BRILL-NOETHER LOCI OF RANK TWO VECTOR BUNDLES ON A GENERAL ν-GONAL CURVE

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Abstract. In this paper we study the Brill Noether locus of rank 2, (semi)stable vector bundles with at least two sections and of suitable degrees on a general ν-gonal curve. We classify its reduced components whose dimensions are at least the corresponding Brill-Noether number. We moreover describe the general member $\mathcal{F}$ of such components just in terms of extensions of line bundles with suitable *minimality properties*, providing information on the birational geometry of such components as well as on the very-ampleness of $\mathcal{F}$.

1. Introduction

Let C denote a smooth, irreducible, complex projective curve of genus $g \geq 2$. As in the statement of [10, Theorem] (cf. also Theorem 1.1 below), C is said to be *general* if C is a curve with general moduli (cf. e.g. [2], pp. 214–215). Let $U_C(n, d)$ be the moduli space of semistable, degree d, rank n vector bundles on C and let $U_C^s(n, d)$ be the open dense subset of stable bundles (when d is odd, more precisely one has $U_C(n, d) = U_C^s(n, d)$). Let $B^k_{n,d} \subseteq U_C(n, d)$ be the *Brill-Noether locus* which consists of vector bundles $\mathcal{F}$ having $h^0(\mathcal{F}) \geq k$, for a positive integer k.

Traditionally, we denote by W^k_d the Brill-Noether locus $B^{k+1}_{1,d}$ of line bundles $L \in \mathrm{Pic}^d(C)$ having $h^0(L) \geq k+1$, for a non-negative integer k. With little abuse of notation, we will sometimes identify line bundles with corresponding divisor classes, interchangeably using multiplicative and additive notation.

For the case of rank 2 vector bundles, we simply put $B^k_d := B^k_{2,d}$, for which it is well-known that the dimension of $B^k_d \cap U^s_C(2, d)$ is at least the Brill-Noether number $\rho^k_d := 4g-3-ik$, where $i := k+2g-2-d$ (cf. [9]). This is no longer true for possible components of B^k_d in $U_C(2, d) \setminus U^s_C(2, d)$, i.e. not containing stable points, which can occur only for d even (cf. [3, Remark 3.3] for more explanations and details).

In the range $0 \leq d \leq 2g-2$, B^1_d has been deeply studied on any curve C by several authors (cf. [9, 6]). Concerning B^2_d, using a degeneration argument, N. Sundaram [9] proved that B^2_d is non-empty for any C and for odd d such that $g \leq d \leq 2g-3$. M. Teixidor I Bigas generalizes Sundaram's result as follows:

Theorem 1.1 ([10]). *Given a non-singular curve C and a d, $3 \leq d \leq 2g-1$, $B^2_d \cap U^s_C(2, d)$ has a component of dimension $\rho^2_d = 2d-3$ and a generic point on it corresponds to a vector bundle whose space of sections has dimension 2 and the generic section has no zeroes. If C is general, this is the*

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only component of $B^2_d \cap U^s_C(2,d)$. *Moreover,* $B^2_d \cap U^s_C(2,d)$ *has extra components if and only if* W^1_n *is non-empty and* $\dim W^1_n \geq d + 2n - 2g - 1$ *for some* n *with* $2n < d$.

Inspiblack by Theorem 1.1, in this paper we focus on B^2_d for C a *general* ν*-gonal curve* of genus g, i.e. C corresponds to a general point of the ν-gonal stratum $\mathcal{M}^1_{g,\nu} \subset \mathcal{M}_g$. Precisely, we prove the following:

Theorem 1.2. *Let* C *be a general* ν*-gonal* ($3 \leq \nu \leq \frac{g+8}{4}$) *curve of genus* g *and let* A *be the unique line bundle of degree* ν *and* $h^0(A) = 2$. *For any positive integer* d *with* $2 + 2\nu \leq d \leq g - 3$, *the reduced components of* B^2_d *having dimension at least* ρ^2_d *are only two, which we denote by* $B_{\rm reg}$ *and* $B_{\rm sup}$*:*

(i) $B_{\rm reg}$ *is generically smooth, of dimension* $\rho^2_d = 2d - 3$ (regular *for short). Moreover,* $\mathcal{F}$ *general in* $B_{\rm reg}$ *is stable, fitting in an exact sequence*

$$0 \to \mathcal{O}_C(p) \to \mathcal{F} \to L \to 0,$$

where $p \in C$ *and* $L \in W^0_{d-1}$ *are general and where* $h^0(\mathcal{F}) = 2$.

(ii) $B_{\rm sup}$ *is generically smooth, of dimension* $d+2g-2\nu-2 > \rho^2_d$ (superabundant *for short). Moreover,* $\mathcal{F}$ *general in* $B_{\rm sup}$ *is stable, fitting in an exact sequence*

$$0 \to A \to \mathcal{F} \to L \to 0,$$

where L *is a general line bundle of degree* $d - \nu$ *and* $h^0(\mathcal{F}) = 2$.

A more precise statement of this result is given in Theorem 3.1 for its *residual* version (i.e. concerning the isomorphic Brill Noether locus B^{2g-d}_{4g-4-d}). Indeed, for any non negative integer i, if one sets $k_i := d - 2g + 2 + i$ and

$$B^{k_i}_d := \{\mathcal{F} \in U_C(2,d) \mid h^0(\mathcal{F}) \geq k_i\} = \{\mathcal{F} \in U_C(2,d) \mid h^1(\mathcal{F}) \geq i\},$$

one has natural isomorphisms $B^{k_i}_d \simeq B^i_{4g-4-d}$, arising from the correspondence $\mathcal{F} \to \omega_C \otimes \mathcal{F}^*$, Serre duality and semistability (cf. Sect. 2.2). The key ingblackients of our approach are the geometric theory of extensions introduced by Atiyah, Newstead, Lange-Narasimhan et al. (cf. e.g.[5]), Theorem 2.3 below and suitable parametric computations involving special and effective quotient line bundles and related families of sections of ruled surfaces, which make sense in the set-up of Theorem 3.1. Finally, by Theorems 1.1 and 1.2, we can also see that a general vector bundle in $B_{\rm reg}$ admits a special section whose zero locus is of degree one while its general section has no zeros (cf. the proof of [10, Theorem] and Remark 3.14 (ii) below).

For standard terminology, we refer the reader to [4].

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2. Preliminaries

2.1. **Preliminary results on general ν-gonal curves.** In this section we will review some results concerning line bundles on general ν-gonal curves, which will be used in the paper.

Lemma 2.1. *(cf.* [7, Corollary 1]*) On a general* ν*-gonal curve of genus* $g \geq 2\nu - 2$, *with* $\nu \geq 3$, *there does not exist a* $g^r_{\nu-2+2r}$ *with* $\nu - 2 + 2r \leq g - 1$, $r \geq 2$.

