Esercizio 1

V sp. vettoriale, $\mathbb{K}$ campo, $\dim_{\mathbb{K}}(V)=n$

(i) $\mathbb{K}=\mathbb{C}$

V con forma hermitiana $\langle , \rangle_h$ definita positiva

Allora F operatore hermitiano, i.e.

$$(*) \quad \boxed{\langle F(\underline{v}), \underline{w}\rangle_h = \langle \underline{v}, F(\underline{w})\rangle_h \quad \forall\, \underline{v}, \underline{w} \in V}$$

$\Leftrightarrow$ $\forall$ $\mathcal{E}$ base $\langle , \rangle_h$-ortonormale, $M_{\mathcal{E},\mathcal{E}}(F)=A$ è matrice hermitiana cioè $\boxed{A^t = \bar{A} \in H(n,n;\mathbb{C})}$

(ii) $\mathbb{K}=\mathbb{R}$

$(V, \langle , \rangle)$ euclideo

Allora F è autoaggiunto, i.e. vale $(*)$ con $\langle , \rangle$ $\Leftrightarrow$

$\forall$ $\mathcal{E}$ base $\langle , \rangle$-ortonormale, $M_{\mathcal{E},\mathcal{E}}(F)=A$ è matrice simmetrica cioè $\boxed{A^t = A}$

(iii) Osservare che la richiesta $\mathcal{E}$ $\langle , \rangle$-ortonormale è fondamentale

Studiare in

$(\mathbb{R}^2, \langle , \rangle_{st})$ e base $\mathcal{G} = \left\{ \underline{g}_1 = \begin{pmatrix}1\\1\end{pmatrix} = \underline{e}_1+\underline{e}_2,\ \underline{g}_2 = \begin{pmatrix}2\\1\end{pmatrix} = 2\underline{e}_1+\underline{e}_2 \right\}$

(a) $F \in \mathrm{End}(\mathbb{R}^2)$ t.c. $F(\underline{g}_1) = 3\underline{g}_1$, $F(\underline{g}_2) = 6\underline{g}_1 - \underline{g}_2$

F è autoaggiunto in $(\mathbb{R}^2, \langle , \rangle_{st})$ ma $M_{\mathcal{G},\mathcal{G}}(F)$ non è simmetrica

(b) $S \in \mathrm{End}(\mathbb{R}^2)$ t.c.

$M_{\mathcal{G},\mathcal{G}}(S) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ è simmetrica

ma S non è autoaggiunto in $(\mathbb{R}^2, \langle , \rangle_{st})$

(iv) se $\mathbb{K}=\mathbb{C}$ e $(V, \langle , \rangle_h)$ hermitiana def. positiva

F è operatore unitario i.e.

$$(**) \quad \boxed{\langle F(\underline{v}), F(\underline{w})\rangle_h = \langle \underline{v}, \underline{w}\rangle_h \quad \forall\, \underline{w}, \underline{v} \in V}$$

$\Leftrightarrow$ $\forall$ base $\mathcal{E}$ $\langle , \rangle_h$-ortonormale $U = M_{\mathcal{E},\mathcal{E}}(F)$ è unitaria

i.e. $U = M_{\mathcal{E},\mathcal{E}}(F)$ e $\boxed{U^t \cdot \bar{U} = I_n}$

(V) Se $\mathbb{K}=\mathbb{R}$ e $(V,\langle,\rangle)$ euclideo

F operatore ortogonale i.e.

$$\boxed{\langle F(\underline{v}), F(\underline{w})\rangle = \langle \underline{v}, \underline{w}\rangle \quad \forall\, \underline{v}, \underline{w} \in V}$$

$\Leftrightarrow$ $\forall$ base $\mathcal{E}$ $\langle,\rangle$-ortonormale, $M_{\mathcal{E},\mathcal{E}}(F)$ è matrice ortogonale, i.e. $M_{\mathcal{E}\mathcal{E}}(F)=M$ e $\boxed{M^t \circ M = I_n}$.

