Scoppiamento

Utilizzo dei prodotti per costruire un'importante mappa birazionale (ma non iso) utile con desingolarizzazioni

- Consideriamo per semplicità

$$\mathbb{P}^2 \quad [x_0, x_1, x_2] \rightsquigarrow \mathbb{P}^n \quad [x_0, \dots, x_n]$$

$$\mathbb{P}^1 \quad [y_1, y_2] \rightsquigarrow \mathbb{P}^{n-1} \quad [y_1, y_2]$$

$$\Downarrow$$

$$\mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow[\cong]{\sigma_{2,1}} S_{2,1} \subset \mathbb{P}(\mathbb{R}^3 \otimes \mathbb{R}^2) = \mathbb{P}^5$$

$$([\underline{x}], [\underline{y}]) \longrightarrow [x_i y_j] = w_{ij}$$

$S_{2,1}$ = varietà di Segre di indici $(2,1)$

$$\dim(S_{2,1}) = \dim(\mathbb{P}^2 \times \mathbb{P}^1) = \dim(\mathbb{A}^2 \times \mathbb{A}^1) = \dim(\mathbb{A}^3) = 3$$

$S_{2,1}$ è Threefold di Segre indici $(2,1)$

$$S_{2,1} = Z_p(w_{ij} w_{k\ell} - w_{i\ell} w_{kj}) \quad 0 \le i \ne k \le 2, \quad 1 \le j \ne \ell \le 2$$

$$S_{2,1} = Z_p\left(\begin{array}{l} w_{01} w_{12} - w_{02} w_{11} = 0, \\ w_{01} w_{22} - w_{02} w_{21} = 0, \\ w_{11} w_{22} - w_{12} w_{21} = 0 \end{array}\right) \subset \mathbb{P}^5$$

Quando in $\mathbb{P}^5$ prendo iperpiano

$$\overline{H} := Z_p(w_{12} - w_{21}) \subset \mathbb{P}^5$$

$\overline{H} = H_{12} - H_{21} \subset \mathbb{P}^5$

$Z_p(w_{ij} - w_{ji}) \quad 1 \le i \ne j \le n$, $i = 0$ mai coinvolto

posso considerare

$$\underbrace{S_{2,1} \cap \overline{H}}_{\text{chiusura sez iperpiano di } S_{2,1}} = Z_p\left(\begin{array}{l} w_{ij} w_{k\ell} - w_{i\ell} w_{kj} \\ w_{12} - w_{21} \end{array}\right)$$

chiusura sez iperpiano di $S_{2,1}$

$S_{2,1} \cong \mathbb{P}^2 \times \mathbb{P}^1$ e su $S_{2,1}$ $w_{ij} = x_i \cdot y_j$

$\Rightarrow$ $S_{2,1} \cap \overline{H} \cong \{ x_1 y_2 - x_2 y_1 = 0 \}$, $i = 0$ non coinvolto

$\widetilde{\mathbb{P}^2} := S_{2,1} \cap \widetilde{H}$ visto su $\mathbb{P}^2 \times \mathbb{P}^1 = Z_p(x_1 y_2 - x_2 y_1)$

$\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^1$ chiuso in $\mathbb{P}^2 \times \mathbb{P}^1$ perché $S_{2,1} \cap \overline{H} \subset S_{2,1}$ chiuso
$S_{2,1} \cong \mathbb{P}^2 \times \mathbb{P}^1$

- $S_{2,1} \subset \mathbb{P}^5$ non degenere

  $S_{2,1} \cap \overline{H} \subset \overline{H} \cong \mathbb{P}^4$ non degenere in $\overline{H} \cong \mathbb{P}^4$

- $P_0 = [1,0,0] \in \mathbb{P}^2$ punto fond. di $\mathbb{P}^2$ ($P_0 = [1,0,\dots,0] \in \mathbb{P}^n$)

$$\begin{tikzcd} & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[dl, "\pi_{\mathbb{P}^2}"'] \arrow[dr, "\pi_{\mathbb{P}^1}"] & \\ \mathbb{P}^2 & & \mathbb{P}^1 \end{tikzcd}$$

$$\begin{tikzcd} & ([x_0,x_1,x_2] \times [y_1,y_2]) \arrow[dl] \arrow[dr] & \\ {[x_0,x_1,x_2]} & & {[y_1,y_2]} \end{tikzcd}$$

morfismi

$$\begin{tikzcd} \widetilde{\mathbb{P}^2} \arrow[r, hook, "i_{\widetilde{\mathbb{P}^2}}", "\text{imm. chiusa}"'] & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[d, "\pi_{\mathbb{P}^2}"'] \arrow[dr, "\pi_{\mathbb{P}^1}"] & \\ & \mathbb{P}^2 & \mathbb{P}^1 \end{tikzcd}$$

$\boxed{b := \pi_{\mathbb{P}^2} \circ i_{\widetilde{\mathbb{P}^2}}}$ morfismo

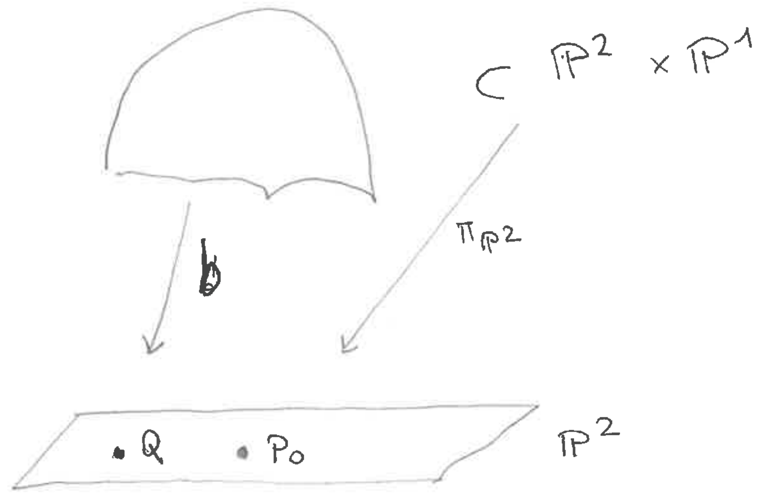
$b : \widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$ scoppiamento di $\mathbb{P}^2$ in P_0

