 0 \end{cases}$

U è chiuso rispetto comb. lineari

$\Rightarrow \dim_{\mathbb{R}}(U) = 3$ perché i parametri liberi sono $a, b, c \in \mathbb{R}$

Per trovare una base di U, considero

$p(x) \in V \iff p(x) = \sum_{i=0}^{4} a_i x^i$ unicamente rispetto alla base canonica $\{1, x, x^2, x^3, x^4\}$

Allora

$p(x) \in U \iff \begin{cases} p(1) = 0 \\ p(2) = 0 \end{cases} \iff \begin{cases} a_0 + a_1 + a_2 + a_3 + a_4 = 0 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 0 \end{cases}$

$\Leftrightarrow$ $\underset{\text{equaz.}}{\mathrm{II}} - \underset{\text{equaz.}}{\mathrm{I}}$ $\begin{cases} a_0 + a_1 + a_2 + a_3 + a_4 = 0 \\ a_1 + 3a_2 + 7a_3 + 15a_4 = 0 \end{cases}$ $\Rightarrow$ sistema a scala con 2 pivots

Poniamo

$a_4 = t$

$a_3 = s$ $\quad$ $t, s, u \in \mathbb{R}$ parametri liberi

$a_2 = u$

$a_1 = -3u - 7s - 15t$

$a_0 = 3u + 7s + 15t - t - s - u = 2u + 6s + 14t$

Perciò $p(x) \in U \Leftrightarrow$

$$p(x) = (2u + 6s + 14t) + (-3u - 7s - 15t)x + ux^2 + sx^3 + tx^4 =$$

$$= u \cdot (\underbrace{2 - 3x + x^2}_{(x-1)(x-2)}) + s(6 - 7x + x^3) + t(14 - 15x + x^4)$$

Una base trovata per U è:

$\{(x-1)(x-2),\ 6 - 7x + x^3,\ (14 - 15x + x^4)\}$

Oppure potevo prendere

$\{(x-1)(x-2),\ (x-1)^2(x-2),\ (x-1)^2(x-2)^2\}$

Ritrovo $\dim_{\mathbb{R}}(U) = 3$ come dedotto all'inizio

(ii) $\dim_{\mathbb{R}}(V/U) = \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(U) = 5 - 3 = 2$

Poiché $\{1, x\}$ sistema libero di V verifico che

$[1] = \pi_U(1)$ e $[x] = \pi_U(x)$ lin. indip. in V/U

Ma $\{\pi_U(1), \pi_U(x)\}$ non libero in V/U $\Leftrightarrow$ $\exists \lambda \in \mathbb{R}^*$ t.c.

visto che $\pi_U(1) \neq [0]$ perché $1 \notin U$

$\pi_U(1) = \lambda \cdot \pi_U(x)$ in V/U $\Leftrightarrow$

$\pi_U(1) - \lambda \pi_U(x) = [0]$ in V/U

$\Leftrightarrow$ $\pi_U(1 - \lambda x) = [0]$ in V/U $\Leftrightarrow$ $1 - \lambda x \in U \subset V$

Imponendo annullamento per $x = 1$ e $x = 2$ si ha:

$\begin{cases} 1 - \lambda(1) = 0 \\ 1 - \lambda(2) = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = 1 \\ \lambda = \frac{1}{2} \end{cases}$ $\#$ $\Rightarrow \{[1], [x]\}$ libero in V/U

Poiché $\dim_{\mathbb{R}}(V/U) = 2 \Rightarrow \{[1], [x]\}$ base di V/U 3

$$[1] = 1 + U = \{1 + p(x) \mid p(x) \in U\}$$
$$[x] = x + U = \{x + p(x) \mid p(x) \in U\}$$

(iii) Ovviamente $\boxed{k=2}$ e pongo perché $\dim(V/U) = 2$

Pongo e.g.

$$\varphi([1]) := \underline{e}_1$$
$$\varphi([x]) := \underline{e}_2$$

estendendolo per linearità

$$\Rightarrow \varphi(\alpha[1] + \beta[x]) = \alpha\varphi([1]) + \beta\varphi([x])$$