The *Clifford index* of a line bundle L on a curve C is defined by

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(L) - 1).$$

Theorem 2.2 ([8], Theorem 2.1)**.** *Let C be a general ν-gonal curve of genus $g \geq 4$, $\nu \geq 4$, and let g^1_ν be the unique pencil of degree ν on C. If C has a line bundle L with $\mathrm{Cliff}(L) \leq \frac{g-4}{2}$ and $degL \leq g-1$, then $|L| = (dim|L|)g^1_\nu + B$, for some effective divisor B.*

2.2. **Segre invariant and semistable vector bundles.** Given a rank 2 vector bundle $\mathcal{F}$ on C, the *Segre invariant* $s_1(\mathcal{F}) \in \mathbb{Z}$ of $\mathcal{F}$ is defined by

$$s_1(\mathcal{F}) = \min_{N \subset \mathcal{F}} \{\deg\mathcal{F} - 2\deg N\},$$

where N runs through all the sub-line bundles of $\mathcal{F}$. It easily follows from the definition that $s_1(\mathcal{F}) = s_1(\mathcal{F} \otimes L)$, for any line bundle L, and $s_1(\mathcal{F}) = s_1(\mathcal{F}^*)$, where $\mathcal{F}^*$ denotes the dual bundle of $\mathcal{F}$. A sub-line bundle $N \subset \mathcal{F}$ is called a *maximal sub-line bundle* of $\mathcal{F}$ if $\deg N$ is maximal among all sub-line bundles of $\mathcal{F}$; in such a case $\mathcal{F}/N$ is a *minimal quotient line bundle* of $\mathcal{F}$, i.e. is of minimal degree among quotient line bundles of $\mathcal{F}$. In particular, $\mathcal{F}$ is *semistable* (resp. *stable*) if and only if $s_1(\mathcal{F}) \geq 0$ (resp. $s_1(\mathcal{F}) > 0$).

2.3. **Extensions, secant varieties and semistable vector bundles.** Let δ be a positive integer. Consider $L \in \mathrm{Pic}^\delta(C)$ and $N \in \mathrm{Pic}^{d-\delta}(C)$. The extension space $\mathrm{Ext}^1(L, N)$ parametrizes isomorphism classes of extensions and any element $u \in \mathrm{Ext}^1(L, N)$ gives rise to a degree d, rank 2 vector bundle $\mathcal{F}_u$, fitting in an exact sequence

$$(u): \quad 0 \to N \to \mathcal{F}_u \to L \to 0. \tag{2.1}$$

We fix once and for all the following notation:

$$\begin{aligned} j &:= h^1(L), \qquad l := h^0(L) = \delta - g + 1 + j, \\ r &:= h^1(N), \quad n := h^0(N) = d - \delta - g + 1 + r \end{aligned} \tag{2.2}$$

In order to get $\mathcal{F}_u$ semistable, a necessary condition is

$$2\delta - d \geq s_1(\mathcal{F}_u) \geq 0. \tag{2.3}$$

In such a case, the Riemann-Roch theorem gives

$$\dim(\mathrm{Ext}^1(L, N)) = \begin{cases} 2\delta - d + g - 1 & \text{if } L \ncong N \\ g & \text{if } L \cong N. \end{cases} \tag{2.4}$$

Since we deal with *special* vector bundles, i.e. $h^1(\mathcal{F}_u) > 0$, they always admit a special quotient line bundle. Recall the following:

Theorem 2.3 ([3], Lemma 4.1)**.** *Let $\mathcal{F}$ be a semistable, special, rank 2 vector bundle on C of degree $d \geq 2g-2$. Then there exist a special, effective line bundle L on C of degree $\delta \leq d$, $N \in \mathrm{Pic}^{d-\delta}(C)$ and $u \in \mathrm{Ext}^1(L, N)$ such that $\mathcal{F} = \mathcal{F}_u$ as in 2.1.*

Tensor (2.1) by N^{-1} and consider $\mathcal{G}_e := \mathcal{F}_u \otimes N^{-1}$, which fits in

$$(e): \quad 0 \to \mathcal{O}_C \to \mathcal{G}_e \to L - N \to 0,$$

where $e \in \mathrm{Ext}^1(L - N, \mathcal{O}_C)$, so $\deg(\mathcal{G}_e) = 2\delta - d$. Then (u) and (e) define the same point in $\mathbb{P} := \mathbb{P}(H^0(K_C + L - N)^*)$. When the map $\varphi := \varphi_{|K_C+L-N|} : C \to \mathbb{P}$ is a morphism, set $X := \varphi(C) \subset \mathbb{P}$. For any positive integer h denote by $\mathrm{Sec}_h(X)$ the h^{st}-secant variety of X, defined as the closure of the union of all linear subspaces $\langle\varphi(D)\rangle \subset \mathbb{P}$, for general divisors D of degree h on C. One has

$$\dim(\mathrm{Sec}_h(X)) = \min\{\dim(\mathbb{P}), 2h-1\}.$$

Theorem 2.4. *(*[5, Proposition 1.1]*) Let $2\delta - d \geq 2$; then φ is a morphism and, for any integer $s \equiv 2\delta - d \pmod 2$ such that $4 + d - 2\delta \leq s \leq 2\delta - d$, one has*

$$s_1(\mathcal{E}_e) \geq s \Leftrightarrow e \notin \mathrm{Sec}_{\frac{1}{2}(2\delta-d+s-2)}(X).$$

3. The main result

In this section C will denote a general ν-gonal curve of genus $g \geq 4$ and A the unique line bundle of degree ν with $h^0(A) = 2$. As explained in the Introduction, from now on we will be concerned with the residual version of Theorem 1.2; therefore we set

$$3 \leq \nu \leq \frac{g+8}{4} \ \text{ and } \ 3g-1 \leq d \leq 4g-6-2\nu, \tag{3.1}$$

where d is an integer. For suitable line bundles L and N on C, we consider rank 2 vector bundles $\mathcal{F}$ arising as extensions. We will give conditions on L and N under which $\mathcal{F}$ is general in a certain component of the Brill-Noether locus $B_d^{k_2}$, where $k_2 = d - 2g + 4$ as in Introduction. We moreover show that L is a quotient of $\mathcal{F}$ with suitable *minimality* properties. Finally, we prove the following theorem.

Theorem 3.1. *The reduced components of $B_d^{k_2}$ having dimension at least $\rho_d^{k_2}$ are only two, which we denote by B_{reg} and B_{sup}:*

(i) *The component B_{reg} is* regular*, i.e. generically smooth and of dimension $\rho_d^{k_2} = 8g - 2d - 11$. A general element $\mathcal{F}$ of B_{reg} is stable, fitting in an exact sequence*

$$0 \to K_C - D \to \mathcal{F} \to K_C - p \to 0, \tag{3.2}$$

where $p \in C$ and $D \in C^{(4g-5-d)}$ are general. Specifically, $s_1(\mathcal{F}) \geq 1$ (resp., 2) if d is odd (resp., even). Moreover, $K_C - p$ is minimal among special quotient line bundles of $\mathcal{F}$ and $\mathcal{F}$ is very ample for $\nu \geq 4$;

(ii) *The component B_{sup} is generically smooth, of dimension $6g - d - 2\nu - 6 > \rho_d^{k_2}$, i.e. B_{sup} is* superabundant*. A general element $\mathcal{F}$ of B_{sup} is stable, very-ample, fitting in an exact sequence*

$$0 \to N \to \mathcal{F} \to K_C - A \to 0, \tag{3.3}$$

for $N \in \mathrm{Pic}^{d-2g+2+\nu}(C)$ general. Moreover, $s_1(\mathcal{F}) = 4g - 4 - d - 2\nu$ and $K_C - A$ is a minimal quotient of $\mathcal{F}$.

Proof. In Sect. 3.1 and 3.2 we will construct the components B_{sup} and B_{reg}, respectively, and prove all the statements in Theorem 3.1 except for the minimality property of $K_C - p$ in (i) and the uniqueness of B_{sup} and B_{reg}, which will be proved in Sect. 3.3. □

Remark 3.2. (i) As explained in the Introduction, Theorem 3.1 and the natural isomorphism $B_d^{k_2} \simeq B_{4g-4-d}^2$ give also a proof of Theorem 1.2.

