Esercizio 1 svolgimento

(i) $\mathbb{A}^3_{\mathbb{R}}$ con (x,y,z)

$r: \quad \underline{X} = P + t\,\underline{v}$ $\qquad P=(1,0,1)$, $\quad \underline{v} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$

$$\mathbb{A}^3_{\mathbb{R}} = A_0 \hookrightarrow \mathbb{P}^3$$
$$(x,y,z) \longrightarrow [1,x,y,z]$$

Approccio proiettivo

$\bar{r}$ passa per $P=(1,0,1) = [1,1,0,1] \in \mathbb{P}^3$

passa per $Q := r_\infty = [0,1,0,-2]$

$$\Rightarrow \quad \bar{r}: \quad (*) \quad \begin{cases} x_0 = \lambda \\ x_1 = \lambda+\mu \\ x_2 = 0 \\ x_3 = \lambda - 2\mu \end{cases} \qquad [\lambda,\mu]\in\mathbb{P}^1$$

$$\bar{r}: \quad \boxed{\underline{X} = \lambda P + \mu Q}$$

param. omogenee $\bar{r}$

$$\mathrm{rg}\begin{array}{c} \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \boxed{\begin{matrix}1 & 1\\ 0 & 1\end{matrix}} & \begin{matrix}0 & 1\\ 0 & -2\end{matrix} \end{pmatrix} \end{array} = 2$$

$\Downarrow$ Kronecker

$$\bar{r} \quad \begin{cases} x_2 = 0 \\ 3x_0 - 2x_1 - x_3 = 0 \end{cases} \qquad (**)$$

cartesiane omogenee $\bar{r}$

Approccio affine e poi chiusura proiettiva

$x = \dfrac{x_1}{x_0} \quad y = \dfrac{x_2}{x_0} \quad z = \dfrac{x_3}{x_0} \quad t = \dfrac{\mu}{\lambda} \qquad [\lambda,\mu]\in\mathbb{P}^1, \ \lambda \neq 0$

eq. param. affini di r

$$\begin{aligned} x &= 1 + t \\ y &= 0 + 0t \qquad t\in\mathbb{R} \\ z &= 1 - 2t \end{aligned}$$

$\Downarrow$

$$\frac{x_1}{x_0} = 1 + \frac{\mu}{\lambda} = \frac{\lambda+\mu}{\lambda}$$
$$\frac{x_2}{x_0} = 0$$
$$\frac{x_3}{x_0} = 1-2t = \frac{\lambda-2\mu}{\lambda}$$

$\Downarrow$

$$(*) \quad \begin{cases} x_0 = \lambda \\ x_1 = \lambda+\mu \\ x_2 = 0 \\ x_3 = \lambda-2\mu \end{cases}$$

eq. cartesiane r

$$\mathrm{rg}\begin{pmatrix} x-1 & y & z-1 \\ 1 & 0 & -2 \end{pmatrix} = 1$$

$\Downarrow$ Kronecker

$$\begin{cases} y = 0 \\ -2(x-1) - (z-1) = 0 \end{cases}$$

$\Downarrow$

$$\begin{cases} y = 0 \\ 2x + z - 3 = 0 \end{cases}$$

omogeneizzo risp. x_0

$$\begin{cases} x_2 = 0 \\ 2x_1 + x_3 - 3x_0 = 0 \end{cases} \qquad (**)$$

(ii)

Prendo $Q = [0, 1, 0, -2] = r_{\infty} \in \overline{r} \Rightarrow \overline{r}$ passa per Q

