Svolgimenti Esercitazione IV

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Esercizio 1

(i) Metodo scarti successivi

- $1-x \neq 0 \Rightarrow 1-x$ è vettore l. i. in V
- $1-x, 1+x$ non proporzionali $\Rightarrow \{1-x, 1+x\}$ sistema l. i. in V
- $1-x^2$ non può essere comb. lin. di $1-x$ e $1+x$ per questioni di grado

$\Rightarrow \{1-x, 1+x, 1-x^2\}$ sistema l. i. in V

- $\exists\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ t. c. $\alpha_1(1-x)+\alpha_2(1+x)+\alpha_3(1-x^2) = x-x^2$?

Tali scalari $\exists$ se e solo se vale

$$(\alpha_1+\alpha_2+\alpha_3)\cdot 1 + (\alpha_2-\alpha_1)x - \alpha_3 x^2 = x - x^2 \quad \text{identità polinomiale}$$

cioè se e solo se vale:

$$\begin{cases} \alpha_1+\alpha_2+\alpha_3 = 0 \\ -\alpha_1+\alpha_2 = 1 \\ \alpha_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2+R_1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{2} & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$\#$ pivots $= 3 \iff \exists!\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ t. c. $x-x^2 = \alpha_1(1-x)+\alpha_2(1+x)+\alpha_3(x-x^2)$

$\Rightarrow \{1-x, 1+x, 1-x^2, x-x^2\}$ non è sistema lin. indip.

$\Rightarrow$ anche S non può essere indipendente.

(ii) Da (i), poiché $x-x^2 \in \mathrm{Span}(\{1-x, 1+x, 1-x^2\}) \subseteq \mathrm{Span}(S)$

posso già considerare

$$S \setminus \{x-x^2\} := \{1-x, 1+x, 1-x^2, x+x^2, 1+x^2+x^3\}$$

- $x+x^2 \in \mathrm{Span}\{1-x, 1+x, 1-x^2\}$? I. e. $\exists\ \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ t. c.

$$\beta_1(1-x)+\beta_2(1+x)+\beta_3(1-x^2) = x+x^2 ?$$

Come prima
$$\begin{cases} \beta_1+\beta_2+\beta_3 = 0 \\ -\beta_1+\beta_2 = 1 \\ -\beta_3 = 1 \end{cases}$$

La stessa trasformazione elementare precedente fornisce che (2)

$\exists!$ di $\beta_1, \beta_2, \beta_3 \Rightarrow$ anche $x+x^2$ si può scartare cioè

$$\mathrm{Span}(S) = \mathrm{Span}(S \setminus \{x-x^2\}) = \mathrm{Span}(S \setminus \{x-x^2, x+x^2\})$$

e $x+x^2$ dipende linearmente da $\{1-x, 1+x, 1-x^2\}$

- infine $1+x^2+x^3$ è lin. indip. da $\{1-x, 1+x, 1-x^2\}$ per questione di grado

$$\Rightarrow \quad S' := \{1-x, 1+x, 1-x^2, 1+x^2+x^3\}$$

dove $\mathrm{span}(S) = \mathrm{span}(S')$ ma S' sistema linearmente indipend.

(iii) Preso $a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in V$ arbitrario, $\exists$ $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ t.c.

$$\gamma_0(1-x) + \gamma_1(1+x) + \gamma_2(1-x^2) + \gamma_3(1+x^2+x^3) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\Leftrightarrow \quad \begin{cases} \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = a_0 \\ -\gamma_0 + \gamma_1 = a_1 \\ -\gamma_2 + \gamma_3 = a_2 \\ \gamma_3 = a_3 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a_0 \\ -1 & 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 1 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & a_0 \\ 0 & \boxed{2} & 1 & 1 & a_0+a_1 \\ 0 & 0 & \boxed{-1} & 1 & a_2 \\ 0 & 0 & 0 & \boxed{1} & a_3 \end{array}\right)$$

$\#$ pivots $= 4 \Rightarrow \exists! \ \gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \Rightarrow$

$\mathrm{Span}(S') = V$ e S' sist. lin. indipendente

(iv) $p(x) = 10 - 7x - x^2 + x^3 = \gamma_0(1-x) + \gamma_1(1+x) + \gamma_2(1-x^2) + \gamma_3(1+x^2+x^3)$

$$\Leftrightarrow \begin{cases} \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 10 \\ -\gamma_0 + \gamma_1 = -7 \\ -\gamma_2 + \gamma_3 = -1 \\ \gamma_3 = 1 \end{cases} \Leftrightarrow \begin{cases} \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 10 \\ 2\gamma_1 + \gamma_2 + \gamma_3 = 3 \\ -\gamma_2 + \gamma_3 = -1 \\ \gamma_3 = 1 \end{cases}$$