(vi)

Svolgimento Esercizio 1

(i) $(\Rightarrow)$ F Hermitiano $\Leftrightarrow$ $\langle F(\underline{v}), \underline{w}\rangle_h = \langle \underline{v}, F(\underline{w})\rangle_h$, $\forall\, \underline{v}, \underline{w} \in V$

se $\forall$ $\mathcal{B}$ base, si hanno

$$C_{\mathcal{B}}(\underline{v}) =: \underline{x}, \quad C_{\mathcal{B}}(\underline{w}) =: \underline{y}, \quad M_{\mathcal{B},\mathcal{B}}(F) =: B \quad e \quad M_{\mathcal{B}}(\langle,\rangle_h) =: G$$

$\Rightarrow$

$$\langle F(\underline{v}), \underline{w}\rangle_h = \underline{x}^t \circ B^t \circ C \circ \overline{\underline{y}}$$
$$\|$$
$$\langle \underline{v}, F(\underline{w})\rangle_h = \underline{x}^t \circ C\, \overline{B} \circ \overline{\underline{y}} \quad \Rightarrow \quad \boxed{B^t \circ C = C \circ \overline{B}}$$

(*) condizione matriciale di F $\langle,\rangle_h$-hermitiano

Se $\mathcal{B}=\mathcal{E}$ base $\langle,\rangle_h$-ortonormale $\Rightarrow$ $G = I_n$ $\Rightarrow$ $\boxed{B^t = \overline{B}}$

cioè $\overline{B^t} = B$ $\Rightarrow$ $\overline{B}^t = B$ $\Rightarrow$ B matrice hermitiana

$(\Leftarrow)$ Sia B Hermitiana su $\mathbb{C}^n$ con $\boxed{\langle \underline{z}, \underline{w}\rangle_{st,h} = \underline{z}^t \circ \overline{\underline{w}}}$ $\Rightarrow$

$\mathcal{E}$ base canonica $\Rightarrow$ $\langle,\rangle_h$-ortonormale (reale)

$$(\mathbb{C}^n, \langle,\rangle_{st,h}, \mathcal{E}) \xleftrightarrow{\ \cong\ } (V, \langle,\rangle_h, \mathcal{E} \text{ ortogonale})$$

$\boxed{F := L_B}$ t.c. $\boxed{F(\underline{v}) = B \circ \underline{x}}$ dove $\underline{v} \in V$ t.c. $\boxed{C_{\mathcal{E}}(\underline{v}) = \underline{x}}$

$$\langle F(\underline{v}), \underline{w}\rangle_h = \underline{x}^t \circ B^t \circ \overline{\underline{y}} \underset{\boxed{B^t = \overline{B}}}{=} \underline{x}^t \circ \overline{B} \circ \overline{\underline{y}} = \underline{x}^t \circ (\overline{B \circ \underline{y}}) =$$

$$= \langle \underline{v}, F(\underline{w})\rangle_h$$

$F = L_B$ è hermitiano ∎

(ii) Nel caso reale stessa procedura come prima

- Se $\mathbb{K} = \mathbb{R}$

  F hermitiano $\Leftrightarrow$ F è autoaggiunto o $\underbrace{\text{simmetrico}}_{\text{[Sernesi]}}$

- Notare che $M_{\mathcal{B},\mathcal{B}}(F) = A$ reale

  Imporre autoaggiunto è come in (*) ma senza coniugio

$$\boxed{B^t \circ C = C \circ B}$$

Se però $\mathcal{E}$ $\langle , \rangle$-ortonormale $\Rightarrow$

$$\boxed{B^t = B}$$

matrice simmetrica

ATTENZIONE N.B

Abbiamo usato pesantemente

$\mathcal{E}$ $\langle , \rangle_h$-ortonormale (se $\mathbb{K} = \mathbb{C}$)

$\langle , \rangle$-ortonormale (se $\mathbb{K} = \mathbb{R}$)

(iii) $(\mathbb{R}^2, \mathcal{G})$, $\langle\,,\rangle_{st}$

$\mathcal{G} = \left\{ \underline{g}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \underline{g}_2 = \begin{pmatrix}2\\1\end{pmatrix} \right\} \Rightarrow M_{\mathcal{G}}(\langle\,,\rangle_{st}) = \begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix} = G$