Notiamo che

$$\mathcal{H}_1 : \{x_1 = 0\} \subset \mathbb{P}^3$$

è sghembo a Q infatti $Q \notin \mathcal{H}_1$

$$\Rightarrow \mathcal{L}(Q, \mathcal{H}_1) = \mathbb{P}^3$$

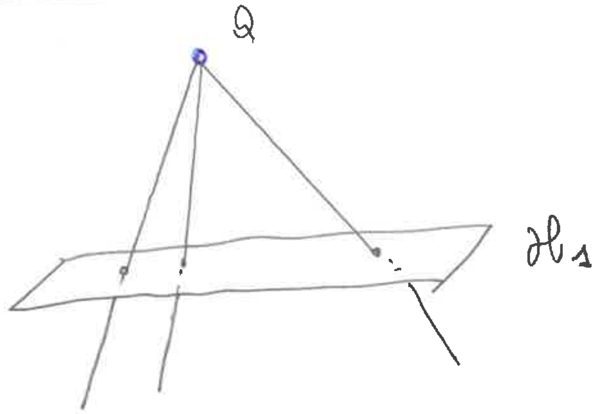
infatti per Grassmann proiettiva

$$\dim(\mathcal{L}(Q, \mathcal{H}_1)) = \dim(\{Q\}) + \dim(\mathcal{H}_1) - \dim(\{Q\} \cap \mathcal{H}_1)$$
$$= 0 + 2 - (-1)$$

infatti $\{Q\} \cap \mathcal{H}_1 = \emptyset = \mathbb{P}(\{0\})$

$$\Rightarrow \dim(\mathcal{L}(Q, \mathcal{H}_1)) = 3 \Rightarrow \mathcal{L}(Q, \mathcal{H}_1) = \mathbb{P}^3$$

(iii) Proiezione da Q su $\mathcal{H}$



Corrispondenza

$$\{\text{rette di } \mathbb{P}^3 \text{ per } Q\} \longleftrightarrow \{\text{punti di } \mathcal{H}_1\}$$

$\downarrow$

$\dim = 2$

$$\dim(\{r_Q \mid \text{retta per } Q\}) = 2$$

$\Rightarrow$ Stella di rette di dimensione 2 per Q in $\mathbb{P}^3$

Infatti

* $\forall$ retta ~~rappresentata~~ r_Q della Q è t.c. $r_Q \not\subseteq \mathcal{H}_1$

perché $Q \notin \mathcal{H}_1$

Poiché $r_Q \cap \mathcal{H}_1 \subseteq r_Q \Rightarrow$

$$\boxed{\dim(r_Q \cap \mathcal{H}_1) \leq \dim(r_Q) = 1} \quad (1)$$

Ora per Grassmann proiettivo

$$\dim(r_Q \cap \mathcal{H}_1) = \underbrace{\dim(r_Q)}_{=1} + \underbrace{\dim(\mathcal{H}_1)}_{=2} - \dim(\mathcal{L}(r_Q, \mathcal{H}_1))$$

Ora $\mathcal{L}(r_Q, \mathcal{H}_1)$ = sottosp. congiungente $\mathcal{H}_1$ e r_Q

$\cap$

$\mathbb{P}^3(\mathbb{R})$

$$\Rightarrow \boxed{\dim(\mathcal{L}(r_Q, \mathcal{H}_1)) \leq \dim(\mathbb{P}^3) = 3}$$

$$\Rightarrow \boxed{\begin{aligned}\dim(r_Q \cap \mathcal{H}_1) &= 1 + 2 - \dim(\mathcal{L}(r_Q, \mathcal{H}_1)) \\ &\geq 3 - 3 = 0\end{aligned}} \quad (2)$$

Perciò $0 \leq \dim(r_Q \cap \mathcal{H}_1) \leq 1$

Ma se fosse $\dim(r_Q \cap \mathcal{H}_1) = 1 \Rightarrow \dim(r_Q \cap \mathcal{H}_1) =$
$= \dim(r_Q) = 1$ i.e. $r_Q \subseteq \mathcal{H}_1$ ⋇ $Q \notin \mathcal{H}_1$

$\Downarrow$

$$\dim(r_Q \cap \mathcal{H}_1) = 0 \quad \forall r_Q$$

cioè ogni retta r_Q individua ! punto su $\mathcal{H}_1$

* Viceversa, $\forall$ punto $K \in \mathcal{H}_1 \Rightarrow \exists !$ retta di $\mathbb{P}^3$

$r_Q := \mathcal{L}(Q, K)$ passante per K e Q

$\Rightarrow$ ogni punto K di $\mathcal{H}_1$ individua ! retta r_Q della stella di rette di centro Q