φ è isomorfismo $V/U \xrightarrow[\varphi]{\cong} \mathbb{R}^2$

(iv) • $\pi_U(x) = \pi_U(x^3)$ in $V/U \Leftrightarrow \pi_U(x - x^3) = [0] \Leftrightarrow$

$x - x^3 \in U \Leftrightarrow x \cdot (1 - x^2) \in U \Leftrightarrow$

$x(1-x)(1+x) \in U$ FALSO non si annulla per $x=2$

$\Rightarrow \pi_U(x) \neq \pi_U(x^3)$

• Analogamente $\alpha \cdot \pi_U(x) + \beta \cdot \pi_U(x^3) = [0] \Leftrightarrow \pi_U(\alpha x + \beta x^3) = [0]$

$\Leftrightarrow \alpha x + \beta x^3 \in U$

$$\begin{cases} \alpha \cdot 1 + \beta \cdot 1 = 0 \\ \alpha \cdot 2 + \beta \cdot 8 = 0 \end{cases} \Rightarrow \begin{cases} \alpha = -\beta \\ -2\beta + 8\beta = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = -\beta \\ 6\beta = 0 \end{cases} \Leftrightarrow (\alpha, \beta) = (0, 0)$$ SIST. LIBERO

$\Rightarrow \{[x], [x^3]\}$ è un'altra base di V/U

(v) In V/U si ha

$$\alpha[x] + \beta[x^2 + 2] = [0] \Leftrightarrow [\alpha \cdot x + \beta(x^2 + 2)] = [0]$$

$$\Leftrightarrow \alpha \cdot x + \beta x^2 + 2\beta \in U$$

Imponendo annullamenti

per $x = 1 \rightarrow$ / per $x = 2 \rightarrow$

$$\begin{cases} \alpha + \beta + 2\beta = 0 \\ 2\alpha + 4\beta + 2\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + 6\beta = 0 \end{cases} \Rightarrow \boxed{\alpha = -3\beta}$$

sist. non libero $\Rightarrow [x]$ e $[x^2+2]$ sono dipendenti in $V/U \Rightarrow$ non formano una base di V/U.

Suggerimento Esercizio 3

$\mathbb{R}^3$ sp. vett. eucl., $\mathcal{E}$ base canonica, (x, y, z) coord, $\langle , \rangle_{st}$.

(a) $U = \{x - 2y + z = 0\} \subset \mathbb{R}^3$ ssp. vett.

$\mathbb{R}^3_{\sim_U}$ - $\underline{v}, \underline{w} \in \mathbb{R}^3$ $\underline{v} \sim_U \underline{w} \iff \underline{v} - \underline{w} \in U$

è di equivalenza

$\mathbb{R}^3 / U$ è sp. vettoriale

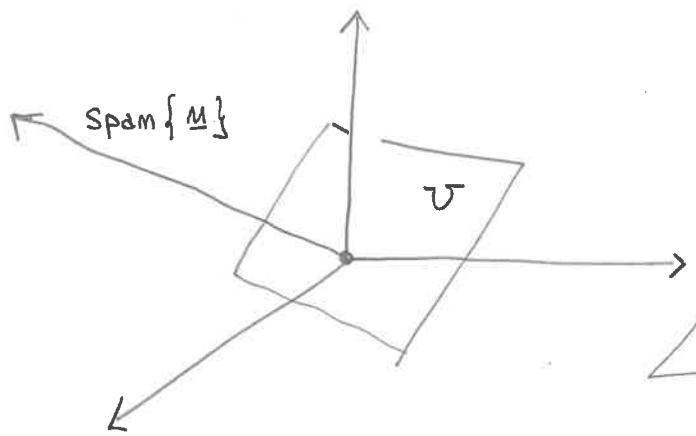
- $[\underline{v}] + [\underline{w}] = [\underline{v} + \underline{w}]$ è ben def. (non dip. dai rapp.)

$$(\underline{v} + \underline{u}) + (\underline{w} + \underline{u}') = \underline{v} + \underline{w} + \underbrace{\underline{u} + \underline{u}'}_{\in U}$$