(a) $F \in End(\mathbb{R}^2)$ t.c. $F(\underline{g}_1) = 3\underline{g}_1$ $\quad F(\underline{g}_2) = 6\underline{g}_1 - \underline{g}_2$

$$\Rightarrow \boxed{M_{\mathcal{G},\mathcal{G}}(F) = \begin{pmatrix}3 & 6\\ 0 & -1\end{pmatrix} = A} \text{ non simmetrica in base } \mathcal{G}$$

eppure

$$\langle F(\underline{x}), \underline{y}\rangle = \underline{x}^t \circ A^t \circ G \circ \underline{y} = \underline{x}^t \circ \begin{pmatrix}6 & 9\\ 9 & 13\end{pmatrix} \circ \underline{y}$$

$$A^t \circ G = \begin{pmatrix}3 & 0\\ 6 & -1\end{pmatrix} \circ \begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix} = \begin{pmatrix}6 & 9\\ 9 & 13\end{pmatrix}$$

$$\langle \underline{x}, F(\underline{y})\rangle = \underline{x}^t \circ G \circ A \circ \underline{y} = \underline{x}^t \circ \begin{pmatrix}6 & 9\\ 9 & 13\end{pmatrix} \circ \underline{y}$$

$$G \circ A = \begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}3 & 6\\ 0 & -1\end{pmatrix} = \begin{pmatrix}6 & 9\\ 9 & 13\end{pmatrix}$$

$\Rightarrow$ F è autoaggiunto in $\mathbb{R}^2, \langle\,,\rangle_{st}$ ma $\mathcal{G}$ non è $\langle\,,\rangle_{st}$-ortonormale

Però: $M := M_{\mathcal{E},\mathcal{G}} = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} \Rightarrow M_{\mathcal{G},\mathcal{E}} = M_{\mathcal{E},\mathcal{G}}^{-1} = M^{-1} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}$

$$\begin{tikzcd}
(\mathbb{R}^2, \mathcal{E}) \arrow[r, "F"] \arrow[d] & (\mathbb{R}^2, \mathcal{E}) \\
(\mathbb{R}^2, \mathcal{G}) \arrow[r, "F"', "A"] & (\mathbb{R}^2, \mathcal{G}) \arrow[u]
\end{tikzcd}$$

$M_{\mathcal{E},\mathcal{G}} \circ M_{\mathcal{G},\mathcal{G}}(F) \circ M_{\mathcal{G},\mathcal{E}} = M \circ A \circ M^{-1} = B = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$

simmetrica in base $\mathcal{E}$ che è $\langle\,,\rangle$-ortonormale

(b) Invece $S \in End(\mathbb{R}^2)$ t.c.

$M_{\mathcal{G},\mathcal{G}}(S) := \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ è simmetrica in base $\mathcal{G}$

ma $\langle S(\underline{g}_1), \underline{g}_2\rangle_{st} = \langle \underline{g}_1 + \underline{g}_2, \underline{g}_2\rangle_{st} = \langle \underline{g}_1, \underline{g}_2\rangle_{st} + \|\underline{g}_2\|^2_{st} = 8$ $\neq$

$\langle \underline{g}_1, S(\underline{g}_2)\rangle_{st} = \langle \underline{g}_1, \underline{g}_1 - \underline{g}_2\rangle_{st} = \|\underline{g}_1\|^2_{st} - \langle \underline{g}_1, \underline{g}_2\rangle_{st} = -1$

$\Rightarrow$ S non è autoaggiunto su $V = \mathbb{R}^2$

(iv)

($\Rightarrow$) F unitario $\Rightarrow$ $\forall\, \underline{v}, \underline{w} \in V$

$$\langle F(\underline{v}), F(\underline{w}) \rangle_h = \langle \underline{v}, \underline{w} \rangle_h$$

Per ogni $\mathcal{B}$ base di V

$C_{\mathcal{B}}(\underline{v}) = \underline{x}$ = coord. in base $\mathcal{B}$ di $\underline{v}$