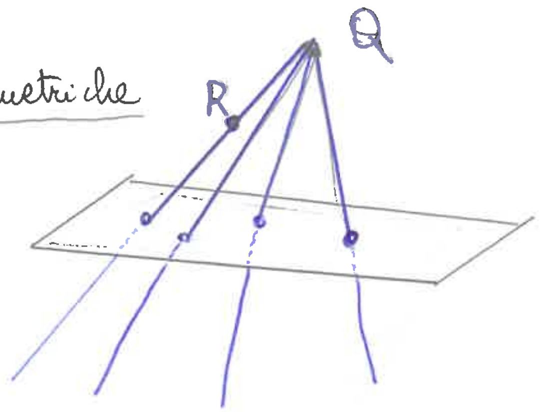
Costruisco proiezione

$\forall\ R = [\alpha, \beta, \gamma, \delta] \in \mathbb{P}^3(\mathbb{R}) \setminus \{Q\}$

$r_Q : \mathcal{L}(Q, R) : \quad \underline{x} = \lambda Q + \mu R$, con $[\lambda, \mu] \in \mathbb{P}^1(\mathbb{R})$

$$(*) \begin{cases} x_0 = \mu \alpha \\ x_1 = \lambda + \mu \beta \\ x_2 = \mu \gamma \\ x_3 = -2\lambda + \mu \delta \end{cases}$$

eq. parametriche r_Q



Taglio con $H_1 : \{x_1 = 0\}$

$(*) \Rightarrow \boxed{\lambda = -\mu \beta}$ $(**)$

se $\mu = 0 \underset{(**)}{\Longrightarrow} \lambda = 0$ ⋇ ma infatti $\mu = 0 \overset{\text{sarà}}{\longleftrightarrow} Q \notin H_1$

Se $\beta = 0 \underset{(**)}{\Rightarrow} \lambda = 0 \Rightarrow$ su $\mathcal{L}(Q, R)$ avevo quindi

$$\begin{cases} x_0 = \mu \alpha \\ x_1 = 0 \\ x_2 = \mu \gamma \\ x_3 = \mu \delta \end{cases}$$

$R = [\alpha, 0, \gamma, \delta] \in H_1 = \{x_1 = 0\}$

R già sul piano H_1

se $\beta \neq 0 \underset{\substack{(**) \\ \lambda = -\mu\beta}}{\Longrightarrow}$

$$\begin{cases} x_0 = \mu \alpha \\ x_1 = -\mu\beta + \mu\beta = 0 \\ x_2 = \mu \gamma \\ x_3 = +2\mu\beta + \mu\delta = \mu(2\beta + \delta) \end{cases}$$

$\boxed{\pi_Q(R) := [\alpha, 0, \gamma, 2\beta + \delta] \in H_1}$ Formule proiezione $\pi_Q : \mathbb{P}^3 \setminus \{Q\} \to H_1$

Perciò $\boxed{\pi_Q : \mathbb{P}^3 \setminus \{Q\} \longrightarrow H_1}$

π_Q non è definita in Q

$\dim(H_1) = 3 - 1 = 2$ piano di $\mathbb{P}^3$

H_1 ... $\mathbb{P}$

Interpretiamo con sp. vettoriali quozienti

Fatto 1

Se $V = U \oplus W \Rightarrow V/_{W} \cong U$

dim Infatti

$\forall\ \underline{v} \in V$ si scrive in modo ! come

$$\underline{v} = \underline{u} + \underline{w}$$

$!\ \underline{u} \in U$ e $!\ \underline{w} \in W$

Siccome $V/_{W} \ni [\underline{v} + W] \underset{\substack{\downarrow \\ \underline{v}-\underline{u} \in W}}{=} [\underline{u} + W] \xrightarrow{\varphi_U \cong} \underline{u} \in U$

infatti $\dim(U) = \dim(V) - \dim(W) = \dim(V/_{W})$ ∎

Fatto 2

Nel nostro caso $V = \mathbb{R}^4 \rightsquigarrow \mathbb{P}^3 = \mathbb{P}(V)$

$W_Q = \mathrm{Span}(\underline{v}_Q)$ dove $Q = [\underline{v}_Q] = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}\right]$

$V = W_Q \oplus U$ con $U \cong V/_{W_Q}$ ottenuto completando $\{\underline{v}_Q\}$ ad una base per $V = \mathbb{R}^4$