$\boxed{\gamma_3 = 1} \Rightarrow \boxed{\gamma_2 = 2} \Rightarrow 2\gamma_1 = 3 - (2) - (1) = 0 \Rightarrow \boxed{\gamma_1 = 0}$

$\gamma_0 = 10 - (0) - (2) - (1) = 7 \Rightarrow \boxed{\gamma_0 = 7}$

Quindi

$$P(x) = 7\cdot(1-x) + 0\cdot(1+x) + 2\cdot(1-x^2) + 1\cdot(1+x^2+x^3).$$

(v) $W = \{ q(x) \in V \mid q(2) = 0 \}$

$\forall\ q_1(x), q_2(x) \in W$ e $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$

$$(\alpha_1 q_1 + \alpha_2 q_2)(2) = \alpha_1 \underset{0}{\underset{\|}{\cancel{q_1(2)}}} + \alpha_2 \underset{0}{\underset{\|}{\cancel{q_2(2)}}} = 0$$

$\Rightarrow$ W è sottospazio di V.

Poiché $1 \in V \setminus W \Rightarrow W$ è sottospazio proprio di V.

$P(x) \in W \iff P(x) = (x-2)(b_0 + b_1 x + b_2 x^2)$, $b_i \in \mathbb{R}$, $0 \le i \le 2$

Pertanto

$$P(x) \in W \iff P(x) = \cancel{b_0 x} + \cancel{b_1 x^2} + b_2 x^3 - \cancel{2b_0} - 2b_1 x - \cancel{2b_2 x^2}$$
$$= b_0(x-2) + b_1(x^2-2x) + b_2(x^3-2x^2)$$

$b_0, b_1, b_2 \in \mathbb{R}$

Quindi $W = \mathrm{Span}(\{x-2,\ x^2-2x,\ x^3-2x^2\})$

ed è un sist. lin. indip. per questioni di gradi.

Esercizio 2

(i) $Tsup_{2,2} \ni A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$= a_{11}E_{11} + a_{12}E_{12} + a_{22}E_{22}$$

$Tsup_{22} = \mathrm{Span}\{E_{11}, E_{12}, E_{22}\}$ sist. lin. indipendente

Analogamente

$Tinf_{22} = \mathrm{Span}\{E_{11}, E_{21}, E_{22}\}$

(ii) Poiché $\forall \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in V$ è $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$

$\Rightarrow V = Tsup_{2,2} + Tinf_{2,2}$

Ma la somma <u>non è diretta</u>, in quanto

$$Tsup_{2,2} \cap Tinf_{2,2} = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \right\} = Diag_{2,2} \subset V.$$

(ii) $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \in V$ ma $A \notin Tsup_{2,2}$ e $A \notin Tinf_{2,2}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} a_{11} + b_{11} = 1 \\ a_{12} = 2 \\ b_{21} = 3 \\ a_{22} + b_{22} = 0 \end{cases} \qquad \begin{array}{l} \text{se } b_{11} = t \in \mathbb{R} \Rightarrow a_{11} = 1 - t \\ \\ \\ b_{22} = s \in \mathbb{R} \Rightarrow a_{22} = -s \end{array}$$

Quindi

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1-t & 2 \\ 0 & -s \end{pmatrix} + \begin{pmatrix} t & 0 \\ 3 & s \end{pmatrix} \quad \forall\, t, s \in \mathbb{R} \Rightarrow$$

A ha <u>∞^2 modi diversi per essere scritto</u> come vettori in $Tsup_{2,2} + Tinf_{2,2}$ (in particolare <u>scritture non uniche</u>)

(iv) $Sym_{2,2} \ni A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$= a_{11} E_{11} + a_{12} (E_{12} + E_{21}) + a_{22} E_{22}$$

$$\Rightarrow \quad Sym_{2,2} = Span \underbrace{\{ E_{11}, \ E_{12} + E_{21}, \ E_{22} \}}_{\text{sist. lin. indipendente}}$$

$$Antisym_{2,2} \ni A = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} = a_{12} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Antisym_{2,2} = Span \{ E_{12} - E_{21} \}$$

(v) $Antisym_{2,2} \cap Sym_{2,2} = \left\{ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$\forall A \in V$

$$A = \underbrace{\tfrac{1}{2}(A + A^t)}_{\in\, Sym_{2,2}} + \underbrace{\tfrac{1}{2}(A - A^t)}_{\in\, Antisym_{2,2}}$$