- $\lambda [\underline{v}] = [\lambda \underline{v}]$, $\forall \lambda \in \mathbb{R}$ è ben definita

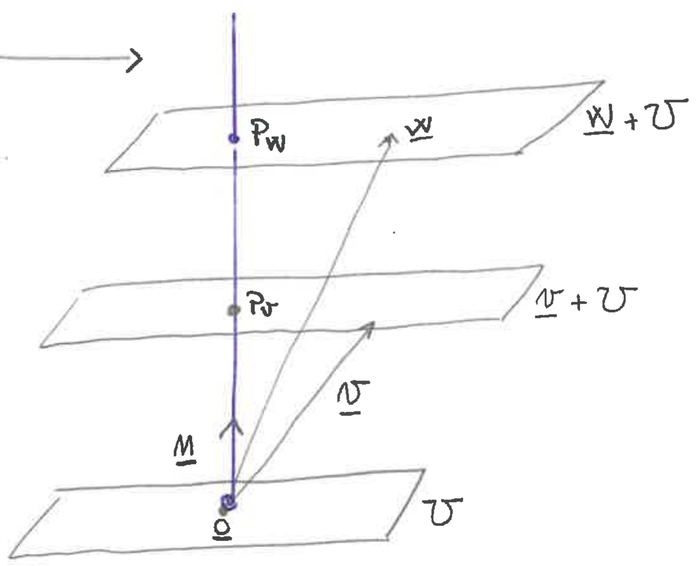
(b) Per studiare $\mathbb{R}^3 / U \ni [\underline{v}] = \underline{v} + U$ sottospazio affine di $\mathbb{R}^3$ parallelo a U

Per rappresentare $[\underline{v}] + [\underline{w}]$ in questa interpretazione:



$[\underline{v}] = \underline{v} + U$

$\underline{M} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$



In $\mathbb{R}^3$ euclideo poiché $\mathrm{Span}\{\underline{M}\} \perp (\underline{w} + U), (\underline{v} + U), U$

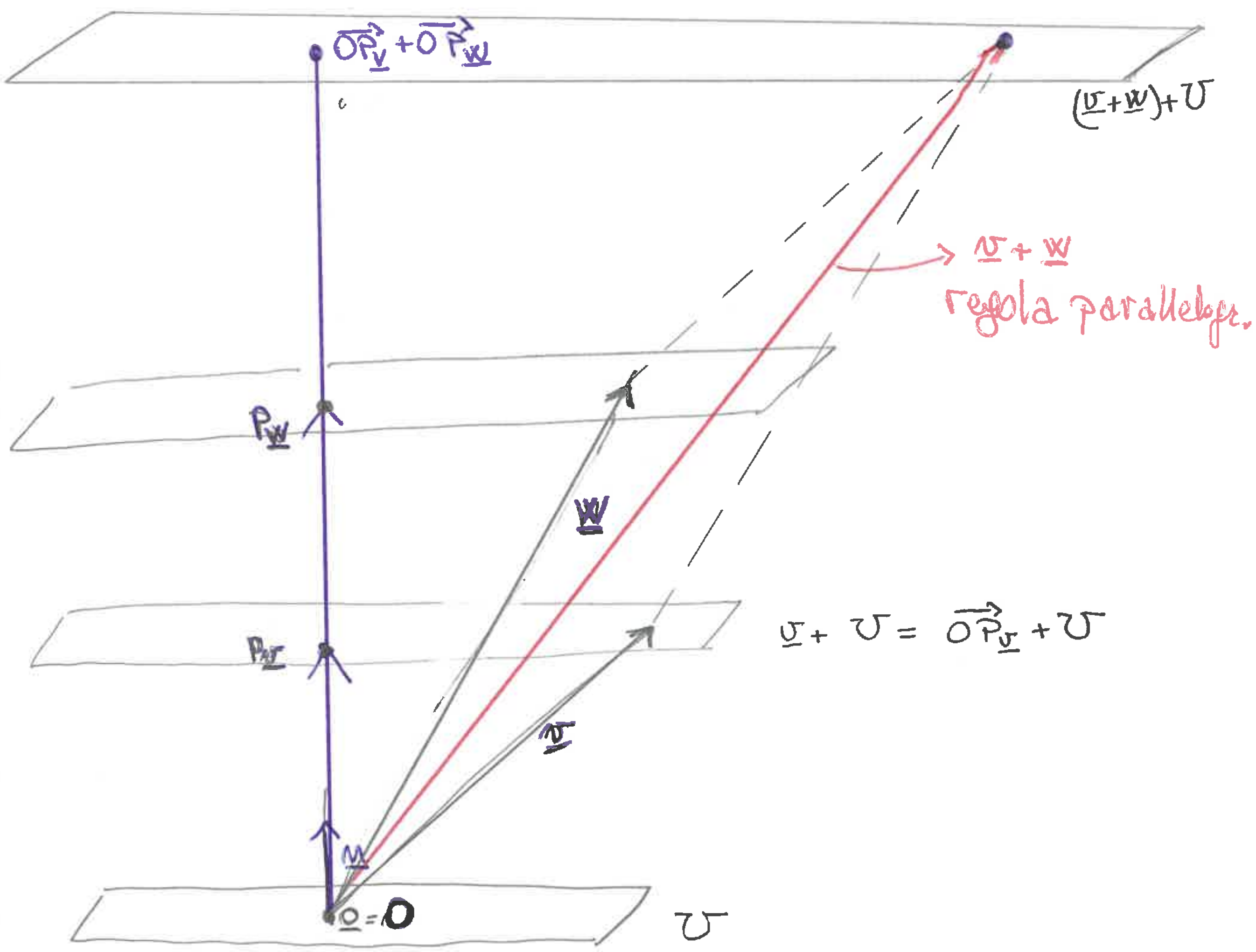
$\Rightarrow \exists !\ P_{\underline{w}} \in \underline{w} + U \cap \mathrm{Span}\{\underline{u}\}$, $\exists !\ P_{\underline{v}} \in \underline{v} + U \cap \mathrm{Span}\{\underline{u}\}$