Ma infatti $\mathcal{H}_1 : \{x_1 = 0\} \subset \mathbb{P}^3$

Corrisponde a Iperpiano $U = \{x_1 = 0\} \subset \mathbb{R}^4 = \mathrm{Span}\{\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$U = \mathrm{Span}\{\underline{e}_0, \underline{e}_2, \underline{e}_3\} \subset \mathbb{R}^4$

e $W_Q \oplus U = \mathrm{Span}\{\underline{v}_Q, \underline{e}_0, \underline{e}_2, \underline{e}_3\}$

Perciò

$$V \xrightarrow{\pi_{W_Q}} V/_{W_Q} \xrightarrow{\varphi_U \cong} U$$

$$\forall\ \underline{v} = t\underline{v}_Q + \underset{\substack{\downarrow \\ \text{comb.} \\ \text{lin. di } \underline{e}_0, \underline{e}_2, \underline{e}_3}}{\underline{u}} \longrightarrow \underset{\substack{\parallel \\ [\underline{v} + W_Q]}}{[\underline{u}]} \longrightarrow \underset{\substack{\downarrow \\ \text{comb. lin} \\ \text{di } \underline{e}_0, \underline{e}_2, \underline{e}_3}}{\underline{u}}$$

- π_{W_Q} applicazione lineare suriettiva
- φ_U isomorfismo

$\Rightarrow \varphi_U \circ \pi_{W_Q}$ suriettiva da V ad U

Ricordo

$$V \xrightarrow{\pi_{W_Q}} V/W_Q$$

$\ker(\pi_{W_Q}) = W_Q$

$$\begin{array}{ccccc} V & \xrightarrow{\pi_{W_Q}} & V/W_Q & \xrightarrow{\varphi_U \cong} & U \\ \cup & & & & \\ W_Q & \longrightarrow & [0] & \longrightarrow & \underline{0} \end{array}$$

Se proietti fico

$$\mathbb{P}^3 = \mathbb{P}(V) \xrightarrow{\mathbb{P}(\varphi_U \circ \pi_{W_Q}) = \pi_Q} \mathbb{P}(U) = \mathcal{H}_1$$

non è applicazione definita su tutto $\mathbb{P}(V)$

perché $\pi_Q(\mathbb{P}(W_Q)) = \pi_Q([\underline{v}_Q]) = \pi_Q(Q) \nexists$

$W_Q = \mathrm{Span}\{\underline{v}_Q\}$

infatti $[\underline{0}] \notin \mathbb{P}(U)$

$\Rightarrow$ • $\mathcal{H}_1 = \mathbb{P}(U) \cong \mathbb{P}(V/W_Q)$ è modello di $\mathbb{P}(V/W_Q)$

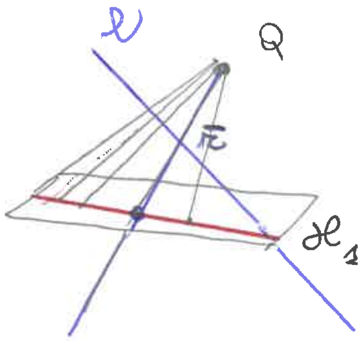
• π_Q è modello di $\mathbb{P}(\pi_{W_Q})$ con $\pi_{W_Q}: V \longrightarrow V/W_Q$.

(iii) Siccome $Q \in \bar{r} \Rightarrow$ $\pi_Q(\bar{r})$ si contrae ad un punto su $\mathcal{H}_1$

$$\bar{r}: \begin{cases} x_0 = \lambda \\ x_1 = \lambda + \mu \\ x_2 = 0 \\ x_3 = \lambda - 2\mu \end{cases} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \Rightarrow \pi_Q(\bar{r}) = [\lambda, 0, 0, 2(\lambda+\mu) + \lambda - 2\mu] = [\lambda, 0, 0, 3\lambda] = [1, 0, 0, 3] \in \mathcal{H}_1$$

Invece la retta

$$\ell: \begin{cases} x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$$



è t.c. $\ell \not\subset \mathcal{H}_1$ e $Q \notin \ell$

$\Rightarrow$ $\pi_Q(\ell)$ si proietta in una retta sul piano $\mathcal{H}_1$

$$\ell: \begin{cases} x_0 = s \\ x_1 = t \\ x_2 = t \\ x_3 = 0 \end{cases} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

con $[s, t] \in \mathbb{P}^1(\mathbb{R}) \Rightarrow$

$\pi_Q(\ell) = [s, 0, t, 2t]$, $[s,t] \in \mathbb{P}^1$ è una retta su $\mathcal{H}_1 : \{x_1 = 0\}$ (7)