In alcuni testi $bl_{P_0} : Bl_{P_0}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ blow-up

$b = \pi_{\mathbb{P}^2}|_{\tilde{\mathbb{P}}^2}$ è morfismo

* Notiamo che $\tilde{\mathbb{P}}^2$ definita da ! equazione bi-omogenea di bigradi $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$ $\Rightarrow$ ipersup. in $\mathbb{P}^2 \times \mathbb{P}^1$

$\tilde{\mathbb{P}}^2$ è superficie proiettiva (superficie chiusa in $S_{2,1}$



① Sia $Q = [q_0, q_1, q_2] \neq P_0$ $\Rightarrow$ $(q_1, q_2) \neq (0,0)$

(Per semplicità supponiamo $q_1 \neq 0$)

$$\Rightarrow \quad Q = [q_0, q_1, q_2] = \left[\frac{q_0}{q_1}, 1, \frac{q_2}{q_1}\right] = \left(\frac{q_0}{q_1}, \frac{q_2}{q_1}\right) \in \underset{\mathbb{A}^2}{\underset{\parallel\wr}{U_1^2}} \subset \mathbb{P}^2$$

su $U_1^2 \ni (u, v) = \left(\frac{q_0}{q_1}, \frac{q_2}{q_1}\right)$ coord. affini

- considero

$$U_1^2 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow S_{2,1} \subset \mathbb{P}^5$$

$$([x_0, \underset{\neq 0}{x_1}, x_2], [y_1 y_2]) \xrightarrow{\sigma_{2,1}} [x_i y_j]$$

$$= [\underset{w_{01}}{x_0 y_1}, \underset{w_{02}}{x_0 y_2}, \underset{w_{11}}{x_1 y_1}, \underset{w_{12}}{x_1 y_2}, \underset{w_{21}}{x_2 y_1}, \underset{w_{22}}{x_2 y_2}]$$

- Poiché $\mathbb{P}^2 \times \mathbb{P}^1 \cong S_{2,1}$

in $S_{2,1}$ vado in

$[y_1, y_2] \neq [0,0]$

$$\underbrace{(S_{2,1} \cap U^5_{1,1}) \cup (S_{2,1} \cap U^5_{1,2})}_{\text{aperto di } S_{2,1}}$$

$U^5_{i,j} \cong \mathbb{A}^5$ aperti di $\mathbb{P}^5$

$$\Rightarrow \quad \underbrace{U_{1,2} \times \mathbb{P}^1}_{\text{aperto } \mathbb{P}^2 \times \mathbb{P}^1} \cong \sigma_{2,1}^{-1}\Big(\underbrace{S_{2,1} \cap (U^5_{1,1} \cup U^5_{1,2})}_{\text{aperto}}\Big)$$

$\sigma_{2,1}$ isomorfismo

$\Rightarrow$ in $U_1^2 \times \mathbb{P}^1$ aperto di $\mathbb{P}^2 \times \mathbb{P}^1 \Rightarrow$

$$\widetilde{\mathbb{P}^2}\Big|_{U_1^2 \times \mathbb{P}^1} \qquad (u, v) \times [y_1, y_2]$$

ha equazione

$$\boxed{\{y_2 - v\, y_1 = 0\} \subset U_1^2 \times \mathbb{P}^1 \cong \mathbb{A}^2 \times \mathbb{P}^1}$$

* Siccome per $Q = (u_Q, v_Q) = \left(\frac{q_0}{q_1}, \frac{q_2}{q_1}\right) \Rightarrow \boxed{v_Q = \frac{q_2}{q_1}}$

otteniamo che

$$b^{-1}(Q) = \{(u_Q, v_Q) \times [y_1, y_2] \mid y_2 = v_Q\, y_1\}$$

Se $q_2 = 0$ $\qquad y_2 = 0 \Rightarrow y_1 = 1$

$b^{-1}(Q) = \{[q_0, q_1, q_2] \times [1, 0]\} = \{Q \times [1, 0]\}$! punto in $\mathbb{P}^2 \times \mathbb{P}^1$

$\Rightarrow$

Se $q_2 \neq 0$ $\qquad y_2 = \frac{q_2}{q_1} y_1 \Leftrightarrow q_1 y_2 - q_2 y_1 = 0$

$\Leftrightarrow \det\begin{pmatrix} y_1 & y_2 \\ q_1 & q_2 \end{pmatrix} = 0 \Leftrightarrow [y_1, y_2] = [q_1, q_2]$

$b^{-1}(Q) = \{[q_0, q_1, q_2] \times [\underset{\neq 0}{q_1}, \underset{\neq 0}{q_2}]\}$

$= \{Q \times [q_1, q_2]\}$! punto in $\mathbb{P}^2 \times \mathbb{P}^1$

$\Rightarrow$ b è $1-1$ su questi punti

Discorso simile se assumevo $q_2 \neq 0$ e in $U_2^2 \subset \mathbb{P}^2$

(2) Se considero $P_0 = [1, 0, 0]$

$P_0 \notin U_1^2, U_2^2 \subset \mathbb{P}^2$

Però $P_0 \in U_0^2 \cong \mathbb{A}^2$ e $P_0 = (0, 0) \in \mathbb{A}^2 \cong U_0^2$

Se in U_0^2 prendo

$$x = \frac{X_1}{X_0} \qquad y = \frac{X_2}{X_0}$$

$\widetilde{\mathbb{P}^2}\Big|_{U_0^2 \times \mathbb{P}^1}$ ha equazione $\qquad (x, y) \times [y_1, y_2]$