$C_{\mathcal{B}}(\underline{w}) = \underline{y}$ = " " " " $\underline{w}$

$M_{\mathcal{B},\mathcal{B}}(F) = B$

$M_{\mathcal{B}}(\langle , \rangle_h) = C$

$\langle , \rangle_h$ hermitiana definita positiva $\Rightarrow$

C hermitiana non degenere, i.e. $C^t = \overline{C}$, $rg(C) = n$

$$\begin{array}{ccc}
\langle F(\underline{v}), F(\underline{w}) \rangle_h & = & \langle \underline{v}, \underline{w} \rangle_h \\
\parallel & & \parallel \\
(B \circ \underline{x})^t \circ C \circ \overline{(B \circ \underline{y})} & & \underline{x}^t \circ C \circ \overline{\underline{y}} \\
\parallel & & \\
\underline{x}^t \circ B^t \circ C \circ \overline{B} \circ \overline{\underline{y}} & & \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n \\
\parallel & & \Downarrow \\
\underline{x}^t \circ (B^t \circ C \circ \overline{B}) \circ \overline{\underline{y}} & & \boxed{B^t \circ C \circ \overline{B} = C}
\end{array}$$

Se però $\mathcal{B} = \mathcal{E}$ base $\langle , \rangle_h$ - ortonormale

$\Rightarrow$ $C = I_n$ $\Rightarrow$ $B^t \circ \overline{B} = I_n$ $\Rightarrow$ B unitaria

($\Leftarrow$) Sia $B \in M(n,n;\mathbb{C})$ unitaria su $\mathbb{C}^n$, i.e. $B^t \circ \overline{B} = I_n$

Consideriamo una $\forall$ $\mathcal{E} = \{\underline{e}_1, -, \underline{e}_n\}$ su $\mathbb{C}^n$ che sia $\langle , \rangle_h$ - ortonormale

$F := L_B \in End_{\mathbb{C}}(\mathbb{C}^n)$

$$(\mathbb{C}^n, \langle , \rangle_{h,st}, \mathcal{E}) \xleftrightarrow{\cong} (V, \langle , \rangle_h, \mathcal{E}_{orton})$$

con

$$F(\underline{v}) = B \circ \underline{x} \quad \forall\ \underline{v} \in V \text{ t.c. } c_{\mathcal{E}}(\underline{v}) = \underline{x}$$

- $\langle F(\underline{v}), F(\underline{w}) \rangle_h = \langle B\circ\underline{x}, B\circ\underline{y} \rangle_{h,st} =$

$$= (B\circ\underline{x})^t \circ \overline{(B\circ\underline{y})} = \underline{x}^t \circ B^t \circ \overline{B} \circ \overline{\underline{y}} = \underline{x}^t \circ (B^t \circ \overline{B}) \circ \overline{\underline{y}} \quad \forall\ \underline{x}, \underline{y} \in \mathbb{C}^n$$

$$\downarrow$$

$$M_{\mathcal{E}}(\langle , \rangle_{h,st}) = I_n$$

- $\langle \underline{v}, \underline{w} \rangle_h = \underline{x}^t \circ \overline{\underline{y}} = \underline{x}^t \circ I_n \circ \overline{\underline{y}} \quad \forall\ \underline{x}, \underline{y} \in \mathbb{C}^n$

Poiché B unitaria $\Rightarrow$ $B^t \circ \overline{B} = I_n$

$\Rightarrow \quad \langle F(\underline{v}), F(\underline{w}) \rangle_h = \langle \underline{v}, \underline{w} \rangle_h \quad \forall\ \underline{v}, \underline{w} \in V$

$\Rightarrow$ $F = L_B$ è unitario su $(V, \langle , \rangle_h)$

(V) Se $(V, \langle , \rangle)$ euclideo con $\mathbb{K} = \mathbb{R}$

La condizione di $M_{\mathcal{E},\mathcal{E}}(F)$ diventa

$M^t \circ M = I_n$ cioè M ortogonale

Esercizio 2

(i) $(\mathbb{R}^4, \langle , \rangle_{st})$, $\mathcal{E} = \{e_1, \dots, e_4\}$ ortonormale

$$F \in End(\mathbb{R}^4) \text{ t.c. } M_{\mathcal{E},\mathcal{E}}(F) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} := A$$