(ii) It is well-known how the study of rank 2 vector bundles on curves is related to that of (surface) scrolls in projective space. Therefore, very-ampleness condition in Theorem 3.1 is a key for the study of components of Hilbert schemes of smooth scrolls, in a suitable projective space, dominating $\mathcal{M}_{g,\nu}^1$. This will be the subject of a forthcoming paper.

3.1. **The superabundant component** $B_{\rm sup}$**.** In this section we first construct the component $B_{\rm sup}$ as in Theorem 3.1 . We consider the line bundle $L := K_C - A \in W^{g-\nu}_{2g-2-\nu}$ and a general $N \in {\rm Pic}^{d-2g+2+\nu}(C)$; since $d-2g+2+\nu \geq g+1+\nu$ from (3.1), in particular $h^1(N) = 0$. We first need the following preliminary result.

Lemma 3.3. *Let* $N \in {\rm Pic}^{d-2g+2+\nu}(C)$ *be general. Then, for a general* $u \in {\rm Ext}^1(K_C - A, N)$*, the corresponding rank* 2 *vector bundle* $\mathcal{F}_u$ *is stable with:*

(a) $h^1(\mathcal{F}_u) = h^1(K_C - A) = 2$*;*
(b) $s_1(\mathcal{F}_u) = 4g - 4 - 2\nu - d$*; more precisely,* $K_C - A$ *is a minimal quotient line bundle of* $\mathcal{F}_u$*;*
(c) $\mathcal{F}_u$ *is very ample.*

Proof. To ease notation, set $L = K_C - A$ and $\delta := {\rm deg} L$. To show that $\mathcal{F}_u$ is stable, note that the upper bound on d in (3.1) implies $2\delta - d = 2(2g - 2 - \nu) - d \geq 2$; so we are in position to apply Theorem 2.4. We consider the natural morphism

$$\varphi := \varphi_{|K_C+L-N|} : C \longrightarrow \mathbb{P} := \mathbb{P}({\rm Ext}^1(L, N)).$$

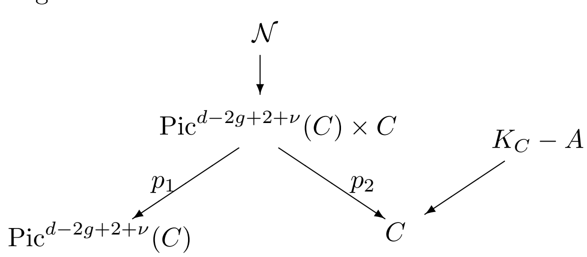
Set $X := \varphi(C)$. Let s be an integer such that $s \equiv 2\delta - d \pmod 2$ and $0 < s \leq 2\delta - d$. Since $s \leq 2\delta - d = 4g - 4 - 2\nu - d < g - 3$, we have

$$\dim\left({\rm Sec}_{\frac{1}{2}(2\delta-d+s-2)}(X)\right) = 2\delta - d + s - 3 < 2\delta - d + g - 2 = \dim(\mathbb{P}),$$

where the last equality follows from (2.4) and $L \ncong N$. One can therefore take $s = 2\delta - d$, so that the general $\mathcal{F}_u$ arising from (3.3) is of degree d, with $h^1(\mathcal{F}_u) = h^1(L) = 2$ and it is stable, since $s_1(\mathcal{F}_u) = 2\delta - d = 4g - 4 - 2\nu - d \geq 2$; the equality $s_1(\mathcal{F}_u) = 2\delta - d$ follows from Theorem 2.4 and from (3.3). This proves the stability of $\mathcal{F}_u$ together with (a) and (b).

Finally, to prove (c), observe first that $K_C - A$ is very ample: indeed, if $K_C - A$ is not very ample, by the Riemann-Roch theorem there exists a $g^2_{\nu+2}$ on C; this is contrary to Lemma 2.1, since the hypothesis $3 \leq \nu \leq \frac{g+8}{4}$ implies $g \geq 2\nu - 2 + (2\nu - 6) \geq 2\nu - 2$. At the same time, since ${\rm deg}(N) = d - 2g + 2 + \nu \geq g + 4$ by (3.1), a general N is also very ample. Thus any $\mathcal{F}_u$ as in (3.3) is very ample too. □

We now want to show that vector bundles constructed in Lemma 3.3 fill up the component $B_{\rm sup}$, as N varies in ${\rm Pic}^{d-2g+2+\nu}(C)$. To do this, we need to consider a parameter space of rank 2 vector bundles on C, arising as extensions of $K_C - A$ by N, as N varies. If $\mathcal{N} \to {\rm Pic}^{d-2g+2+\nu}(C) \times C$ is a Poincaré line bundle, we have the following diagram:



Set $\mathcal{E}_{d,\nu} := R^1 p_{1*}(\mathcal{N} \otimes p_2^*(A - K_C))$. By [2, pp. 166-167], $\mathcal{E}_{d,\nu}$ is a vector bundle on a suitable open, dense subset $S \subseteq {\rm Pic}^{d-2g+2+\nu}(C)$ of rank $\dim {\rm Ext}^1(K_C - A, N) = 5g - 5 - 2\nu - d$ as in (2.4), since $K_C - A \ncong N$. Consider the projective bundle $\mathbb{P}(\mathcal{E}_{d,\nu}) \to S$, which is the family of $\mathbb{P}\left({\rm Ext}^1(K_C - A, N)\right)$'s as N varies in S. One has

$$\dim \mathbb{P}(\mathcal{E}_{d,\nu}) = g + (5g - 5 - 2\nu - d) - 1 = 6g - 6 - 2\nu - d.$$

Consider the natural (rational) map

$$\begin{array}{rcl} \mathbb{P}(\mathcal{E}_{d,\nu}) & \stackrel{\pi_{d,\nu}}{\dashrightarrow} & U_C(2,d) \\ (N,u) & \to & \mathcal{F}_u; \end{array}$$

from Lemma 3.3 we know that $\mathrm{im}(\pi_{d,\nu}) \subseteq B_d^{k_2} \cap U_C^s(2,d)$.

Proposition 3.4. *The closure* $B_{\sup}$ *of* $\mathrm{im}(\pi_{d,\nu})$ *in* $U_C(2,d)$ *is a generically smooth component of* $B_d^{k_2}$*, having dimension* $6g-6-2\nu-d$*. In particular* $B_{\sup}$ *is* superabundant*.*

Proof. The result will follow once we prove that

$$\dim T_{\mathcal{F}}(B_d^{k_2}) = \dim B_{\sup},$$

for a general $\mathcal{F}$ in $\mathrm{im}(\pi_{d,\nu})$. First we claim that $\dim B_{\sup} = 6g-6-2\nu-d$. Indeed, let $\Gamma \subset F = \mathbb{P}(\mathcal{F}_u)$ be the section corresponding to the quotient $\mathcal{F}_u \twoheadrightarrow K_C - A$. Its normal bundle is $N_{\Gamma/F} \simeq K_C - A - N$ (cf. [4, Sect. V, Prop. 2.9]); since N is general of degree at least $g+4$ by (3.1), we have $h^0(K_C - A - N) = 0$; in other words Γ is an algebraically isolated section of F. This guarantees that $\pi_{d,\nu}$ is generically finite (for more details see the proof of [3, Lemma 6.2] and apply the same arguments). Hence we get $\dim \mathrm{im}(\pi_{d,\nu}) = 6g-6-2\nu-d$.