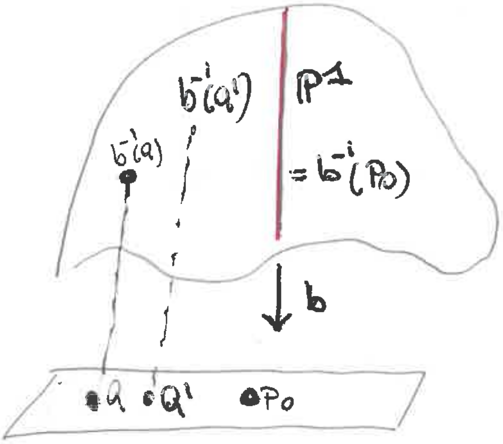
$$\boxed{\{x\, y_2 - y\, y_1 = 0\} \subset U_0^2 \times \mathbb{P}^1 \cong \mathbb{A}^2 \times \mathbb{P}^1}$$

Siccome $P_0 = O = (0,0) \in U_0^2$

$$b^{-1}(P_0) = \{(0,0)\times[y_1,y_2] \mid 0y_2 - 0y_1 = 0\} \cong \mathbb{P}^1$$

$$= \{P_0\}\times\mathbb{P}^1$$

ho tutta una retta proiettiva



$b(\mathbb{P}^1) = P_0$

$\Rightarrow$ tutto $\mathbb{P}^1$ si contrae a P_0

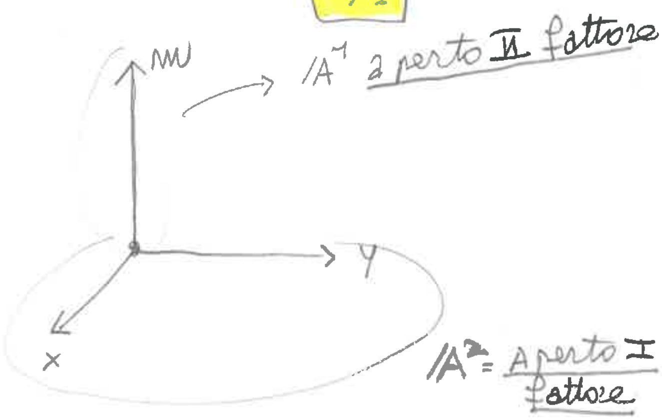
Viceversa b fa scoppiare P_0

Cosa è questo $\mathbb{P}^1$?

In U_0^2 $\quad x = \frac{x_1}{x_0}$ e $y = \frac{x_2}{x_0}$ coord. affini in $\mathbb{A}^2$

In $U_1^1 = \{[y_1,y_2]\in\mathbb{P}^1 \mid y_1\neq 0\}$ prendo $\boxed{m := \frac{y_2}{y_1}}$ coord. affine in $\mathbb{A}^1$

$$U_0^2 \times U_1^1 \subset \mathbb{P}^2\times\mathbb{P}^1$$
$$\cong$$
$$\mathbb{A}^2\times\mathbb{A}^1$$
$$\cong$$
$$\mathbb{A}^3 \quad (x,y,m)$$



Se considero

$$\widetilde{\mathbb{P}}^2\big|_{U_0^2\times U_1^1} = Z_a(xm - y) \subset \mathbb{A}^3 \cong \mathbb{A}^2\times\mathbb{A}^1$$

$(x,y,m) = (x,y)\times(m)$

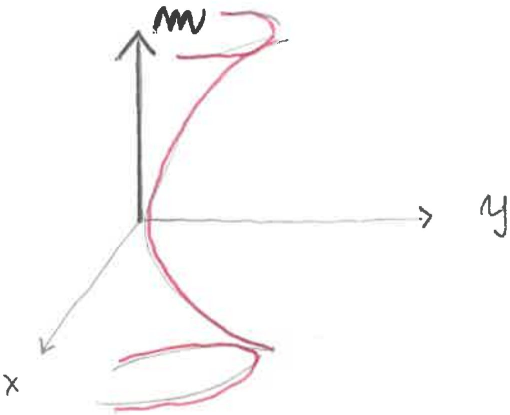
$\downarrow$ Superficie in $\mathbb{A}^3$

$\downarrow$ $Z_p(x_1y_2 - x_2y_1)$

Quindi

⑥

$$\widetilde{\mathbb{P}}^2\big|_{U_0^2\times U_1^1} = \{(x,y,m)\in \mathbb{A}^3 \mid y = mx\} \subset \mathbb{A}^3$$



paraboloide a sella in $\mathbb{A}^3$

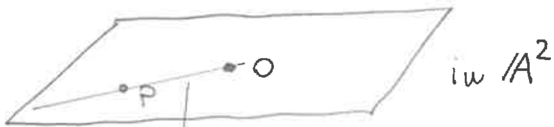
$$Z_a(y-mx) = \widetilde{\mathbb{P}}^2\big|_{U_0^2\times U_1^1}$$

$$\downarrow b|_{U_0^2\times U_1^1} \qquad\qquad \downarrow b$$

$$U_0^2 = \mathbb{A}^2_{x,y}$$

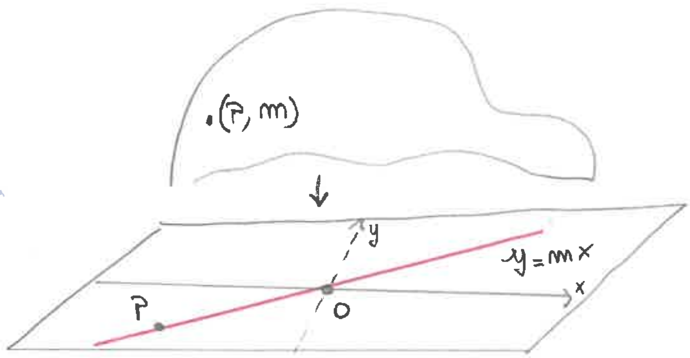
- $\boxed{O = (x,y) = (0,0)}$ $\quad b^{-1}(O) = \{(0,0,m) \mid m \in \mathbb{A}^1\} \simeq \mathbb{A}^1_m$ → tutto asse m
- $\boxed{P = (P_1,P_2) \neq (0,0)}$ $\quad b^{-1}(P) = \{(P_1,P_2,m) \mid P_2 = m\cdot P_1\}$

se $P_1 \neq 0$ $\quad \Rightarrow \quad m = \dfrac{P_2}{P_1}$ = coeff angolare retta $\langle O, P\rangle$ → in $\mathbb{A}^2$