$\Rightarrow [\underline{v}] + [\underline{w}] = \overrightarrow{OP_{\underline{v}}} + U + \overrightarrow{OP_{\underline{w}}} + U$

$\overrightarrow{OP_{\underline{v}}} \sim_U \underline{v}$ perché individuano stesso ssp. affine $\underline{v} + U$

$\overrightarrow{OP_{\underline{w}}} \sim_U \underline{w}$ " " " " $\underline{w} + U$

$$[\underline{v}] + [\underline{w}] = \overrightarrow{OP_{\underline{v}}} + \overrightarrow{OP_{\underline{w}}} + U = \underline{v} + \underline{w} + U$$



: Quoziente

$V/_U$ è sp. vett. e $\dim(V/_U) = \dim(V) - \dim(U)$

perché

$$V \xrightarrow{\pi_U} V/_U \quad \text{suriettiva e } \ker(\pi_U) = U$$

$\Rightarrow \dim(V/_U) = 1$

Pertanto $V/_U \underset{e.g.}{\cong} \mathrm{Span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$ con interpretazione precedente

(d) Considerato $\varphi \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$ t.c.

$\varphi = \pi_{\mathrm{Span}\{\underline{u}\}} \circ \psi$ (proiez. ortog. su span $(\underline{u})$) con $M_{\mathcal{E},\mathcal{E}}(\psi) = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} = A$

Determinare $! \ \overline{\varphi} : \dfrac{\mathbb{R}^3}{U} \longrightarrow \mathbb{R}^3$ per cui

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\varphi"] \arrow[dr, "\pi_U"'] & \mathbb{R}^3 \\ & \mathbb{R}^3/U \arrow[u, "\overline{\varphi}"'] \end{tikzcd}$$

↺ commutativo

Notiamo che

$$\psi(\underline{u}) = A \circ \begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}2\\-4\\2\end{pmatrix} = 2\,\underline{u} \Rightarrow \underline{u} \text{ autovett. di autovalore } \lambda = 2 \text{ per } \psi$$

$\det(A) = 2 \neq 0 \Rightarrow \psi \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3)$