Now we prove that $\dim T_{\mathcal{F}}(B_d^{k_2}) = 6g-6-2\nu-d$. To show this, consider the Petri map of a general $\mathcal{F} \in \mathrm{im}(\pi_{d,\nu})$:

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*).$$

By (3.3) and $h^1(N) = 0$, we have

$$H^0(\mathcal{F}) \simeq H^0(N) \oplus H^0(K_C - A) \quad \text{and} \quad H^0(\omega_C \otimes \mathcal{F}^*) \simeq H^0(A).$$

Thus $\mu_{\mathcal{F}}$ reads as

$$\big(H^0(N) \oplus H^0(K_C - A)\big) \otimes \ H^0(A) \ \stackrel{\mu_{\mathcal{F}}}{\longrightarrow} \ H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*).$$

Consider the following natural multiplication maps:

$$\mu_{A,N} : \quad H^0(N) \otimes H^0(A) \to H^0(N+A) \tag{3.4}$$

$$\mu_{0,A} : \quad H^0(K_C - A) \otimes H^0(A) \to H^0(K_C). \tag{3.5}$$

Claim 3.5. $\ker(\mu_{\mathcal{F}}) \simeq \ker(\mu_{0,A}) \oplus \ker(\mu_{A,N})$.

Proof of Claim 3.5. Consider the exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N+A-K_C & \to & \mathcal{F}\otimes(A-K_C) & \to & \mathcal{O}_C & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N\otimes\mathcal{F}^* & \to & \mathcal{F}\otimes\mathcal{F}^* & \to & (K_C-A)\otimes F^* & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_C & \to & \mathcal{F}\otimes N^{-1} & \to & (K_C-A)\otimes N^{-1} & & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ;
\end{array}$$

which arises from (3.3) and its dual sequence $0 \to A-K_C \to \mathcal{F}^* \simeq \mathcal{F}(A-K_C-N) \to N^{-1} \to 0$. If we tensor the column in the middle by ω_C, we get $H^0(\mathcal{F}\otimes A) \hookrightarrow H^0(\omega_C\otimes\mathcal{F}\otimes\mathcal{F}^*)$.

Observe moreover that $H^0(N+A)\oplus H^0(K_C) \simeq H^0(\mathcal{F}\otimes A)$, which follows from (3.3) tensored by A and the fact that $h^1(N+A)=0$. Therefore there is no intersection between $\mathrm{im}(\mu_{0,A})$ and $\mathrm{im}(\mu_{A,N})$ and the statement is proved. □

By Claim 3.5,

$$\begin{aligned}
\dim T_{\mathcal{F}}(B_d^{k_2}) &= 4g-3-h^0(\mathcal{F})h^1(\mathcal{F})+\dim(\ker\mu_{\mathcal{F}})\\
&= 4g-3-2(d-2g+4)+\dim(\ker(\mu_0(A)))+\dim(\ker(\mu_{A,N})).
\end{aligned}$$

From (3.4) and (3.5), we have

$$\ker(\mu_{0,A}) \simeq H^0(K_C-2A) \cong H^1(2A)^* \ \text{ and } \ \ker(\mu_{A,N}) \simeq H^0(N-A),$$

as it follows from the base point free pencil trick. Under the numerical assumption $\nu \leq \frac{g+8}{4}$, from Theorem 2.2 we have $h^0(2A)=3$, which implies $h^1(2A)=g+2-2\nu$. The inequality $\deg N \geq g+1+\nu$ given by (3.1) and the generality of N show that $h^1(N-A)=0$, which yields $h^0(N-A)=d-3g+3$. So we have

$$\begin{aligned}
\dim T_{\mathcal{F}}(B_d^{k_2}) &= 4g-3-2(d-2g+4)+(g+2-2\nu)+(d-3g+3)\\
&= 6g-6-2\nu-d = \dim B_{\mathrm{sup}}.
\end{aligned}$$

To complete the proof, it suffices to observe that $\rho_d^{k_2} = 8g-11-2d \leq 5g-10-d < 6g-6-2\nu-d$, as it follows by (3.1). □

3.2. **The regular component** B_{reg}**.** In this subsection we construct the regular component B_{reg} as in Theorem 3.1. In what follows, we use notation as in (2.2), i.e. $l=h^0(L)$, $j=h^1(L)$, $r=h^1(N)$ which will be considered all positive (cf. Theorem 2.3 for L). For any exact sequence (u) as in (2.1), let $\partial_u : H^0(L) \to H^1(N)$ be the corresponding coboundary map. For any integer $t>0$, consider

$$\mathcal{W}_t := \{u \in \mathrm{Ext}^1(L,N) \mid \mathrm{corank}(\partial_u) \geq t\} \subseteq \mathrm{Ext}^1(L,N), \tag{3.6}$$

which has a natural structure of determinantal scheme; its expected codimension is $t(l-r+t)$ (cf. [3, Sect. 5.2]). In this set–up, one has:

Theorem 3.6. *([3, Theorem 5.8 and Corollary 5.9]) Let C be a smooth curve of genus $g \geq 3$. Let*

$$r = h^1(N) \geq 1, \; l = h^0(L) \geq \max\{1, r-1\}, \; m := \dim(\mathrm{Ext}^1(L,N)) \geq l+1.$$

Then, we have:

(i) $l - r + 1 \geq 0$;
(ii) $\mathcal{W}_1$ *is irreducible of (expected) dimension* $m - c(l, r, 1)$;
(iii) *if* $l \geq r$, *then* $\mathcal{W}_1 \subset \mathrm{Ext}^1(L,N)$. *Moreover for general* $u \in \mathrm{Ext}^1(L,N)$, ∂_u *is surjective whereas for general* $w \in \mathcal{W}_1$, $\mathrm{corank}(\partial_w) = 1$.

To construct B_{reg}, observe firts that by (3.1) W^0_{4g-5-d} is not empty, irreducible and $h^0(D) = 1$, for general $D \in W^0_{4g-5-d}$. We will prove the following preliminary result.

Lemma 3.7. *Let $D \in W^0_{4g-5-d}$ and $p \in C$ be general and let $\mathcal{W}_1 \subseteq \mathrm{Ext}^1(K_C - p, K_C - D)$ be as in (3.6). Then, for $u \in \mathcal{W}_1$ general, the corresponding rank 2 vector bundle $\mathcal{F}_u$ is stable, with:*

(a) $h^1(\mathcal{F}_u) = 2$;
(b) $s_1(\mathcal{F}) \geq 1$ *(resp., 2) if d is odd (resp., even);*
(c) $\mathcal{F}_u$ *is very ample when* $\nu \geq 4$.