$y = mx$

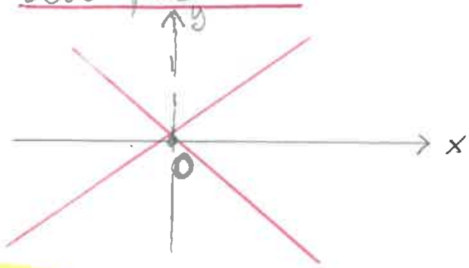
al variare dei $(P_1,P_2) \in \mathbb{A}^2$ t.c. $P_2 \neq 0$



* Per i punti $P \neq 0$ ma $p_1 = 0 \Rightarrow$ sto su asse $y \Rightarrow$ basta cambiare carta $\frac{y_1}{y_2} = m'$ coeff angolare ha senso (7)

• Pertanto

$$b^{-1}(0) = b^{-1}((0,0)) = \text{retta } \mathbb{A}^1_m \text{ dei coeff. angolari di rette del fascio affine rette per } O$$



si completa a fascio proiettivo

$$[y_1, y_2] = \{[1, m]\} \cup \{[0,1]\}$$

asse y

* Ricordiamo che paraboloide a sella era legato pure a $S_{1,1}$: $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma_{1,1}} S_{1,1} \subset \mathbb{P}^3 \Rightarrow$

$$\begin{array}{ccc} \widetilde{\mathbb{P}}^2 = S_{2,1} \cap \overline{H} & \xrightarrow{\cong} & S_{1,1} \\ \cap & & \cap \\ \mathbb{P}^4 = \overline{H} & & \mathbb{P}^3 \\ \cap & & \\ \mathbb{P}^5 & & \end{array}$$

Prop

$\widetilde{\mathbb{P}}^2 \setminus b^{-1}(P_0)$ è varietà quasi proiettiva isomorfa mediante b a $\mathbb{P}^2 \setminus \{P_0\}$

dim

* b è morfismo perché restrizione morfismo $\pi_{\mathbb{P}^2}$
* $b|_{\mathbb{P}^2 \setminus b^{-1}(P_0)}$ è 1-1 tra $\mathbb{P}^2 \setminus b^{-1}(P_0)$ e $\mathbb{P}^2 \setminus \{P_0\}$
* Verifichiamo che inversa insiemistica è morfismo

$$\tau = \left(b|_{\mathbb{P}^2 \setminus b^{-1}(P_0)}\right)^{-1} \quad \mathbb{P}^2 \setminus \{P_0\} \xrightarrow{\tau} \mathbb{P}^2 \times \mathbb{P}^1 \cong S_{2,1} \text{ var. proiettiva}$$

$$Q = [q_0, q_1, q_2] \longmapsto ([q_0, q_1, q_2], [q_1, q_2])$$

$[q_0,q_1,q_2] \neq [1,0,0]$ — ben definita

Ma allora se compongo con isomorfismo $\sigma_{2,1}$

$$\mathbb{P}^2 \setminus \{P_0\} \xrightarrow{\tau} \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\sigma_{2,1}} S_{2,1} \subset \mathbb{P}^5$$

$$[x_0, \underbrace{x_1, x_2}_{[x_1,x_2] \in \mathbb{P}^1}] \longmapsto ([x_0,x_1,x_2],[x_1,x_2]) \longmapsto [\underset{W_{01}}{x_0x_1}, \underset{W_{02}}{x_0x_2}, \underset{W_{11}}{x_1^2}, \underset{W_{12}}{x_1x_2}, \underset{W_{21}}{x_2x_1}, \underset{W_{22}}{x_2^2}]$$

$\gamma := \sigma_{2,1} \circ \tau$ è morfismo

perché polinomi omogenei non cont. nulli

* Poiché $\sigma_{2,1}$ isomorfismo $\Rightarrow$ $\boxed{\tau = (\sigma_{2,1})^{-1} \circ \gamma}$

$\Rightarrow$ τ morfismo

* siccome inversa insiemistica di $b|_{\mathbb{P}^2 \setminus b^{-1}(P_0)}$

$\Rightarrow$ $\boxed{b|_{\mathbb{P}^2 \setminus b^{-1}(P_0)} \text{ isomorfismo}}$

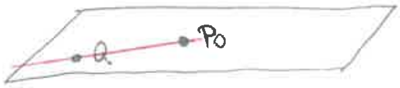
$\Rightarrow$ $\widetilde{\mathbb{P}}^2 \setminus b^{-1}(P_0)$ è varietà quasi-proiettiva (aperto di $\widetilde{\mathbb{P}}^2$)

∎

$b^{-1}(P_0) \cong \mathbb{P}^1 =$ divisore eccezionale $= E_{P_0}$

(Curve $= \mathcal{O}_C(U)$ anelli valutaz. discrete
punti $= \{$divisori su $C\}$ perchè $\mathcal{O}_C(U)_{\psi_P} \cong \mathbb{k}[t]_{\substack{t=0}} / I_a(C_0)$ e $\mathfrak{m}_{C,P} \cong t\,\mathcal{O}_{C,P}$
t era divisore)

$*$ Se in $\mathbb{P}^2 \setminus \{P_0\} \subset \mathbb{P}^2$
$Q = [0, q_1, q_2] \in \mathbb{P}^2 \setminus \{P_0\}$ $\Rightarrow$ $r := \mathcal{L}(P_0, Q)$ è UNICA