$\pi_{\mathrm{Span}\{\underline{u}\}}$ ha rango 1 $\mathrm{Im}(\pi_{\mathrm{Span}\{\underline{u}\}}) = \mathrm{Span}\{\underline{u}\}$

$\Rightarrow \varphi(\underline{u}) = \pi_{\mathrm{Span}(\underline{u})}(\psi(\underline{u})) = \pi_{\mathrm{Span}(\underline{u})}(2\,\underline{u}) = 2\,\underline{u}$

$\Rightarrow \mathrm{Im}(\varphi) = \mathrm{Span}\{\underline{u}\}$

$\varphi(t\,\underline{u}) = 2t\,\underline{u}, \quad \forall\, t \in \mathbb{R}$

$$\Rightarrow \begin{tikzcd} \mathbb{R}^3 \arrow[r, two heads, "\varphi"] \arrow[d, "\pi_U"'] & \mathrm{Im}(\varphi) = \mathrm{Span}(\underline{m}) \subsetneq \mathbb{R}^3 \\ \mathbb{R}^3/U \cong \mathrm{Span}(\underline{m}) \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

$\overline{\varphi}$ = moltiplicazione per 2 $\Rightarrow$ isomorfismo

Svolgimento esercizio 4

(i) $(\mathbb{C}^n, \langle\, , \rangle_{h,st})$, Z matrice hermitiana, cioè $\overline{Z}^t = Z$
$A \in M(n,n;\mathbb{C})$ qualsiasi

$$B = Z - A \circ Z \in M(n,n;\mathbb{C}) \cong End_{\mathbb{C}}(\mathbb{C}^n)$$

$B^* =$ matrice aggiunta di B se

$$\langle B \circ \underline{x}, \underline{y} \rangle_{h,st} = \langle \underline{x}, B^* \circ \underline{y} \rangle_{h,st} \quad \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n$$

Allora

$$\langle B \circ \underline{x}, \underline{y} \rangle_{h,st} = (B \circ \underline{x})^t \circ \overline{\underline{y}} = \underline{x}^t \circ B^t \circ \overline{\underline{y}} \quad \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n$$

$$\parallel$$

$$\langle \underline{x}, B^* \circ \underline{y} \rangle_{h,st} = \underline{x}^t \circ \overline{B^* \circ \underline{y}} = \underline{x}^t \circ \overline{B^*} \circ \overline{\underline{y}}$$

$$\Rightarrow \quad B^t = \overline{B^*} \quad \Rightarrow \quad \overline{B^t} = \overline{\overline{B^*}} \quad \Rightarrow \quad \overline{B^t} = B^*$$

$$\Rightarrow B^* = \overline{B^t} = \overline{(Z - A \circ Z)^t} = \overline{(Z \circ (I_n - A))^t} =$$

$$= \overline{(I_n - A)^t \circ Z^t} = \overline{(I_n - A)^t} \circ \overline{Z^t} = (\overline{I_n^t - A^t}) \circ Z$$

$$\overline{Z^t} = \overline{Z}^t = Z$$
perché Z hermitiana

$$= (\overline{I_n - A^t}) \circ Z = (I_n - \overline{A^t}) \circ Z$$

$$= I_n \circ Z - \overline{A^t} \circ Z = Z - \overline{A}^t \circ Z$$

$$\Rightarrow \quad B^* = Z - \overline{A}^t \circ Z \quad \text{come richiesto}$$

(ii) se $(\mathbb{R}^n, \langle\, , \rangle_{st})$ e $X \in Sym(n,n;\mathbb{R})$ cioè $X^t = X$
$A \in M(n,n;\mathbb{R})$ qualsiasi

$$C = X - A \circ X \in M(n,n;\mathbb{R}) \cong End_{\mathbb{R}}(\mathbb{R}^n)$$

$C^* =$ matrice aggiunta di C se

$$\langle C \circ \underline{x}, \underline{y} \rangle_{st} = \langle \underline{x}, C^* \circ \underline{y} \rangle_{st} \quad \forall\, \underline{x}, \underline{y} \in \mathbb{R}^n$$

Stessi conti precedenti dimostrano che

$$C^* = X - A^t \circ X \quad \text{come richiesto.}$$