Proof. From the assumptions we have:

$$(3.7) \qquad \begin{array}{lccccccc} (u)\colon 0 \to & K_C - D & \to & \mathcal{F} & \to & K_C - p & \to 0 \\ \deg & d-2g+3 & & d & & 2g-3 & \\ h^0 & d-3g+5 & & & & g-1 & \\ h^1 & 1 & & & & 1 & \end{array}$$

By (3.1) $\deg D = 4g - d - 5 \geq 2\nu + 1$, therefore $K_C - D \ncong K_C - p$; thus, using (2.4) and notation as in Theorem 3.6, one has

$$l = g-1, \; r = 1 \; \text{ and } \; m = \dim \mathrm{Ext}^1(K_C - p, K_C - D) = 5g - 7 - d.$$

By (3.1), one has $d \leq 4g - 7$ so $m \geq l + 1 = g$. Hence we can apply Theorem 3.6 to

$$\mathcal{W}_1 = \{u \in \mathrm{Ext}^1(K_C - p, K_C - D) \mid \mathrm{corank}(\partial_u) \geq 1\},$$

which therefore is irreducible, of (expected) dimension $\dim \mathcal{W}_1 = m - 1(l - r + 1) = 4g - 6 - d$. Moreover, by Theorem 3.6 (iii) and formula (3.7), for general $u \in \mathcal{W}_1$ one has $h^1(\mathcal{F}_u) = 2$, which proves (a).

We now want to show that $\mathcal{F}_u$ satisfies also (b), for $u \in \mathcal{W}_1$ general; in particular it is stable. To do this, set $\mathbb{P} := \mathbb{P}\left(\mathrm{Ext}^1(K_C - p, K_C - D)\right)$ and consider the projective scheme $\widehat{\mathcal{W}_1} := \mathbb{P}(\mathcal{W}_1) \subset \mathbb{P}$, which has therefore dimension $4g - 7 - d$. Posing $\delta := 2g - 3$ and considering (3.1), one has $2\delta - d \geq 2\nu \geq 6$. We are therefore in position to apply Theorem 2.4. We consider the natural morphism $C \xrightarrow{\varphi} \mathbb{P}$, given by the complete linear system $|K_C + D - p|$. Set $X = \varphi(C)$, as in the proof of Lemma 3.3. Let s be an integer such that $s \equiv 2\delta - d \pmod 2$ and $0 \leq s \leq 2\delta - d$. Then we have

$$\dim \mathrm{Sec}_{\frac{1}{2}(2\delta - d + s - 2)}(X) = 2\delta - d + s - 3 = 4g - 9 - d + s \leq 4g - 7 - d = \dim \widehat{\mathcal{W}_1}$$

if and only if $s \leq 2$, where the equality holds if and only if $s = 2$.

Therefore, for d odd, by Theorem 2.4 one has $s_1(\mathcal{F}_u) \geq 1$ for $u \in \mathcal{W}_1$ general; in particular $\mathcal{F}_u$ is stable and (b) is proved in this case.

For d even, if one dualizes the exact sequence (3.2) and tensors via ω_C, one gets

$$(e): \quad 0 \to p \to \mathcal{E}_e := \mathcal{F}_u^* \otimes \omega_C \to D \to 0,$$

where (e) defines the same point as (u) in the projective space $\mathbb{P}$; in particular $s_1(\mathcal{F}_u) = s_1(\mathcal{E}_e)$ (cf. Sect. 2.2) and $h^0(\mathcal{E}_e) = 2$, by Serre duality and the fact that $(u) \in \widehat{\mathcal{W}}_1$. Following the same strategy as in the first part of the proof of [10, Theorem], one deduces that (e) belongs to the linear span $\langle \varphi(D) \rangle \subset \mathbb{P}$. On the other hand, any point $x \in \langle \varphi(D) \rangle$ gives rise to an extension:

$$(x): \quad 0 \to p \to \mathcal{E}_x \to D \to 0$$

which belongs to $\widehat{\mathcal{W}}_1$, since $h^0(\mathcal{E}_x) = 2$ (cf. diagram (2) and the subsequent details in the proof of [10, Theorem]). Thus $\langle \varphi(D) \rangle \subseteq \widehat{\mathcal{W}}_1$. By the Riemann-Roch theorem,

$$\dim \langle \varphi(D) \rangle = h^0(K_C + D - p) - h^0(K_C - p) - 1 = 4g - 7 - d = \dim \widehat{\mathcal{W}}_1.$$

Since they are both closed and irreducible, one gets $\widehat{\mathcal{W}}_1 = \langle \varphi(D) \rangle$. On the other hand

$$\mathrm{Sec}_{\frac{1}{2}(2\delta - d + 2 - 2)}(X) = \mathrm{Sec}_{\frac{1}{2}(4g-6-d)}(X),$$

which is of dimension $4g - 7 - d$ too, is non-degenerate in $\mathbb{P}$ as $X \subset \mathbb{P}$ is not. Thus, we conclude that $\widehat{\mathcal{W}}_1 \neq \mathrm{Sec}_{\frac{1}{2}(4g-6-d)}(X)$. In particular, from Theorem 2.4, for a general $u \in \widehat{\mathcal{W}}_1$ one has $s_1(\mathcal{F}_u) \geq 2$, so $\mathcal{F}_u$ is stable and (b) is proved also in this case.

To prove (c) observe first that, since $\nu \geq 4$ by assumption, then $K_C - p$ is very ample as it follows by the Riemann-Roch theorem. Now:

Claim 3.8. *For general $D \in W^0_{4g-5-d}$, $K_C - D$ is very ample if $\nu \geq 4$.*

Proof of Claim 3.8. Assume by contradiction that $K_C - D$ is not very ample for general $D \in W^0_{4g-5-d}$. For a non-negative integer τ, define the following:

$$\Xi_\tau := \{(D, p+q) \in W^0_{4g-5-d} \times W^0_2 \mid h^0(D + p + q) = \tau + 1\}.$$

If $\Xi_\tau \neq \emptyset$, then we have the diagram:

$$\begin{array}{ccc} & \Xi_\tau & \\ {\scriptstyle \pi_\tau} \swarrow & & \searrow {\scriptstyle \wp_\tau} \\ W^0_{4g-5-d} & & W^\tau_{4g-3-d} \end{array}$$

which is given by $\pi_\tau(D, p+q) := D$ and $\wp_\tau(D, p+q) := D + p + q$. The assumption implies that, for some $\tau \in \{1, 2\}$, the image of π_τ is dense in W^0_{4g-5-d}. Considering the map $\wp_\tau$, we get $\dim \Xi_\tau \leq \dim W^\tau_{4g-3-d} + \tau$. By Martens' and Mumford's Theorems (cf. [2, Thm. (5.1), (5.2)]), we have $\dim W^\tau_{4g-3-d} \leq 4g - 5 - d - 2\tau$, since C is a general ν-gonal curve with $\nu \geq 4$ and $4g - 3 - d \leq g - 2$ by (3.1). In sum, it turns out that

$$\dim W^0_{4g-5-d} \leq \dim \Xi_r \leq 4g - 5 - d - \tau,$$

which cannot occur. This completes the proof of the claim. □

The above arguments prove (c) and complete the proof of the Lemma. □

To construct the component $B_{\rm reg}$ notice that, as in Sect. 3.1, one has a projective bundle $\mathbb{P}(\mathcal{E}_d) \to S$ where $S \subseteq W^0_{4g-5-d} \times C$ is a suitable open dense subset: $\mathbb{P}(\mathcal{E}_d)$ is the family of $\mathbb{P}(\mathrm{Ext}^1(K_C-p, K_C-D))$'s as $(D,p) \in S$ varies. Since, for any such $(D,p) \in S$, $\widehat{\mathcal{W}}_1$ is irreducible of constant dimension $4g-7-d$, one has an irreducible subscheme $\widehat{\mathcal{W}}_1^{Tot} \subset \mathbb{P}(\mathcal{E}_d)$ which has therefore dimension

$$\dim \widehat{\mathcal{W}}_1^{Tot} = \dim S + 4g - 7 - d = 4g - d - 4 + 4g - 7 - d = 8g - 2d - 11 = \rho_d^{k_2}.$$

From Lemma 3.7, one has the natural (rational) map

$$\begin{array}{ccc} \widehat{\mathcal{W}}_1^{Tot} & \stackrel{\pi}{\dashrightarrow} & U_C(d) \\ (D,p,u) & \longrightarrow & \mathcal{F}_u; \end{array}$$

and $\mathrm{im}(\pi) \subset B_d^{k_2} \cap U_C^s(2,d)$.