Prendo $U_0^2 \subset \mathbb{P}^2$ $\rightsquigarrow$ $P_0 = [1,0,0] = 0 = (0,0)$
$Q = [0, q_1, q_2] =$ improprio
corrisponde a vettore direttore $v_Q = (q_1, q_2)$

$r_0 = r \cap U_0^2 \in \mathbb{A}^2$ con $x = \frac{x_1}{x_0}$ $y = \frac{x_2}{x_0}$

$r_0 : \begin{cases} x = 0 + q_1 t \\ y = 0 + q_2 t \end{cases}$ retta affine per P_0

$\overline{r_0} = r : \begin{cases} x_0 = \lambda \\ x_1 = q_1 \mu \\ x_2 = q_2 \mu \end{cases}$ $t = \frac{\mu}{\lambda}$, $\lambda \neq 0$ $[\lambda, \mu] \in \mathbb{P}^1$
retta proiettiva

$\mathbb{P}^2 \setminus \{P_0\} \xrightarrow[\cong]{\tau} \widetilde{\mathbb{P}}^2 \setminus b^{-1}(P_0) = \widetilde{\mathbb{P}}^2 \setminus E_{P_0}$

ISOMORFISMO da prima

$\mathbb{P}^2 \setminus \{P_0\} \xleftrightarrow{\tau} \widetilde{\mathbb{P}}^2 \setminus E_{P_0}$

$r \setminus \{P_0\} \subset \mathbb{P}^2 \setminus \{P_0\}$

$r \setminus \{P_0\} \xrightarrow{\tau|_{r \setminus \{P_0\}}}$

$\downarrow$

deve essere isomorfismo

$$\tau([\lambda, \mu q_1, \mu q_2]) \xrightarrow{\tau|_{\pi\setminus\{P_0\}}} ([\lambda, \mu q_1, \mu q_2], [\mu q_1, \mu q_2])$$

$\mu \neq 0$ altrimenti avrei $P_0 \in \pi$

$\widetilde{\mathbb{P}}^2_{E_{P_0}} \subset \mathbb{P}^2 \times \mathbb{P}^1$

$$M_\delta([\lambda, \mu q_1, \mu q_2], [\mu q_1, \mu q_2]) = ([\lambda, \mu q_1, \mu q_2], [q_1, q_2])$$

$\mu \neq 0$

$$\Rightarrow \boxed{\tau|_{\pi\setminus\{P_0\}}([\lambda, \mu q_1, \mu q_2]) \underset{\mu\neq 0}{=} ([\lambda, \mu q_1, \mu q_2], [q_1, q_2])} \quad (*)$$

sparito μ

$*$ $\pi \setminus \{P_0\} \cong \mathbb{P}^1 \setminus \{P\} \cong \mathbb{A}^1 \Rightarrow$

$$\mathbb{P}^1\setminus\{P\} \xrightarrow[\cong]{\psi_*} \pi\setminus\{P_0\} \xrightarrow[\cong]{\tau|_{\pi\setminus\{P_0\}}} \mathrm{Im}(\tau|_{\pi\setminus\{P_0\}}) \subset \underbrace{\mathbb{P}^2 \times \mathbb{P}^1}_{\text{proiettiva}} \underset{\sigma_{2,1}}{\cong} S_{2,1} \subset \mathbb{P}^5$$

$\Downarrow$

$$\mathbb{P}^1 \dashrightarrow^{\psi := \tau_{\pi\setminus\{P_0\}} \circ \psi_*} \mathrm{Im}(\tau_{\pi\setminus\{P_0\}}) \subset S_{2,1} \subset \mathbb{P}^5$$

$\psi := (\tau_{\pi\setminus\{P_0\}} \circ \psi_*)$ è applicazione razionale su $\mathbb{P}^1$ birazionale sull'immagine

Siccome ho $\mathbb{P}^1$ (dominio) $\Rightarrow *$ si estende ad un morfismo su $\mathbb{P}^1$

$$\mathbb{P}^1 \xrightarrow{\psi} \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow[\sigma_{2,1}]{\cong} S_{2,1} \hookrightarrow \mathbb{P}^5$$

$*$ $\mathbb{P}^1$ proiettiva, ψ morfismo

$\Rightarrow \psi(\mathbb{P}^1) \subset \mathbb{P}^2 \times \mathbb{P}^1 \cong S_{2,1} \subset \mathbb{P}^5$

chiuso proiettivo (completezza var proiettive)

$*$ $\psi(\mathbb{P}^1)$ irriducibile

perché $\mathbb{P}^1$ irrid e ψ morfismo $\Rightarrow \psi$ continua $\Rightarrow$ immagine di irriducibile è irriducibile

Come definita questa estensione?

$$\begin{tikzcd}
{[\lambda,\mu]}\ \mathbb{P}^1 \arrow[rr, "\psi"] \arrow[d, "p"'] \arrow[d, bend right, "p"'] \arrow[d, "\cong"] & & \psi(\mathbb{P}^1) \subset \mathbb{P}^2\times\mathbb{P}^1 \arrow[r, "\cong", "\sigma_{2,1}"'] & S_{2,1}\subset\mathbb{P}^5 \\
r = \{X = \lambda P_0 + \mu Q\} \arrow[urr, "\tau|_r"'] & & &
\end{tikzcd}$$

$r = \{X = \lambda P_0 + \mu Q\}$ retta proiettiva in $\mathbb{P}^2$

parametrizzazione omogenea di r

$$\mathbb{P}^1 \xrightarrow{p} r \xrightarrow{\tau|_r} \psi(\mathbb{P}^1) \subset \mathbb{P}^2\times\mathbb{P}^1$$

(*)

$$[\lambda,\mu] \neq [1,0] \xrightarrow{p} ([\lambda, \mu q_1, \mu q_2]) \xrightarrow{\tau|_r} ([\lambda, \mu q_1, \mu q_2], [q_1, q_2])$$

$$[\lambda,\mu] = [1,0] \xrightarrow{p} ([1, \underset{P_0}{\overset{\shortparallel}{0}}, 0]) \longrightarrow ([1,0,0], \underbrace{[q_1, q_2]}_{\text{sempre stesso punto}})$$