$\Rightarrow$ eq. cartesiane omogenee $\boxed{\pi_Q(\ell): \begin{cases} x_1 = 0 & \to \text{piano } \mathcal{H}_1 \text{ di } \mathbb{P}^3(\mathbb{R}) \\ x_3 - 2x_0 = 0 & \to \text{piano } \alpha \text{ di } \mathbb{P}^3(\mathbb{R}) \end{cases}}$

(V) Come prima $V = \mathbb{R}^4$ così per dualità poniamo

$$(\mathbb{P}^3)^* := \mathbb{P}(V^*) \quad \text{spazio proiettivo duale di } \mathbb{P}^3$$

Visto che $\pi_Q(\ell) = \begin{cases} \alpha \longleftrightarrow \text{piano di } \mathbb{P}^3 \\ \mathcal{H}_1 \longleftrightarrow \text{piano di } \mathbb{P}^3 \end{cases}$

$$\Rightarrow \boxed{\pi_Q(\ell) = \mathbb{P}(\ker(e_1^*)) \cap \mathbb{P}(\ker(2e_0^* - e_3^*))}$$

i.e. $\boxed{\pi_Q(\ell) = \mathbb{P}\left(\mathrm{Ann}_V\{\underbrace{e_1^*}_{\mathcal{H}_1}, \underbrace{2e_0^* - e_3^*}_{\alpha}\}\right) \subset \mathbb{P}(V) = \mathbb{P}^3(\mathbb{R})}$

$\dim(\mathrm{Ann}_V(e_1^*, 2e_0^* - e_3^*)) = 4 - 2 = 2 \Rightarrow \dim(\pi_Q(\ell)) = 1$ ok

e $\pi_Q(\ell)$ visto come asse del fascio di iperpiani in $\mathbb{P}^3$ generato da $\mathcal{H}_1$ e α

Contemporaneamente $W_\ell = \begin{cases} x_1 = 0 \\ x_3 - 2x_0 = 0 \end{cases}$ è ssp. vett. di $V = \mathbb{R}^4$

con $W_\ell = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ e $\boxed{\underset{\pi_Q(\ell)}{\mathbb{P}(W_\ell)} \subset \mathbb{P}(V) = \mathbb{P}^3}$ sottosp. proiettivo di $\mathbb{P}(V)$

(VI) $\mathcal{H}_1 : \{x_1 = 0\}$ è $\mathbb{P}(\ker(e_1^*))$ con $\mathcal{E}^* = \{e_0^*, e_1^*, e_2^*, e_3^*\}$ base duale di $\mathcal{E} = \{e_0, e_1, e_2, e_3\}$

$$\Rightarrow P_{\mathcal{H}_1} = [0, 1, 0, 0] \in \mathbb{P}(V^*) = (\mathbb{P}^3)^* \text{ punto}$$

$$\alpha : \{2x_0 - x_3 = 0\} \text{ è } \mathbb{P}(\ker(2e_0^* - e_3^*))$$

$$\Rightarrow P_\alpha = [2, 0, 0, -1] \in \mathbb{P}(V^*) = (\mathbb{P}^3)^* \text{ punto}$$

Pertanto i 2 iperpiani di $\mathbb{P}^3$ $\mathcal{H}_1$ e α

corrispondono a due punti di $(\mathbb{P}^3)^*$

$P_{\mathcal{H}_1}$ e P_α

In $(\mathbb{P}^3)^*$ ho la retta proiettiva congiungente questi 2 punti

$\mathcal{L}(P_{H_1}, P_\alpha) \subset (\mathbb{P}^3)^*$ retta proiettiva

$[a_0, a_1, a_2, a_3]$ coord. punto in $(\mathbb{P}^3)^*$ $\longleftrightarrow$ $\{a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ iperpiano di $\mathbb{P}^3$