Proposition 3.9. *The closure $B_{\rm reg}$ of $\mathrm{im}(\pi)$ in $U_C(2,d)$ is a generically smooth component of $B_d^{k_2}$ with dimension $\rho_d^{k_2} = 8g - 11 - 2d$, i.e. $B_{\rm reg}$ is regular.*

Proof. From the fact that $\mathrm{im}(\pi)$ contains stable bundles, any component of $B_d^{k_2}$ containing it has dimension at least $\rho_d^{k_2}$. We concentrate in computing $\dim T_{\mathcal{F}}(B_d^{k_2})$, for general $\mathcal{F} \in \mathrm{im}(\pi)$. Consider the Petri map

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*)$$

for a general $\mathcal{F} \in \mathrm{im}(\pi)$. From diagram (3.7) and the fact that $\mathcal{F} = \mathcal{F}_u$, for some u in some fiber $\widehat{\mathcal{W}}_1$ of $\widehat{\mathcal{W}}_1^{Tot}$, one has that the corresponding coboundary map ∂_u is the zero-map; in other words

$$H^0(\mathcal{F}) \cong H^0(K_C - D) \oplus H^0(K_C - p) \quad \text{and} \quad H^1(\mathcal{F}) \cong H^1(K_C - D) \oplus H^1(K_C - p).$$

This means that, for any such bundle, the domain of the Petri map $\mu_{\mathcal{F}}$ coincides with that of $\mu_{\mathcal{F}_0}$, where $\mathcal{F}_0 := (K_C - D) \oplus (K_C - p)$ corresponds to the zero vector in $\mathcal{W}_1 \subset \mathrm{Ext}^1(K_C - p, K_C - D)$. We will concentrate on $\mu_{\mathcal{F}_0}$; observe that

$$\begin{array}{rcl} H^0(\mathcal{F}_0) \otimes H^0(\omega_C \otimes \mathcal{F}_0^*) & \cong & \big(H^0(K_C - D) \otimes H^0(D)\big) \oplus \big(H^0(K_C - D) \otimes H^0(p)\big) \oplus \\ & & \big(H^0(K_C - p) \otimes H^0(D)\big) \oplus \big(H^0(K_C - p) \otimes H^0(p)\big). \end{array}$$

Moreover

$$\omega_C \otimes \mathcal{F}_0 \otimes \mathcal{F}_0^* \cong K_C \oplus (K_C + p - D) \oplus (K_C + D - p) \oplus K_C.$$

Therefore, for Chern classes reason,

$$\mu_{\mathcal{F}_0} = \mu_{0,D} \oplus \mu_{K_C - D, p} \oplus \mu_{K_C - p, D} \oplus \mu_{0,p}$$

where the maps

$$\begin{array}{rc} \mu_{0,D} : & H^0(D) \otimes H^0(K_C - D) \to H^0(K_C), \\ \mu_{K_C - D, p} : & H^0(K_C - D) \otimes H^0(p) \to H^0(K_C - D + p) \\ \mu_{K_C - p, D} : & H^0(K_C - p) \otimes H^0(D) \to H^0(K_C + D - p) \\ \mu_{0,p} : & H^0(p) \otimes H^0(K_C - p) \to H^0(K_C) \end{array}$$

are natural multiplication maps. Since $h^0(D) = h^0(p) = 1$, the maps $\mu_{0,D},\ \mu_{K_C-D,p},\ \mu_{K_C-p,D}\ \mu_{0,p}$ are all injective and so is $\mu_{\mathcal{F}_0}$. By semicontinuity on $\mathcal{W}_1$, one has that $\mu_{\mathcal{F}}$ is injective, for $\mathcal{F}$ general in $\widehat{\mathcal{W}}_1$.

The previous argument shows that a general $\mathcal{F} \in \mathrm{im}(\pi)$ is contained in only one irreducible component, say B_{reg}, of $B_d^{k_2}$ for which

$$\begin{aligned}\dim B_{\mathrm{reg}} = \dim T_{\mathcal{F}}(B_{\mathrm{reg}}) &= 4g-3-h^0(\mathcal{F})h^1(\mathcal{F}) \\ &= 4g-3-2(d-2g+4) = 8g-11-2d,\end{aligned}$$

i.e. B_{reg} is generically smooth and of dimension $\rho_d^{k_2}$.

To conclude that B_{reg} is the closure of $\mathrm{im}(\pi)$, it suffices to show that the rational map π is generically finite onto its image. To do this, let $F = \mathbb{P}(\mathcal{F}_u)$ be the ruled surface, for general $\mathcal{F}_u \in \widehat{\mathcal{W}}_1^{Tot}$, and let Γ be the section corresponding to the quotient $\mathcal{F}_u \twoheadrightarrow K_C - p$. Then its normal bundle is $N_{\Gamma/F} \simeq D - p$ which has no sections. Thus, one deduces the generically finiteness of π by reasoning as in the proof of Proposition 3.4. $\square$

3.3. **No other reduced components of dimension at least $\rho_d^{k_2}$.** In this section, we will show that no other reduced components of $B_d^{k_2}$, having dimension at least $\rho_d^{k_2} = 8g-11-2d$, exist except for B_{reg} and B_{sup} constructed in the previous sections.

Let $B \subset B_d^{k_2}$ be any reduced component with $\dim B \geq \rho_d^{k_2} = 8g-11-2d$; from Theorem 2.3, $\mathcal{F} \in B$ general fits in an exact sequence of the form

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{3.8}$$

where L is a special, effective line bundle of degree $\delta \leq d$, i.e. $l,\ j > 0$ and $h^1(\mathcal{F}) \geq 2$.

We first focus on the case of $h^1(\mathcal{F}) = 2$. We start with the following:

Proposition 3.10. *Let B be any reduced component of $B_d^{k_2}$, with $\dim B \geq \rho_d^{k_2}$. For $\mathcal{F}$ general in B, assume that it fits in an exact sequence like (3.8), with $h^1(\mathcal{F}) = h^1(L) = 2$. Then, B coincides with the component B_{sup} as in Sect. 3.1.*

Proof. Since $\mathcal{F}$ is semistable, from (2.3) and (3.1) one has $\deg L \geq \frac{3g-1}{2}$. Moreover, since C is a general ν-gonal curve and $h^1(L) = 2$, from [1, Theorem 2.6] we have $|\omega_C \otimes L^{-1}| = g^1_\nu + B_b$, where B_b is a base locus of degree b. Hence $L \simeq K_C - A - B_b$, where $b \leq \frac{g-3}{2} - \nu$. For simplicity, put $\delta := \deg L = 2g-2-\nu-b$ so $\deg N = d - \delta$.

Since B is reduced, one must have

$$\dim B = \dim T_{\mathcal{F}} B$$

for general $\mathcal{F} \in B$. We will prove the Proposition by showing that $\dim B = \dim T_{\mathcal{F}} B$ can occur only if $L = K_C - A$ and N is non-special, general of its degree.