(a) Riconoscere che F è autoaggiunto su $\mathbb{R}^4$

(b) dedurre forma diagonale D di F e spettro di (F)

(c) Trovare M matrice ortogonale per cui $M^t \circ A \circ M$ dà la forma diagonale

(ii) $V = \mathbb{C}^3$, con prodotto hermitiano canonico

$$\langle \underline{z}, \underline{w} \rangle_h := \underline{z}^t \circ \overline{\underline{w}}$$

$\mathcal{E} = \{e_1, e_2, e_3\}$ base canonica reale h-ortonormale

Consideriamo $T \in End(\mathbb{C}^3)$ t.c. $M_{\mathcal{E}\mathcal{E}}(T) =: B$

$$B = \begin{pmatrix} 3/4 & i/4 & \frac{-1+i}{4} \\ -\frac{i}{4} & 3/4 & \frac{1}{4}(-1-i) \\ \frac{1}{4}(-1-i) & \frac{1}{4}(-1+i) & 1/2 \end{pmatrix}$$

(a) Riconoscere che T è hermitiano

(b) Determinare Spettro e forma diagonale D di T.

(c) Determinare una base $\langle , \rangle_h$-ortog. di autovettori di T e calcolare matrice di $\langle , \rangle_h$ in tale base

(d) Trovare U unitaria per cui $\overline{U}^t \circ B \circ U = D$

(ii) $(\mathbb{R}^4, \langle , \rangle_{st})$ e $F \in End(\mathbb{R}^4)$ t.c. ②

$$A := M_{\mathcal{E},\mathcal{E}}(F) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$\mathcal{E}$ ortonormale risp. a $\langle , \rangle_{st}$ $\Rightarrow$ F auto aggiunto $\Rightarrow$ Spettro$(A) \subseteq \mathbb{R}$ e A diag.

$$P_A(x) = \det(A - x I_4) \in \mathbb{R}[x]$$

Ma ricordo

$$P_A(x) = (-1)^n x^n + (-1)^{n-1} tr(A) x^{n-1} + (-1)^{n-2} \ldots + \det(A)$$

coefficiente di x^k $= (-1)^k \cdot \left(\sum \det(\text{minori } k \times k) \text{ centrati su Diagonale di } A \right)$

$$\Rightarrow P_A(x) = x^4 - tr(A)x^3$$

Infatti visto che $rg(A) = rg(T) = 1 \Rightarrow \dim(\ker(T)) = 3 \Rightarrow$

$\boxed{\lambda_1 = 0}$ è autovalore con $\boxed{\mu_a(0) = \mu_g(0) = 3}$ → perché T diag. certamente

$$\Rightarrow P_A(x) = P_T(x) = x^3 \cdot (x - tr(A)) \text{ con } \boxed{tr(A) = 4}$$

$\Rightarrow$ l'altro autovalore $\boxed{\lambda_2 = 4}$ è altro autovalore

$\Rightarrow \exists \ \mathcal{F} = \{f_1, f_2, f_3, f_4\}$ orto normale di autovettori t.c.

$$M_{\mathcal{E},\mathcal{F}} = M \quad e \quad M^t \circ A \circ M = D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Come trovare M?

$$\ker(T) = \ker(A) = \{x_1 + x_2 + x_3 + x_4 = 0\}$$

$$\underbrace{v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}_{\langle , \rangle_{st}\text{-ortogonali}} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$w_1 = v_1, \quad w_2 = v_2, \quad w_3 = v_3 - \frac{\langle v_3, v_1 \rangle_{st}}{\langle v_1, v_1 \rangle_{st}} v_1 - \frac{\langle v_3, v_2 \rangle_{st}}{\langle v_2, v_2 \rangle_{st}} v_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \frac{(-1)}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \frac{(-1)}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{w}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \underline{w}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$\{\underline{w}_1, \underline{w}_2, \underline{w}_3\} = \mathcal{W}$ <u>base ortogonale per $V_0(T) = \ker(T)$</u>