$(\mathbb{P}^3)^*$ $\shortparallel$

$\Rightarrow$ iperpiani di $\mathbb{P}^3$ formano spazio proiettivo $(\mathbb{P}^3)^*$

$$\Rightarrow \begin{cases} a_0 = 2\lambda + 0\mu \\ a_1 = 0\lambda + \mu \\ a_2 = 0 \\ a_3 = -\lambda \end{cases} \qquad \begin{cases} a_0 = 2\lambda \\ a_1 = \mu \\ a_2 = 0 \\ a_3 = -\lambda \end{cases}$$

eq. parametriche omogenee di retta $\mathcal{L}(P_{H_1}, P_\alpha) \subset (\mathbb{P}^3)^*$

$$\mathcal{L}(P_{H_1}, P_\alpha): \begin{cases} a_0 + 2a_3 = 0 \\ a_2 = 0 \end{cases}$$

eq. cartesiane omogenee di retta $\mathcal{L}(P_{H_1}, P_\alpha) \subset (\mathbb{P}^3)^*$

Come le interpreto nell'insieme $\{$Iperpiani di $\mathbb{P}^3\} = (\mathbb{P}^3)^*$?

$\lambda P_\alpha + \mu P_H$ in $(\mathbb{P}^3)^*$ $\qquad [\lambda, \mu] \in \mathbb{P}^1$

$\Updownarrow$

$\lambda\alpha + \mu H$ in $\mathbb{P}^3$ $\qquad [\lambda, \mu] \in \mathbb{P}^1$

$\lambda(2x_0 - x_3) + \mu x_1 = 0$

$$\Rightarrow \boxed{2\lambda x_0 + \mu x_1 - \lambda x_3 = 0}$$

equazione fascio di piani di asse $\pi_Q(\ell)$ in $\mathbb{P}^3$

$\dim(\mathcal{F}_{\lambda,\mu}) = 1 =$
$= \dim(\mathcal{L}(P_{H_1}, P_\alpha))$

Viceversa

$$\pi_Q(\ell) = \mathbb{P}\Big(\underbrace{\mathrm{Span}\left\{\begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}}_{W_\ell \subset V}\Big) \subset \mathbb{P}^3$$

$$\dim\left(\mathrm{Ann}_{V^*}(W_\ell)\right) = 2 \quad e \quad \begin{cases} \mathrm{Ann}_{V^*}(W_\ell) = \mathrm{Span}\{H_1, \alpha\} \\ W_\ell = \mathrm{Ann}_V(\mathrm{Span}\{H_1, \alpha\}). \end{cases}$$

Svolgimento es 2

(i) $\mathbb{P}^2(\mathbb{R})$ con

$P_0 = [2,-3,1]$ $\quad P_1 = [0,1,1]$, $\quad P_2 = [0,1,-1]$

$M = [2,3,1]$

$$\det\begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \underset{\substack{\text{Laplace} \\ \text{I riga}}}{=} 2 \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -4 \neq 0$$

$\Rightarrow$ P_0, P_1, P_2 punti indipendenti in $\mathbb{P}^2(\mathbb{R})$

Sia $\underline{m} \in \mathbb{R}^3$ t.c. $M = [\underline{m}]$ $\Rightarrow$ Se $P_0 = [\underline{v}_0]$, $P_1 = [\underline{v}_1]$, $P_2 = [\underline{v}_2]$

$$\underline{m} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \lambda_0 \underline{v}_0 + \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 = \lambda_0 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2\lambda_0 \\ -3\lambda_0 + \lambda_1 + \lambda_2 \\ \lambda_0 + \lambda_1 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Leftrightarrow \begin{cases} \lambda_0 = 1 \\ -3\lambda_0 + \lambda_1 + \lambda_2 = 3 \\ \lambda_0 + \lambda_1 - \lambda_2 = 1 \end{cases}$$

$$\begin{cases} \lambda_1 + \lambda_2 = 6 \\ \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

$$\Rightarrow \underline{m} = 1\,\underline{v}_0 + 1\,(3\underline{v}_1) + 1\,(3\underline{v}_2)$$

M punto unità per riferimento P_0, P_1, P_2, M

$\Rightarrow$ se prendo

$$B = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$[B]$ proiettività che manda

$P_0 \longrightarrow F_0$
$P_1 \longrightarrow F_1$
$P_2 \longrightarrow F_2$
$U \longrightarrow M$

$$\Rightarrow A = B^{-1} \sim \mathcal{C}_B^{\,t} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \alpha \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

è proiettività che dà cambiamento rif. proiettivo.