$\Rightarrow V = Sym_{2,2} \oplus Antisym_{2,2}$ e <u>scritture uniche</u>

ora

$$A=\begin{pmatrix}1&2\\3&0\end{pmatrix} \Rightarrow A^t=\begin{pmatrix}1&3\\2&0\end{pmatrix}$$

$$A+A^t=\begin{pmatrix}2&5\\5&0\end{pmatrix}$$

$$A-A^t=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\Rightarrow A=\frac{1}{2}\begin{pmatrix}2&5\\5&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}1&5/2\\5/2&0\end{pmatrix}}_{\in\, Sym_{2,2}}+\underbrace{\begin{pmatrix}0&-1/2\\1/2&0\end{pmatrix}}_{\in\, Antisym_{2,2}}$$

<u>scrittura unica.</u>

<u>Esercizio 3</u>

(i) • $\underline{v}_1$ e $\underline{v}_2$ non proporzionali

• $\underline{v}_3 \notin \text{Span}\{\underline{v}_1, \underline{v}_2\}$ infatti

$$\alpha_1 \underline{v}_1+\alpha_2\underline{v}_2=\underline{v}_3 \iff \alpha_1\begin{pmatrix}0\\1\\-1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\3\end{pmatrix} \iff$$

$$\begin{cases}\alpha_2=1\\ \alpha_1=-1\\ -\alpha_1+\alpha_2=3\end{cases}$$

sistema incompatibile

Alternativamente per verificare che $\{\underline{v}_1,\underline{v}_2,\underline{v}_3\}$ <u>sist. lin. indip.</u>

$$\beta_1\underline{v}_1+\beta_2\underline{v}_2+\beta_3\underline{v}_3=\underline{0} \iff$$

$$\beta_1\begin{pmatrix}0\\1\\-1\end{pmatrix}+\beta_2\begin{pmatrix}1\\0\\1\end{pmatrix}+\beta_3\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \iff$$

$$\begin{cases}\beta_2+\beta_3=0\\ \beta_1-\beta_3=0\\ -\beta_1+\beta_2+\beta_3=0\end{cases}$$

$$\begin{pmatrix}0&1&1\\1&0&-1\\-1&1&1\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_3}\begin{pmatrix}-1&1&1\\1&0&-1\\0&1&1\end{pmatrix}\xrightarrow{R_2\to R_2+R_1}$$

$$\begin{pmatrix}-1&1&1\\0&1&0\\0&1&1\end{pmatrix}\xrightarrow{R_3\to R_3-R_2}\begin{pmatrix}\boxed{-1}&1&1\\0&\boxed{1}&0\\0&0&\boxed{1}\end{pmatrix}$$

Il sistema omogeneo ha ! soluzione banale
$(\beta_1,\beta_2,\beta_3)=(0,0,0) \Rightarrow \{\underline{v}_1,\underline{v}_2,\underline{v}_3\}$ sist. lin. indipendente

$\forall \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \underline{E}_1 + c_2 \underline{E}_2 + c_3 \underline{E}_3 \in \mathbb{R}^3$

$\exists\ \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ t.c. $\quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \beta_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

equivalentemente

$$c_1 \underline{E}_1 + c_2 \underline{E}_2 + c_3 \underline{E}_3 = \beta_1 (\underline{E}_2 - \underline{E}_3) + \beta_2 (\underline{E}_1 + \underline{E}_3) + \beta_3 (\underline{E}_1 - \underline{E}_2 + 3\underline{E}_3)$$

Entrambi i metodi forniscono

$$\begin{cases} \beta_2 + \beta_3 = c_1 \\ \beta_1 - \beta_3 = c_2 \\ -\beta_1 + \beta_2 + 3\beta_3 = c_3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & c_1 \\ 1 & 0 & -1 & c_2 \\ -1 & 1 & 1 & c_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 1 & 1 & c_3 \\ 1 & 0 & -1 & c_2 \\ 0 & 1 & 1 & c_1 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & c_3 \\ 0 & 1 & 0 & c_3 + c_2 \\ 0 & 1 & 1 & c_1 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} -1 & 1 & 1 & c_3 \\ 0 & 1 & 0 & c_3 + c_2 \\ 0 & 0 & 1 & c_1 - c_2 + c_3 \end{array}\right)$$

$\Rightarrow \boxed{\beta_3 = c_1 - c_2 + c_3 \quad e \quad \beta_2 = c_2 + c_3}$

$$-\beta_1 = c_3 - (c_2 + c_3) - (c_1 - c_2 + c_3) = \cancel{c_3} - \cancel{c_2} - \cancel{c_3} - c_1 + \cancel{c_2} - c_3$$
$$= -c_1 - c_3$$

$\Rightarrow \boxed{\beta_1 = c_1 + c_3}$

Quindi

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = (c_1 + c_3)\, \underline{v}_1 + (c_2 + c_3)\, \underline{v}_2 + (c_1 - c_2 + c_3)\, \underline{v}_3$$
$$= (c_1 + c_3) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (c_2 + c_3) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (c_1 - c_2 + c_3) \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