$\left\{ \underline{w}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \to$ <u>base di $V_0(T)^\perp = V_4(T)$</u> Ma $\lambda_1 = 0 \neq 4 = \lambda_2$

$$\Rightarrow \boxed{V_0(T)^\perp = V_4(T)}$$

$$\mathcal{F} = \left\{ \underline{f}_1 = \frac{\underline{w}_1}{\|\underline{w}_1\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \ \underline{f}_2 = \begin{pmatrix} 0 \\ 0 \\ +1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \ \underline{f}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \ \underline{f}_4 = \begin{pmatrix} \frac{1}{2} \\ 1/2 \\ 1/2 \\ \frac{1}{2} \end{pmatrix} \right\}$$

<u>base ortonormale per $\mathbb{R}^4, \langle , \rangle_{st}$ e di autovettori di T</u>

base ker

$$M_{\mathcal{E},\mathcal{F}}^{-1} = M_{\mathcal{E},\mathcal{F}}^{t}$$ <u>matrice ortogonale</u> $M_{\mathcal{E},\mathcal{F}} = M$

$$M^{-1} \circ A \circ M = M^{t} \circ A \circ M = D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

(ii) Notare che

(a) $B^t = \overline{B} \Rightarrow$ <u>B hermitiana su $\mathbb{C}^n$</u>

e $\mathcal{E}$ è $\langle , \rangle_{h,st}$ ortonormale $\Rightarrow \underset{L_B}{\overset{\|}{T}}$ è <u>operatore hermitiano</u> su $(\mathbb{C}^n, \langle , \rangle_{h,st})$

$\Rightarrow$ Spettro $(T) \subseteq \mathbb{R}$

$$P_T(x) = \det(B - x I_3)$$

Ma $\det(B) = 0 \Rightarrow P_T(x) = -x^3 + Tr(B)x^2 - \alpha x = -x(x^2 - Tr(B)x^2 + \alpha)$

$$Tr(B) = \frac{3}{4} + \frac{3}{4} + \frac{1}{2} = \frac{6}{4} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2$$

$$\alpha = \left[\det\begin{pmatrix} 3/4 & i/4 \\ -i/4 & 3/4 \end{pmatrix} + \det\begin{pmatrix} 3/4 & \frac{1}{4}(-1+i) \\ \frac{1}{4}(-1-i) & \frac{1}{2} \end{pmatrix} + \det\begin{pmatrix} 3/4 & \frac{1}{4}(-1-i) \\ \frac{1}{4}(-1+i) & \frac{1}{2} \end{pmatrix}\right] = 1$$

$\Rightarrow P_T(x) = -x \cdot (x^2 - 2x + 1) \Rightarrow$

$x = 0 = \lambda_1$ <u>autovalore semplice</u>

$x = 1 = \lambda_2$ <u>autovalore $\mu_a(1) = 2$</u>

Spettro $(T) = \{0, 1\} \subseteq \mathbb{R}$

(b) $$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) $V_0(T) = \ker(T) = \begin{cases} 3x + iy + (-1+i)z = 0 \\ -ix + 3y + (-1-i)z = 0 \end{cases} \Rightarrow$

$$\Rightarrow \boxed{V_0(T) = \operatorname{Span}\left\{ v_3 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\}}$$

Poiché $\lambda_2 = \lambda_3 = 1 \neq \lambda_1 = 0$ (5)

$$\Rightarrow \quad V_1(T) \underset{\substack{\downarrow \\ T \text{ hermitiano}}}{=} (V_0(T))^{\perp} = (\mathrm{Span}(v_3))^{\perp}$$

$$0 = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_3 \right\rangle_{hist} \Leftrightarrow \quad (x\ y\ z) \circ \overline{v_3} = 0$$

$$(x\ y\ z) \circ \begin{pmatrix} 1 \\ -i \\ 1-i \end{pmatrix} = 0$$

$$\boxed{x - iy + (1-i)z = 0} \quad \text{eq. cartesiana di } V_1(T)$$

* Se $x = 0 \Rightarrow$ $\boxed{v_1 = \begin{pmatrix} 0 \\ 1-i \\ i \end{pmatrix} \in V_1(T)}$