②

$$f(P_0) = \alpha \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6\alpha \\ 0 \\ 0 \end{pmatrix} \Rightarrow f(P_0) = F_0 = [1,0,0]$$

$$f(P_1) = \alpha \left(\quad \right) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\alpha \\ 0 \end{pmatrix} \Rightarrow f(P_1) = F_1 = [0,1,0]$$

$$f(P_2) = \alpha \left(\quad \right) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\alpha \end{pmatrix} \Rightarrow f(P_2) = F_2 = [0,0,1]$$

$$f(M) = \alpha \left(\quad \right) \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6\alpha \\ 6\alpha \\ 6\alpha \end{pmatrix} \Rightarrow f(M) = U = [1,1,1]$$

$$f = [A] \quad \text{cioè} \quad f \in PGL(3,\mathbb{R})$$

(ii) Se $P_1 = [0,1,1]$ e $Q = [1,1,2]$

$$r = \mathcal{L}(P_1, Q); \quad \det \begin{pmatrix} x_0 & x_1 & x_2 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 0$$

$$r := \quad x_0 + x_1 + x_2 = 0$$

Prendo f di prima, siccome

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \alpha \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} y_0 = 3\alpha x_0 \\ y_1 = \alpha x_0 + \alpha x_1 + \alpha x_2 \\ y_2 = 2\alpha x_0 + \alpha x_1 - \alpha x_2 \end{cases}$$

$$\Rightarrow \boxed{f(\{x_0 = 0\}) = \{y_0 = 0\}}$$

$$\Downarrow$$

$$f(A_{0,\underline{x}}) = f(A_{0,\underline{y}})$$

Ora $r = \mathcal{L}(P_1, Q)$

$$f(P_1) = [0,1,0] = F_1$$

$$f(Q) = \alpha A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = [3,4,1]$$

Nel riferimento $[y_0, y_1, y_2]$ r si scrive

$$\det \begin{pmatrix} y_0 & y_1 & y_2 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} = y_0 - 3y_2 = 0$$

(iii) In $A_{0,\underline{x}}$ $\quad x = \frac{x_1}{x_0} \quad y = \frac{x_2}{x_0}$

$r_{0,x} := r \cap A_{0,\underline{x}}$: $\boxed{x - y + 1 = 0}$

se considero $A_{0,\underline{y}}$ $\quad x' = \frac{y_1}{y_0} \quad y' = \frac{y_2}{y_0}$

$r_{0,y} := r \cap A_{0,\underline{y}}$: $\quad 1 - 3y' = 0 \Rightarrow \boxed{y' = \frac{1}{3}}$

Ma f era $\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3\alpha x_0 \\ \alpha x_0 + \alpha x_1 + \alpha x_2 \\ 2\alpha x_0 + \alpha x_1 - \alpha x_2 \end{pmatrix}$ e $f(A_{0,\underline{y}}) = A_{0,\underline{x}}$

$$\Rightarrow \quad x' = \frac{y_0}{y_1} = \frac{\alpha x_0 + \alpha x_1 + \alpha x_2}{3\alpha x_0} = \frac{1}{3} + \frac{1}{3}x + \frac{1}{3}y$$

$$y' = \frac{y_2}{y_0} = \frac{2\alpha x_0 + \alpha x_1 - \alpha x_2}{3\alpha x_0} = \frac{2}{3} + \frac{1}{3}x - \frac{1}{3}y$$

Ho affinità di $\mathbb{A}^2_{(x,y)} \quad \mathbb{A}^2_{(x',y')}$

$$\varphi: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \boxed{C\,\underline{x} + \underline{d} = \varphi(\underline{x})}$$

I punti $P_1 = [0,1,1]$ e $F_1 = [0,1,0]$ non sono nelle carte $A_{0,\underline{x}}$ e $A_{0,\underline{y}}$ risp.