$r = \mathcal{L}(P_0, Q)$ e τ si ricorda di $Q = [0, q_1, q_2]$ per tutti i punti di r

$$\Rightarrow \quad \psi : \mathbb{P}^1 \longrightarrow \psi(\mathbb{P}^1) \xrightarrow[\cong]{\sigma_{2,1}} S_{2,1} \subset \mathbb{P}^5$$

$$[\lambda,\mu] \longrightarrow (\underbrace{[\lambda, \mu q_1, \mu q_2]}_{X}, \underbrace{[q_1, q_2]}_{Y}) \longrightarrow [x_i y_j]$$

a meno di $\sigma_{2,1} \Rightarrow \psi$ è isomorfismo da $\mathbb{P}^1$ a $\psi(\mathbb{P}^1)$

infatti $\boxed{\psi' := \sigma_{2,1} \circ \psi}$

$$[\lambda,\mu] \xrightarrow{\psi'} [\lambda q_1, \lambda q_2, \mu q_1^2, \mu q_1 q_2, \mu q_1 q_2, \mu q_2^2]$$

ψ' morfismo perché polinomi omog. lineari in $[\lambda,\mu] \in \mathbb{P}^1$

ψ' è isomorfismo $\Rightarrow$

$$\boxed{\psi = (\sigma_{2,1})^{-1} \circ \psi' \quad \text{isomorfismo}}$$

⇒ siccome

$\psi = \tau|_{r} \circ p$ e p isomorfismo da $\mathbb{P}^1$ a r

⇒ $\tau|_{r} = \psi \circ p^{-1}$ è isomorfismo perché ψ e p^{-1} lo sono

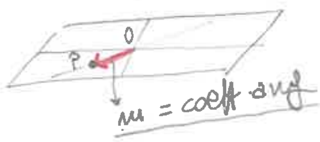
$$\tau|_{r} : r \xrightarrow{\cong} \tau(r) \subset \widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

Posta $\boxed{\tilde{r} := \tau(r)}$

- $\tilde{r} \cong r \cong \mathbb{P}^1$
- $\tilde{r} \cap E_{P_0} = \tilde{r} \cap b^{-1}(P_0) = \tau|_{r}(P_0) = ([1,0,0], [q_1, q_2])$

coeff. ang. (P, m)

parametri direttori di r_0



Proposizione

$\widetilde{\mathbb{P}^2}$ è irriducibile in $\mathbb{P}^2 \times \mathbb{P}^1$

dim

$$\widetilde{\mathbb{P}^2} = \underbrace{(\widetilde{\mathbb{P}^2} \setminus E_{P_0})}_{\text{irrid. } \cong \mathbb{P}^2 \setminus \{P_0\}} \sqcup E_{P_0}$$

E_{P_0}: irrid. $\cong \mathbb{P}^1$ e chiuso di $\mathbb{P}^2 \times \mathbb{P}^1$ perché $b^{-1}(P_0)$, b morf. e $P_0 \in \mathbb{P}^2$ chiuso

* Se dimostriamo che $\widetilde{\mathbb{P}^2} \setminus E_{P_0}$ è denso in $\widetilde{\mathbb{P}^2}$ sarà irriducibile perché $\widetilde{\mathbb{P}^2} \setminus E_{P_0}$ è irriducibile.

* ∀ punto di E_{P_0} è nella chiusura in $\widetilde{\mathbb{P}^2}$ di $\tau(r \setminus \{P_0\}) = \tau(\overline{r \setminus \{P_0\}}) = \tau(r)$

Infatti $\tau(r) \cap E_{P_0} = \{[P_0], [q_1, q_2]\}$

⇒ $\widetilde{\mathbb{P}^2} \setminus E_{P_0}$ è denso in $\widetilde{\mathbb{P}^2}$ ∎

Ricapitolando

$$b : \widetilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$$

- b birazionale (perché isomorfa $\widetilde{\mathbb{P}}^2 \setminus E_{P_0}$ e $\mathbb{P}^2 \setminus \{P_0\}$)
- b morfismo
- b^{-1} non definito in $\{P_0\}$ $\Rightarrow$ b no isomorfismo
- $\forall\, r \subset \mathbb{P}^2$ per P_0 $\exists$ $(b|_r)^{-1}$ invece definito
- $\widetilde{\mathbb{P}}^2$ VARIETÀ PROIETTIVA RAZIONALE perché birazionale a $\mathbb{P}^2$

Utilizzo scoppiamento?

$C \subset \mathbb{P}^2$ curva irriducibile proiettiva per P_0

$$b : \widetilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2 \quad \text{scoppiamento}$$

$$\overline{b^{-1}(C \setminus \{P_0\})} \subseteq \widetilde{\mathbb{P}}^2 \quad \text{chiusura}$$

* $C \setminus \{P_0\}$ è sottovarietà aperta di C curva irriducibile

* $b^{-1}(C \setminus \{P_0\}) \cong C \setminus \{P_0\}$ curva irriducibile in $\widetilde{\mathbb{P}}^2$ varietà loc. chiusa

* $\widetilde{C} := \overline{b^{-1}(C \setminus \{P_0\})} \subseteq \widetilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \cong S_{2,1} \subset \mathbb{P}^5$

$\widetilde{C}$ curva irrid. proiettiva in $\widetilde{\mathbb{P}}^2$ perché chiusura proiettiva e $\widetilde{\mathbb{P}}^2$ proiettiva

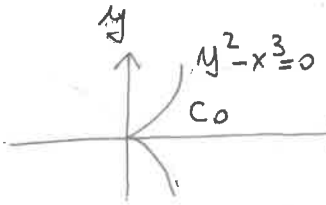
$\widetilde{C}$ birazionale a $C \subset \mathbb{P}^2$

$$\widetilde{C} \setminus (\widetilde{C} \cap b^{-1}(P_0)) \cong C \setminus \{P_0\}$$