(iii) Poichè $\underline{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1\, \underline{E}_1 + 2\, \underline{E}_3$, dai conti precedenti si ha

$$\underline{w} = 3 \underline{v}_1 + 2 \underline{v}_2 + 3 \underline{v}_3$$

(iv) $\underline{u} = \underline{v}_1 - 2\underline{v}_2 + \underline{v}_3 = (\underline{E}_2 - \underline{E}_3) - 2(\underline{E}_1 + \underline{E}_3) + (\underline{E}_1 - \underline{E}_2 + 3\underline{E}_3)$

$$= -\underline{E}_1 + 0\,\underline{E}_2 + 0\,\underline{E}_3 = -\underline{E}_1$$

Esercizio 4

(i) • 1 è la funzione identicamente uguale ad 1 mentre $\sin(x)$ è periodica di periodo 2π $\Rightarrow$ $\{1, \sin(x)\}$ è sist. lin. indip.

• $\cos(x) \notin \mathrm{Span}\{1, \sin x\}$, altrimenti $\exists\ \alpha_1, \alpha_2 \in \mathbb{R}$ t.c.

$$\cos(x) = \alpha_1 + \alpha_2 \sin(x)$$

Poiché $\cos(0) = 1 \Rightarrow \alpha_1 = 1$ visto che $\sin(0) = 0$

$\cos(\pi/2) = 0 \Rightarrow 0 = 1 + \alpha_2 \cdot \sin(\pi/2) \Leftrightarrow 0 = 1 + \alpha_2$

quindi $\alpha_2 = -1$

se fosse

$$\cos x = 1 - \sin(x)$$

$x = \pi/4$ avrebbe $\frac{\sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2}$ ⨳

$\Rightarrow$ $\{1, \sin(x), \cos(x)\}$ sist. lin. indipendente

• $\alpha_1 \cdot 1 + \alpha_2 \sin x + \alpha_3 \cos x + \alpha_4 \sin x \cdot \cos x \overset{?}{=} 0$

$\downarrow$ f.ne identicamente nulla

$x = 0 \qquad \alpha_1 + \alpha_3 = 0$

$x = \pi/2 \qquad \alpha_1 + \alpha_2 = 0$

$x = \frac{\pi}{4} \qquad \alpha_1 + \frac{\sqrt{2}}{2}\alpha_2 + \frac{\sqrt{2}}{2}\alpha_3 + \frac{1}{2}\alpha_4 = 0$

$x = \frac{3}{4}\pi \qquad \alpha_1 + \frac{\sqrt{2}}{2}\alpha_2 - \frac{\sqrt{2}}{2}\alpha_3 - \frac{1}{2}\alpha_4 = 0$

$$\begin{cases} \alpha_1 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 \\ 2\alpha_1 + \sqrt{2}\alpha_2 + \sqrt{2}\alpha_3 + \alpha_4 = 0 \\ 2\alpha_1 + \sqrt{2}\alpha_2 - \sqrt{2}\alpha_3 - \alpha_4 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_3 = -\alpha_1 \\ \alpha_2 = -\alpha_1 \\ 2\alpha_1 - 2\sqrt{2}\alpha_1 + \alpha_4 = 0 \\ 2\alpha_1 - \alpha_4 = 0 \end{cases}$$

$$\begin{cases} \alpha_3 = -\alpha_1 \\ \alpha_2 = -\alpha_1 \\ (2-2\sqrt{2})\alpha_1 + \alpha_4 = 0 \\ 2\alpha_1 - \alpha_4 = 0 \end{cases}$$

sistema con unica soluz. banale
$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0,0,0,0)$

$\Rightarrow \{1, \sin(x), \cos(x), \sin(x)\cdot\cos(x)\}$ sist. lin. indipendente

- Ora, poiché $\sin(2x) = 2\sin(x)\cdot\cos(x) \Rightarrow$

$$\sin(2x) \in \mathrm{Span}\{1, \sin(x), \cos(x), \sin(x)\cdot\cos(x)\}$$

visto che

$$\sin(2x) = 0\cdot 1 + 0\cdot\sin(x) + 0\cdot\cos(x) + 2\cdot(\sin(x)\cdot\cos(x))$$

$$\Downarrow$$

$$S' = \{1, \sin(x), \cos(x), \sin(x)\cos(x)\}.$$

(ii) Poiché $\cos(2x) = \cos^2(x) - \sin^2(x)$ con strategia simile alla precedente si verifica che $\cos(2x) \notin \mathrm{Span}(S')$

(iii) Analogo per $tg(x)$ che non è definita su $\mathbb{R}$ a differenza degli elementi in S'.