* Per trovare $v_2 \in V_1(T)$ t.c. $v_2 \in v_1^{\perp_h}$ considero:

$$\begin{cases} 0 = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_1 \right\rangle_h = (x\ y\ z) \circ \begin{pmatrix} 0 \\ 1+i \\ -i \end{pmatrix} = (1+i)y - iz = 0 & \rightarrow \text{eq. } v_1^{\perp} \\ x - iy + (1-i)z = 0 & \rightarrow \text{eq. di } V_1(T) \end{cases}$$

$\Rightarrow$ trovo $\boxed{v_2 = \begin{pmatrix} 3i-3 \\ 1+i \\ 2 \end{pmatrix} \in V_1(T) \text{ e } v_2 \perp v_1}$

$$\Rightarrow \quad V_0(T) = \mathrm{Span}\{v_3\} = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}}_{v_3} \right\}$$

$$V_1(T) = \mathrm{Span}\{v_1, v_2\} = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 0 \\ 1-i \\ i \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} -3+3i \\ 1+i \\ 2 \end{pmatrix}}_{v_2} \right\}$$

$$\mathcal{V} = \left\{ v_1 = \begin{pmatrix} 0 \\ 1-i \\ i \end{pmatrix}, \; v_2 = \begin{pmatrix} 3i-3 \\ i+1 \\ 2 \end{pmatrix}, \; v_3 = w = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\}$$

base $\langle , \rangle_h$-ortogonale per $\mathbb{C}^3$

$$M_{\mathcal{V},\mathcal{V}}(\langle , \rangle_h) = \left(\langle v_i, v_j \rangle_h \right) \quad 1 \le i,j \le 3$$

Ma $\mathcal{V}$ è base $\langle , \rangle_h$-ortogonale $\Rightarrow$

$$M_{\mathcal{V},\mathcal{V}}(\langle , \rangle_h) = \begin{pmatrix} h(v_1, v_1) & 0 & 0 \\ 0 & h(v_2, v_2) & 0 \\ 0 & 0 & h(v_3, v_3) \end{pmatrix}$$

dove

$$h(v_i, v_i) = \| v_i \|_h^2, \quad 1 \le i \le 3 \Rightarrow$$

$$\| v_1 \|_h^2 = v_1^t \circ \overline{v_1} = (0 \quad 1-i \quad i) \circ \begin{pmatrix} 0 \\ 1+i \\ -i \end{pmatrix} = 1 - (i)^2 - (i)^2 = 3$$

$$\| v_2 \|_h^2 = (-3+3i \quad 1+i \quad 2) \circ \begin{pmatrix} -3-3i \\ 1-i \\ 2 \end{pmatrix} = 9 - (3i)^2 + 1 - (i)^2 + 4 = 24$$

$$\| v_3 \|_h^2 = (1 \quad i \quad 1+i) \circ \begin{pmatrix} 1 \\ -i \\ 1-i \end{pmatrix} = 1 - (i)^2 + 1 - (i)^2 = 4$$

$$\Rightarrow \quad M_{\mathcal{V},\mathcal{V}}(\langle , \rangle_h) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 4 \end{pmatrix} = P$$

$P^t = P = \overline{P} \Rightarrow P$ effettivamente è Hermitiana

(d) Considero

$$\mathcal{W} = \left\{ \frac{v_1}{\|v_1\|_h} = \begin{pmatrix} 0 \\ \frac{1-i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \end{pmatrix} = u_1, \quad \frac{v_2}{\|v_2\|_h} = \begin{pmatrix} \frac{3i-3}{2\sqrt{6}} \\ \frac{i+1}{2\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = u_2, \quad \frac{v_3}{\|v_3\|_h} = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \frac{1+i}{2} \end{pmatrix} = u_3 \right\}$$

base $\langle , \rangle_h$-ortonormale di $V_1(T)$

base $\langle , \rangle_h$-ortonormale di $V_0(T)$

$$\Rightarrow \quad U = M_{\mathcal{E},\mathcal{W}} = \begin{pmatrix} 0 & \frac{3i-3}{2\sqrt{6}} & \frac{1}{2} \\ \frac{1-i}{\sqrt{3}} & \frac{i+1}{2\sqrt{6}} & i/2 \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1+i}{2} \end{pmatrix}$$ unitaria e

$$U^{-1} \circ B \circ U = \overline{U}^t \circ B \circ U = D$$