$b^{-1}(C) =$ Trasformata totale di $C \subset \mathbb{P}^2$, $C \ni P_0$

$\tilde{C} =$ Trasformata propria o stretta di $C \subset \mathbb{P}^2$, $C \ni P_0$

Esempi



(1) Parabola semicubica

$$\mathbb{A}^2_{x,y} \cong U_0^2$$

$C_0 = Z_a(y^2 - x^3)$ passa per $O = (0,0)$ singolare per C_0

$\Rightarrow$ $C := \overline{C_0} = Z_p(x_0 x_2^2 - x_1^3) \subset \mathbb{P}^2$

passa per $P_0 = [1,0,0]$ sing. per C

$b : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ blow-up in P_0

$b^{-1}(C) = ?$ $\qquad \tilde{C} = ?$

Sia $\boxed{T_C := C \times \mathbb{P}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1}$ che è definito con stessa eq. di C

$$T_C := Z_p(x_0 x_2^2 - x_1^3) \subset \mathbb{P}^2 \times \mathbb{P}^1 \ ([x_0, x_1, x_2], [y_1, y_2])$$

$\Rightarrow$ T_C è $C \times \mathbb{P}^1$ superficie rigata banalmente

$$\boxed{b^{-1}(C) := T_C \cap \tilde{\mathbb{P}}^2} = Z_p(x_0 x_2^2 - x_1^3,\ x_1 y_2 - x_2 y_1)$$

In $U_0^2 \times U_1^1 \cong \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$

$x = \dfrac{x_1}{x_0}$, $y = \dfrac{x_2}{x_0}$ in $U_0^2 \subset \mathbb{P}^2$

$m = \dfrac{y_2}{y_1}$ in $U_1^1 \subset \mathbb{P}^1$

$$b^{-1}(C)\cap(\mathcal{U}_0^2\times\mathcal{U}_1^1) = \begin{cases} y - mx = 0 & \rightsquigarrow \text{Paraboloide a sella in } (x,y,m)\in\mathbb{A}^3 \\ y^2 - x^3 = 0 & \rightsquigarrow C_0\subset\mathbb{A}^2 \text{ ma in } \mathbb{A}^3 \text{ è sup. rigata su } C_0 \end{cases}$$

$$\Rightarrow \begin{cases} y = mx \\ y^2 - x^3 = 0 \end{cases} \Rightarrow \begin{cases} y = mx \\ m^2x^2 - x^3 = 0 \end{cases} \Rightarrow \begin{cases} y = mx \\ x^2(m^2 - x) = 0 \end{cases}$$

Il sistema si spezza in

$$\begin{cases} y = mx \\ x = m^2 \end{cases}$$

per $\forall\, m \in \Bbbk$

$\Downarrow$

$(x,y,m) = (m^2, m^3, m) \in \mathbb{A}^3$

$\downarrow$

trasformata affine della cubica gobba affine in $\mathbb{A}^3$ standard

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^3$$
$$m' \longmapsto (m', (m')^2, (m')^3) = (m', x', y')$$

ho usato affinità

$$\begin{cases} x = y' \\ y = z' \\ m = m' \end{cases}$$

$\Downarrow$

$\tilde{C}_0 \cong$ Cubica gobba affine in $\mathbb{A}^3$

$\Downarrow$

$\tilde{C} = \overline{\tilde{C}_0}^{pr} \cong$ cubica gobba proiettiva

$$\begin{cases} x^2 = 0 \\ y = 0 \end{cases} \Downarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \forall\, m \in \Bbbk \rightarrow b^{-1}(0,0) = E_{P_0}$$

contato doppiamente

asse m in $\mathbb{A}^3_{(x,y,m)}$

P_0 era punto doppio (cuspidale) per C_0

$\Downarrow$

E_{P_0} contato come retta doppia

$\Downarrow$

$$\Rightarrow \boxed{b^{-1}(C) = \tilde{C} \cup \{2E_{P_0}\}}$$

$$b|_{\tilde{C}\cap(\mathcal{U}_0^2\times\mathcal{U}_1^1)} : \quad \mathbb{A}^2\times\mathbb{A}^1 \simeq \mathbb{A}^3 \xrightarrow{\pi_{\mathbb{A}^2}} \mathbb{A}^2 \cong \mathcal{U}_0^2$$
$$\tilde{C}\cap\mathbb{A}^3$$
$$(m^2, m^3, m) \longmapsto (m^2, m^3)$$

$\Rightarrow b = \pi_{\mathbb{A}^2} = \pi_{(1,2)}$ è proiezione in $\mathbb{A}^3$ su piano coordinato

$\downarrow$

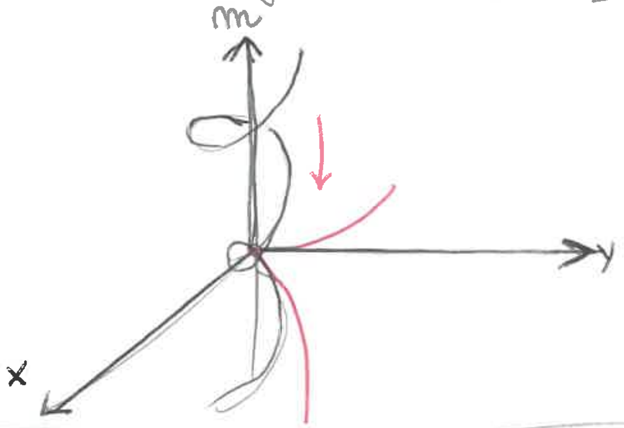
$I = (1,2)$ multiindice

$(x,y,m) \longrightarrow (x,y)$ cioè su $m=0$

$$\tilde{C} \cap E_{P_0}\big|_{U_0^1 \times U_1^1} := \begin{cases} x = m^2 \\ y = m^3 \\ \begin{pmatrix} x=0 \\ y=0 \end{pmatrix} \end{cases} \Rightarrow \underline{\underline{(0,0,0) \text{ con molt. } 2}}$$

l'asse m in $\mathbb{A}^3$ t.c. E_{P_0} divisore eccezionale

$\Rightarrow$ asse m è tg. a $\tilde{C}\big|_{U_0^1 \times U_1^1}$ in $O = (0,0,0)$



