Svolgimento Appello 3 Geometria 1 (Giugno 2021)

Esercizio 1

(i)

$F(1) = 1$

$F(x) = x+1$

$F(x^2) = x^2+2x+2$

$F(x^3) = x^3+3x^2+6x+6$

$$\Rightarrow M_{\mathcal{E}}(F) = \begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\det(M_{\mathcal{E}}(F)) = 1 \neq 0$

(i) $\ker(F) = \{\underline{0}\}$ quindi $\underline{x} = \underline{0}$ sono le sue eq. parametriche e $\underline{x} = \underline{0}$ le sue equazioni cartesiane. Non ha basi

(ii) $\operatorname{Im}(F) = V$

$$\underline{x} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} 6 \\ 6 \\ 3 \\ 1 \end{pmatrix} \quad \text{eq. param.}$$

eq. cartesiane $0 = 0$

(iii) $V = \operatorname{Im}(F)$ quindi $h(x) = \underline{0}$ e $i(x) = q(x)$

Esercizio 2

(i) Notiamo che $\{P_t \mid t \in \mathbb{R}\}$ descrive la retta di eq. parametriche

$$r: \begin{cases} x_1 = -t \\ x_2 = 0 \\ x_3 = (-1+t) \end{cases} \qquad \text{cioè } r: \underline{x} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Pertanto, poiché $r \not\ni Q \Rightarrow \mathcal{L}$ è il piano per r e Q

Eq. cartesiane di r sono

$$r: \begin{cases} x_1 + x_3 = -1 \\ x_2 = 0 \end{cases} \quad \to \text{ fascio di piani di asse } r$$

$$\lambda(x_1 + x_3 + 1) + \mu x_2 = 0$$

Imponendo passaggio per $Q = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ si ha

$$\lambda(2 - \cancel{1} + \cancel{1}) - 2\mu = 0 \Rightarrow 2\lambda - 2\mu = 0 \Rightarrow \lambda = \mu$$

Pertanto $\quad \mathcal{L}: \ x_1 + x_2 + x_3 = -1$

(ii) $P_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad P_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \overrightarrow{QP_0} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

$$\overrightarrow{P_1P_0} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{2} \| \overrightarrow{QP_0} \wedge \overrightarrow{P_1P_0} \| = \frac{1}{2} \left\| \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} \right\| = \frac{\sqrt{12}}{2} = \sqrt{3}$$

(iii) $$d(0, \mathcal{L}) = \frac{|0 + 0 + 0 + 1|}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Esercizio 3

(i) $\underline{w}_1 = \underline{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$$\underline{w}_2 = \underline{u}_2 - \frac{\underline{u}_1 \cdot \underline{u}_2}{\|\underline{w}_1\|} \cdot \underline{u}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{(-2)}{5} \underline{u}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -1/5 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Poniamo

$$\underline{f}_1 = \frac{\underline{w}_1}{|\underline{w}_1|} = \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \\ 0 \end{pmatrix} \qquad \underline{f}_2 = \begin{pmatrix} 2/\sqrt{30} \\ -1/\sqrt{30} \\ 5/\sqrt{30} \end{pmatrix}$$

(ii) $\underline{u}_1 \wedge \underline{u}_2 \sim \underline{w}_1 \wedge \underline{w}_2 \sim \underline{f}_1 \wedge \underline{f}_2$

$$\underline{u}_1 \wedge \underline{u}_2 = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \underline{f}_3 = \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix} \text{ è t.c. } \langle \underline{f}_3 \rangle = U^\perp$$

(iii) Posta $\mathcal{F} = \{\underline{f}_1, \underline{f}_2, \underline{f}_3\}$

$$M_{\mathcal{F}}(\pi_{U^\perp}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$

Se $M = M_{\mathcal{E},\mathcal{F}}$, dal diagramma

$$\begin{tikzcd} \mathbb{R}^3, \mathcal{E} \arrow[r, "\pi_{U^\perp}"] \arrow[d, leftrightarrow] \arrow[r, bend right] & \mathbb{R}^3, \mathcal{E} \arrow[d, leftrightarrow] \\ \mathbb{R}^3, \mathcal{F} \arrow[r, "\pi_{U^\perp}"] & \mathbb{R}^3, \mathcal{F} \end{tikzcd}$$

e dal fatto che $M \in O(3, \mathbb{R})$

$$M_{\mathcal{E}}(\pi_{U^\perp}) = M_{\mathcal{E},\mathcal{F}} \circ M_{\mathcal{F}}(\pi_{U^\perp}) \circ M_{\mathcal{F},\mathcal{E}} =$$

$$= M \circ A\, M^t$$