Claim 3.11. $\dim B \leq \begin{cases} 6g-d-2\nu-6-b & \textit{if } h^1(N) = 0 \\ 9g-2d-3\nu-2r-2b-7 & \textit{if } h^1(N) \geq 1 \end{cases}$

Proof of Claim 3.11. We will use notation as in (2.2). Since B is irreducible, all integers in (2.2) are constant for a general $\mathcal{F} \in B$. From (3.8) combined with $L = K_C - A - B_b$, it follows there exists an open dense subset S of a closed subvariety of $\mathrm{Pic}^{d-\delta} \times C^{(b)}$ and a projective bundle $\mathcal{P} \to S$, whose general fiber identifies with $\mathbb{P} = \mathbb{P}(H^0(K_C + L - N)^*) = \mathbb{P}(\mathrm{Ext}^1(L, N)) \cong \mathbb{P}^{m-1}$, where $m := \dim(\mathrm{Ext}^1(L, N))$.

Since $h^1(\mathcal{F}) = h^1(L)$, as in [3, Sect. 6], the component B has to be the image of $\mathcal{P}$ via a dominant rational map

$$\begin{array}{ccc} \mathcal{P} & \stackrel{\pi}{\dashrightarrow} & B \subset B_d^{k_2} \\ \downarrow & & \\ S & & \end{array}$$

(cf. [3, Sect. 6] for details). Therefore we obtain $\dim B \leq \dim \mathcal{P} = \dim S + m - 1$ since $\mathcal{P}$ is a projective bundle over S whose general fiber is $(m-1)$-dimensional. Specifically, if $r \geq 1$ then S is a subset of $W_{d-\delta}^{d-\delta-g+r} \times C^{(b)}$, the latter being equivalent to $W_{2g-2+\delta-d}^{r-1} \times C^{(b)}$ by Serre duality, and $\dim W_{2g-2+\delta-d}^{r-1} \leq 2g-2+\delta-d-2(r-1)$ by using Martens' theorem (cf. [2, Theorem 5.1]) for $r \geq 2$. Therefore, we get

$$\dim S \leq \begin{cases} g+b & \text{if } r=0 \\ 2g-2+\delta-d-2r+2+b & \text{if } r \geq 1. \end{cases}$$

This inequality, combined with (2.4), gives

$$\dim B \leq \begin{cases} (g+b)+2\delta-d+g-2 & \text{if } r=0 \\ (2g-2+\delta-d-2r+2+b)+2\delta-d+g-1 & \text{if } r \geq 1, \end{cases}$$

since a non-special line bundle cannot be isomorphic to a special one. By substituting $\delta = 2g-2-\nu-b$, we get the conclusion of Claim 3.11. □

Claim 3.12. $\dim T_{\mathcal{F}}(B) \geq 6g-d-2\nu-2r-6$

Proof of Claim 3.12. The tangent space $T_{\mathcal{F}}(B)$ is the orthogonal space to the image of the Petri map:

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F}^* \otimes \mathcal{F}),$$

so $\dim T_{\mathcal{F}}(B) = \dim(\mathrm{im}(\mu_{\mathcal{F}})^{\perp}) = h^0(K_C \otimes F^* \otimes F) - h^0(\mathcal{F})h^1(\mathcal{F}) + \dim\ker\mu_{\mathcal{F}}$.

From the exact sequence (3.8), we get $H^0(\mathcal{F}) \simeq H^0(N) \oplus W$ where $W := \mathrm{im}(H^0(\mathcal{F}) \to H^0(L))$. Since $H^1(\mathcal{F}) \simeq H^1(L)$, the connecting homomorphism in (3.8) is surjective, hence $\dim W = l - r = h^0(L) - h^1(N)$. Let $\mu_{N,\omega_C\otimes L^{-1}}$ and $\mu_{0,W}$ be the maps defined as follows:

$$\begin{aligned} \mu_{N,\omega_C\otimes L^{-1}} \ &: \ H^0(N) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(N \otimes \omega_C \otimes L^{-1}) \\ \mu_{0,W} \ &: \ W \otimes H^0(\omega_C \otimes L^{-1}) \hookrightarrow H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C) \end{aligned}$$

Then we have

$$\dim\ker\mu_{\mathcal{F}} \geq \dim\ker\mu_{N,\omega_C\otimes L^{-1}} + \dim\ker\mu_{0,W} \tag{3.9}$$

by the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) & \xrightarrow{\mu_{\mathcal{F}}} & H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*) \\ & & \uparrow \alpha \qquad\qquad \uparrow \beta \\ \wr\| & & H^0(\omega_C \otimes L^{-1} \otimes N) \qquad H^0(\omega_C) \\ & & \uparrow \mu_{N,\omega_C\otimes L^{-1}} \qquad \uparrow \mu_{0,W} \\ (H^0(N) \oplus W) \otimes H^0(\omega_C \otimes L^{-1}) & \xrightarrow{\cong} & (H^0(N) \otimes H^0(\omega_C \otimes L^{-1})) \oplus (W \otimes H^0(\omega_C \otimes L^{-1})), \end{array}$$

where the map β comes from the trivial section of $H^0(\mathcal{F}\otimes\mathcal{F}^*)$ after tensoring via ω_C; to explain the map α, if one takes the diagram determined by the exact sequence (3.8) and its dual sequence and tensor it by ω_C, one gets:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \omega_C\otimes N\otimes L^{-1} & \to & \omega_C\otimes\mathcal{F}\otimes L^{-1} & \to & \omega_C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \omega_C\otimes N\otimes\mathcal{F}^* & \to & \omega_C\otimes\mathcal{F}\otimes\mathcal{F}^* & \to & \omega_C\otimes L\otimes\mathcal{F}^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \omega_C & \to & \omega_C\otimes\mathcal{F}\otimes N^{-1} & \to & \omega_C\otimes L\otimes N^{-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & ; \end{array}$$

the map α is the composition of the two injections

$$H^0(\omega_C\otimes N\otimes L^{-1})\hookrightarrow H^0(\omega_C\otimes\mathcal{F}\otimes L^{-1})\hookrightarrow H^0(\omega_C\otimes\mathcal{F}\otimes\mathcal{F}^*).$$

Since $K_C-L=A+B_b$, by the base point free pencil trick, we have

$$\begin{aligned}\dim\ker\mu_{N,\omega_C\otimes L^{-1}}=h^0(N-A) &= \deg N-\deg A-g+h^0(K_C-N+A)+1\\ &\geq d-\delta-\nu-g+1=d-3g+3+b.\end{aligned}$$

From $\dim W=h^0(L)-r$, it follows that $\dim\ker\mu_{0,W}\geq\dim\ker\mu_0(L)-2r$, where

$$\mu_0(L):H^0(L)\otimes H^0(K_C-L)\to H^0(K_C).$$

To compute $\dim\ker\mu_0(L)$, we apply once again the base point free pencil trick which gives

$$\begin{aligned}\dim\ker\mu_0(L) &= h^0(L-A)=h^0(K_C-2A-B_b)\\ &= 2g-2-2\nu-b-g+h^0(2A+B_b)+1\\ &\geq g-2\nu-b+2,\end{aligned}$$

the latter inequality following from the fact that $h^0(2A+B_b)\geq 3$. Hence, from (3.9), one has:

$$\begin{aligned}\dim\ker\mu_{\mathcal{F}} &\geq d-3g+3+b+g-2\nu-b+2-2r\\ &= d-2g-2\nu-2r+5.\end{aligned}$$