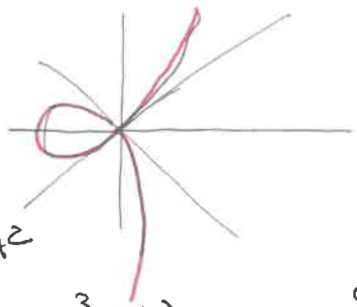
$\tilde{C}$ desingolarizzata $C \subset \mathbb{P}^2$ e $\sigma^{-1}(C)$ tiene traccia di sing. di C in P_0 come punto doppio

(ii) Se invece

$T'_0 \subset \mathbb{A}^2$

$T'_0 = Z_a\,(y^2 = x^2(x+1))$

$x^3 + x^2 - y^2 = 0$



$$\mathbb{A}^1 \xrightarrow{\varphi} \mathbb{A}^2$$

$$t \longrightarrow (t^2-1,\ t^3-t)$$

$\varphi'(t) \neq (0,0)\ \forall t$

però $\varphi(1) = \varphi(-1)$

$O = (0,0)$ è $\Rightarrow$ $\underline{\text{NODO}}$ per T'_0

$T' := \overline{T'_0} = Z_p\,(x_1^3 + x_1^2 x_0 - x_2^2 x_0) \subset \mathbb{P}^2$ P_0 è $\underline{\text{NODO}}$ per T'

(17)

$$b^{-1}(T') = \begin{cases} x_1^3 + x_0(x_1^2 - x_2^2) = 0 & \to T_{T'} = T' \times \mathbb{P}^1 \\ x_1 y_2 - x_2 y_1 = 0 & \to \widetilde{\mathbb{P}}^2 \end{cases}$$

In $U_0^2 \times U_1^1 \cong \mathbb{A}^3$

$x = \frac{x_1}{x_0} \quad y = \frac{x_2}{x_0}$ in U_0^2

$s = \frac{y_2}{y_1}$ in U_1^1

$\widetilde{\mathbb{P}}^2 \cap (U_0^2 \times U_1^1)$ in $\mathbb{A}^3$ (x,y,s) è $y - xs = 0$ paraboloide a sella in $\mathbb{A}^3_{(x,y,s)}$

$$b^{-1}(T') \cap (U_0^2 \times U_1^1) = \begin{cases} x^3 + x^2 - y^2 = 0 \\ y = xs \end{cases}$$

$$\begin{cases} x^2(x+1-s^2) \\ y = xs \end{cases}$$

$\nearrow \begin{cases} x^2 = 0 \\ y = 0 \end{cases}$ {asse s} = E_{P_0} doppio

$\searrow \begin{cases} x = s^2 - 1 \\ y = s^3 - s \end{cases} \to \widetilde{T}'_0 = \widetilde{T}' \cap (U_0^2 \times U_1^1)$

Ma allora ho

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow \mathbb{A}^3 \\ s &\longrightarrow (s^2-1, s^3-s, s) = (x, y, s) \end{aligned}$$

~~isomorfa~~ a cubica gobba affine usando affinità

$x = x' + 1$
$y = y' + z'$
$s = s'$

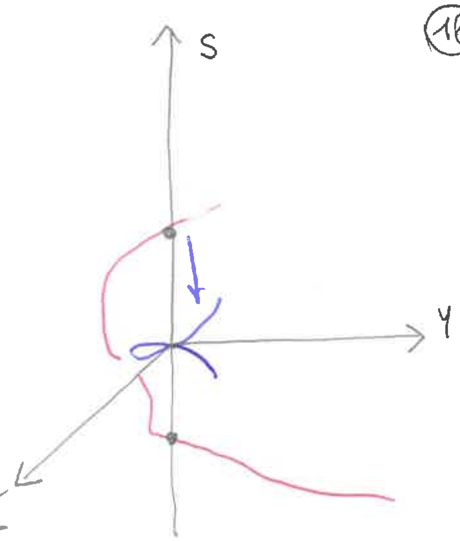
$$\begin{pmatrix} x \\ y \\ s' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ s' \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

ottengo $x' = s^2$, $y' = s^3$, $s' = s$ che è affinemente equiv. a cubica gobba affine standard

$\Rightarrow$ $\boxed{\widetilde{T}' \cong \text{Cubica gobba proiettiva}}$

Inoltre:

$$(\widetilde{T}' \cap E_{P_0}) \cap (U_0^2 \times U_1^1) = \begin{cases} x = s^2 - 1 \\ y = s^3 - s \\ x = 0 \\ y = 0 \end{cases}$$



$s^2 - 1 = 0 \iff s = \pm 1$

$\Downarrow$

$(0,0,1) \cup (0,0,-1)$

* Proiettando su $\mathbb{A}^2 = v_0^2$, $\pi_{/\mathbb{A}^2} = \pi_{(1,2)}$

$(s^2-1,\ s^3-s,\ s) \xrightarrow{\pi_{/\mathbb{A}^2}} (s^2-1,\ s^3-s) \subset \mathbb{A}^2$

$\Uparrow$

Γ_0

$(\underbrace{0,0}_{0},\ \underbrace{1}_{\text{coef. ang. } y = x})$ e $(\underbrace{0,0}_{0},\ \underbrace{-1}_{\text{coeff. ang. } y = -x})$

tg. principali a Γ_0 in O

$y^2 - x^2 = 0$

$y - x = 0$ $\quad$ $y + x = 0$