Però $\quad Q = [1,1,2] = (1,2) \in A_{0,x}$

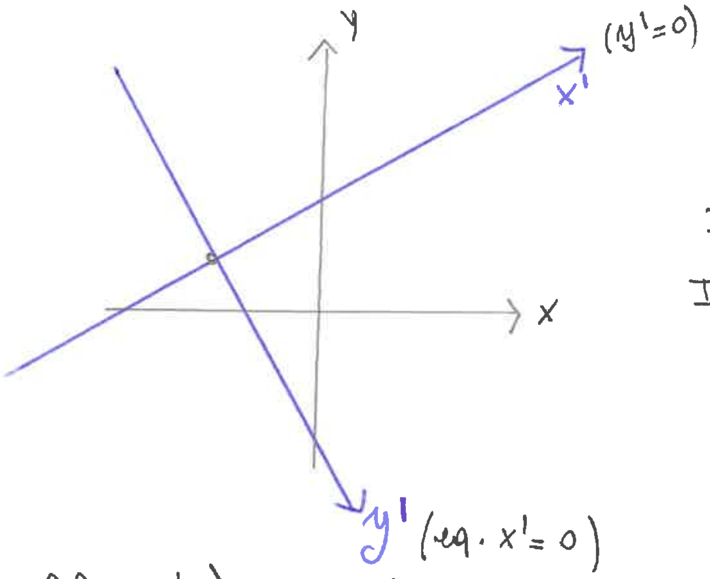
$f(Q) = [3,4,1] = \left(\frac{4}{3}, \frac{1}{3}\right) \in A_{0,\underline{y}}$

Noto che: $Q \in r_{0,\underline{x}} = r \cap A_{0,x}$ $\quad \Rightarrow r_{0,\underline{x}}: \begin{pmatrix} x \\ y \end{pmatrix} = Q + t\underline{v}$

$r_{0,\underline{x}}$ aveva vettore direttore $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longleftrightarrow P_1 = [0,1,1]$

$\begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix} = f(Q) \in r_{0,\underline{y}} = r \cap A_{0,\underline{y}}$ perché $y' = \frac{1}{3}$

e vettore direttore $\underline{e}_1 = (1,0) \longleftrightarrow [0,1,0] = F_1$



$$\begin{cases} x' = \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3} = 0 \\ y' = \frac{1}{3}x - \frac{1}{3}y + \frac{2}{3} = 0 \end{cases}$$

$\mathrm{I}+\mathrm{II}$: $x = -\frac{3}{2}$

$\mathrm{I}-\mathrm{II}$: $y = \frac{1}{2}$

$\Rightarrow O' = \left(-\frac{3}{2}, \frac{1}{2}\right)$

Noto che affinità $\varphi(\underline{x})$ è t.c.

$$\varphi\begin{pmatrix}1\\2\end{pmatrix} = \varphi(Q) = \begin{pmatrix}\frac{4}{3}\\ \frac{1}{3}\end{pmatrix} = f(Q)$$

Mentre $\varphi_0(\underline{x}) = C\,\underline{x}$ — è affinità lineare

$$\varphi_0(\underline{v}) = C\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/3 & 1/3\\ 1/3 & -1/3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2/3\\0\end{pmatrix} \sim \underline{e}_1$$

$$\Rightarrow \varphi_0(\underline{v}) = \frac{2}{3}\underline{e}_1$$

Perciò

r_0 in (x,y) è retta per $Q = \begin{pmatrix}1\\2\end{pmatrix}$ e $\underline{v} = \begin{pmatrix}1\\1\end{pmatrix}$

in (x',y') " " " $f(Q) = \varphi(Q)$ e $\varphi(\underline{v}) = f(P_1) = \begin{pmatrix}1\\0\end{pmatrix}$

In effetti

r_0 in (x,y) $\Rightarrow \begin{cases} x = 1+t \\ y = 2+t \end{cases}$ $t \in \mathbb{R}$

r_0 in (x',y')

$$x' = \frac{1}{3} + \frac{1}{3}(1+t) + \frac{1}{3}(2+t) = \frac{4}{3} + \frac{2}{3}t$$

$$y' = \frac{2}{3} + \frac{1}{3}(1+t) - \frac{1}{3}(2+t) = \frac{1}{3}$$

$$\Rightarrow \begin{cases} x' = \frac{4}{3} + \frac{2}{3}t \\ y' = \frac{1}{3} \end{cases}, \ t \in \mathbb{R} \quad \text{cioè} \quad \boxed{y' = \frac{1}{3}}$$