The previous inequality gives $\dim T_{\mathcal{F}}(B)\geq 6g-d-2\nu-2r-6$, proving Claim 3.12. □

Assume that $h^1(N)\geq 1$. Then, Claims 3.11, 3.12 and (3.1) imply that

$$\dim T_{\mathcal{F}}B-\dim B\geq d-3g+\nu+2b+1\geq\nu+2b.$$

Thus the equality $\dim B=\dim T_{\mathcal{F}}B$ cannot occur for $h^1(N)\geq 1$; therefore, N must be non-special. In this case, $\dim B=\dim T_{\mathcal{F}}B$ holds if and only if $b=0$ and N is general of its degree. Consequently, the Proposition is proved. □

Thus, the only remaining case is the following:

Proposition 3.13. *Let B be any reduced component of $B^{k_2}_d$, with $\dim B \geq \rho^{k_2}_d$. Assume that a general element $\mathcal{F}$ of B fits in the following exact sequence;*

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{3.10}$$

where $h^1(\mathcal{F}) = 2$ and $h^1(L) = 1$. Then, B coincides with the component $B_{\rm reg}$ as in Sect. 3.2.

Proof. We will use notation as in (2.2). Since B is irreducible, all integers in (2.2) are constant for a general $\mathcal{F} \in B$. Then $\frac{3g-1}{2} \leq \delta \leq 2g-2$, since L is special and $\mathcal{F}$ is semistable. Hence

$$g - 1 \leq \deg N = d - \delta \leq d/2 \leq 2g - 3\nu. \tag{3.11}$$

By (3.10), the line bundle N is special and the corresponding coboundary map ∂ is of corank one. As in the proof of Proposition 3.10, for a suitable open dense subset S of $W^{r-1}_{2g-2+\delta-d} \times C^{(2g-2-\delta)}$, one has a projective bundle $\mathbb{P}(\mathcal{E}) \to S$, whose general fiber is $\widehat{\mathcal{W}_1} := \mathbb{P}(\mathcal{W}_1)$, where $\mathcal{W}_1 := \{u \in \mathrm{Ext}^1(L,N) \mid \mathrm{corank}(\partial_u) \geq 1\}$. Then the component B is the image of $\mathcal{P}$ via a dominant rational map $\mathcal{P} \stackrel{\pi}{\dashrightarrow} B \subset B^{k_2}_d$ (cf. [3, Sect. 6] for details). Hence

$$\dim B \leq \dim W^{r-1}_{2g-2-d+\delta} + 2g - 2 - \delta + \dim \widehat{\mathcal{W}_1}.$$

Since from (3.11) $\deg(K_C - N) \leq g - 1$, by Martens' theorem [2, Thm. (5.1)] we obtain

$$\dim W^{r-1}_{2g-2+\delta-d} \leq \begin{cases} 2g-2-d+\delta = \deg(K_C - N) & \text{if } r = 1 \\ 2g-2-d+\delta-2r+1 & \text{if } r \geq 2. \end{cases}$$

Note that $m \geq g + 2\delta - d - 1$ by (2.4), where $m := \dim(\mathrm{Ext}^1(L,N))$. Thus it follows that $l \geq r$ and $m \geq l + 1$ since $l = h^0(L) = \delta - g + 2 \geq \frac{g+3}{2}$ and $r - 1 \leq \frac{\deg(K_C - N)}{2}$. Applying Theorem 3.6, we get $\dim \widehat{\mathcal{W}_1} = m - l + r - 2 = m - \delta + g + r - 4$, whence

$$\begin{aligned} \dim B &\leq \dim W^{r-1}_{2g-2-d+\delta} + (2g-2-\delta) + m - \delta + g + r - 4 \\ &\leq \begin{cases} 5g - d - \delta - 7 + m & \text{if } r = 1 \\ 5g - d - \delta - r - 7 + m & \text{if } r \geq 2, \end{cases} \end{aligned}$$

Assume that $r \geq 2$; this implies that N cannot be isomorphic to L. Therefore (2.4) gives $m = 2\delta - d + g - 1$. Thus we have

$$\rho^{k_2}_d \leq \dim B \leq 6g - 2d + \delta - r - 8,$$

which cannot occur since $\rho^{k_2}_d = 8g - 2d - 11$ and $\delta \leq 2g - 2$. Therefore, we must have $r = 1$. Then by (2.4) we get

$$\dim B \leq \begin{cases} (5g - d - \delta - 7) + 2\delta - d + g - 1 & \text{if } L \not\cong N \\ (5g - d - \delta - 7) + g & \text{if } L \cong N. \end{cases} \tag{3.12}$$

If $L \cong N$ then we have $8g - 2d - 11 \leq \dim B \leq 6g - d - \delta - 7$ which yields $\deg N = d - \delta \geq 2g - 4$. This is a contradiction to (3.11). Accordingly, we have $L \not\cong N$ and hence by (3.12)

$$8g - 2d - 11 \leq \dim B \leq 6g - 2d + \delta - 8,$$

which implies $\delta \geq 2g-3$. Since L is a special line bundle, it turns out that either $L \simeq K_C$ or $L \simeq K_C(-p)$ for some $p \in C$.

If $L \simeq K_C$, let Γ be the section of the ruled surface $F = \mathbb{P}(\mathcal{F})$ corresponding to the quotient $\mathcal{F} \twoheadrightarrow K_C$; then $\dim |\mathcal{O}_F(\Gamma)| = 1$ by [3, (2.6)] and the fact that $h^1(\mathcal{F}) = 2$. By [3, Prop. 2.12] any such

$\mathcal{F}$ admits therefore $K_C - p$ as a quotient line bundle, for some $p \in C$. This completes the proof since N is special. □

Remark 3.14. (i) From the proof of Proposition 3.13, it also follows that $K_C - p$ is minimal among special quotient line bundles for $\mathcal{F}$ general in the component B_{reg}, completely proving Theorem 3.1 (i).

(ii) Notice moreover that, from the same proof, $\mathcal{F}$ general in B_{reg} admits also a *presentation* via a canonical quotient, i.e. $0 \to K_C - D - p \to \mathcal{F} \to K_C \to 0$, which on the other hand is not via a quotient line bundle of $\mathcal{F}$ of minimal degree among special quotients and whose residual presentation coincides with that in the proof of [10, Theorem], i.e. $0 \to \mathcal{O}_C \to \mathcal{E} \to L \to 0$, where $\mathcal{E} = \omega_C \otimes \mathcal{F}^*$ and $L = \mathcal{O}_C(D+p)$. In other words, the component B_{reg} coincides with that in [10, Theorem]; the minimality of $K_C - p$ for $\mathcal{F}$ reflects in our presentation Theorem 1.2 (i) via a special section of $\mathcal{E}$ whose zero locus is of degree one.

We now consider the case $h^1(\mathcal{F}) = i \geq 3$.

Proposition 3.15. *There is no reduced component of $B_d^{k_2}$ whose general member $\mathcal{F}$ is of speciality $i \geq 3$.*

Proof. If $\mathcal{F} \in B_d^{k_2}$ is such that $h^1(\mathcal{F}) = i \geq 3$, then by the Riemann-Roch theorem $h^0(\mathcal{F}) = d-2g+2+i = k_2 + (i-2) = k_i > k_2$. Thus $\mathcal{F} \in \mathrm{Sing}(B_d^{k_2})$ (cf. [2, p. 189]). Therefore the statement follows. □

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