

GONALITY OF CURVES ON GENERAL HYPERSURFACES

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ABSTRACT. This paper concerns the existence of curves with low gonality on smooth hypersurfaces of sufficiently large degree. It has been recently proved that if $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $d \geq n + 2$, and if $C \subset X$ is an irreducible curve passing through a general point of X , then its gonality verifies $\text{gon}(C) \geq d - n$, and equality is attained on some special hypersurfaces. We prove that if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n + 2$, the least gonality of an irreducible curve $C \subset X$ passing through a general point of X is $\text{gon}(C) = d - \lfloor \frac{\sqrt{16n+1}-1}{2} \rfloor$, apart from a series of possible exceptions, where $\text{gon}(C)$ may drop by one.

1. INTRODUCTION

In this paper, we consider smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of sufficiently large degree, and we are interested in the existence of irreducible curves $C \subset X$ having low gonality and passing through a general point of X . We recall that the *gonality* $\text{gon}(C)$ of an irreducible projective curve C is the least degree of a non-constant morphism $\tilde{C} \rightarrow \mathbb{P}^1$, where \tilde{C} is the normalization of C .

The study of curves on varieties is at the basis of the birational classification. As for rational curves on projective hypersurfaces or complete intersections, their existence has been investigated long since (see e.g. [14, 17, 18]), and it has been understood in a series of seminal works (see [8, 11, 12, 19, 20]). In particular, it turns out that no rational curve lies on a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n$. When instead X has degree $d \leq 2n - 1$, it contains lines varying in a $(d - 2n - 1)$ -dimensional family (see e.g. [5, 9]), and if $n \geq 5$, lines are the only rational curves on a very general hypersurface of degree $d = 2n - 1$ (cf. [15]).

In order to deal with (possibly moving) curves having higher gonality, it is profitable to consider the following birational invariant introduced in [4], and coinciding with the gonality in dimension 1. Given an irreducible variety Y , we define the *covering gonality* of Y to be the integer

$$\text{cov. gon}(Y) := \min \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given a general point } y \in Y, \exists \text{ an irreducible} \\ \text{curve } C \subseteq Y \text{ such that } y \in C \text{ and } \text{gon}(C) = c \end{array} \right. \right\}.$$

Since $\text{cov. gon}(Y) = 1$ is equivalent to Y being uniruled, we can think of the covering gonality as a measure of the failure of Y to be uniruled.

The following theorem is probably the most general result governing the gonality of moving curves in a very general hypersurface of large degree.

Theorem ([4, Proposition 3.8]). *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. If $Y \subset X$ is an irreducible subvariety of dimension $s \geq 1$, one has*

$$\text{cov. gon}(Y) \geq d - 2n + s. \tag{1.1}$$

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In particular,

$$\text{cov. gon}(X) \geq d - n. \quad (1.2)$$

Actually (1.2) holds even if X has at worst canonical singularities and $d \geq n + 2$; under these assumptions, the bound (1.2) is sharp (cf. [4, Corollary 1.11 and Example 1.7]).

When $n = 1$ and $X \subset \mathbb{P}^2$ is a smooth plane curve of degree $d \geq 3$, famous result by M. Noether yields $\text{gon}(X) = d - 1$, and all the morphisms $X \rightarrow \mathbb{P}^1$ of degree $d - 1$ are projections from a point of X (see e.g. [7]). The case of smooth surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 5$ has been studied in [13], where one shows that $\text{cov. gon}(X) = d - 2$, and all families of curves computing the covering gonality are classified. In particular, if $X \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 5$ and $C \subset X$ is a $(d - 2)$ -gonal curve passing through a general point $x \in X$, then C is a plane curve cut out on X by some tangent plane $T_p X$, so that C has a double point at $p \in X$ and the map $C \dashrightarrow \mathbb{P}^1$ of degree $d - 2$ is the projection from p (cf. [13, Corollary 1.8]).

The main result in this paper is the following Theorem 1.1, which determines the covering gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of sufficiently large degree and arbitrary dimension, apart from a series of exceptions for which, as we will see, the covering gonality is *almost* determined (see Remark 1.2 below).

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then*

$$d - \left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor \leq \text{cov. gon}(X) \leq d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor. \quad (1.3)$$

If $n \in \mathbb{N} \setminus \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 \mid \alpha \in \mathbb{N}^*\}$, then

$$\text{cov. gon}(X) = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor, \quad (1.4)$$

and for general $x \in X$, there exists an irreducible plane curve $C \subset X$ passing through x , which computes the covering gonality via the projection $C \dashrightarrow \mathbb{P}^1$ from a singular point $p \in C$ of multiplicity $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$.

Remark 1.2. For $n \in \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 \mid \alpha \in \mathbb{N}\}$, the integers $\left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor$ and $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$ differ by 1, so (1.3) almost determines the covering gonality (cf. Lemma 4.2).

To prove Theorem 1.1, we combine various approaches and techniques developed in several works (see [3, 4, 6, 11, 13, 15, 19]). The key idea is to relate irreducible curves of low gonality contained in $X \subset \mathbb{P}^{n+1}$ to the cones $V_p^h \subset \mathbb{P}^{n+1}$ swept out by tangent lines having intersection multiplicity at least $h \geq 2$ at $p \in X$.

To this aim, we use the argument of [1, 13], and we deduce from [3, Theorem 2.5] that, if $C \subset X$ is an irreducible curve which passes through a general point $x \in X$ and admits a map

$$\varphi: C \dashrightarrow \mathbb{P}^1$$

of small degree $c \leq d - 3$, then any fiber of φ consists of collinear points (see Proposition 2.10). By arguing as in [4, Theorem C], we prove further that a curve $C \subset X$ as above lies on a cone V_p^{d-c} , and the map $\varphi: C \dashrightarrow \mathbb{P}^1$ is the projection from the vertex $p \in C$ (cf. Proposition 2.11).

Then we follow the approach of [15], relying on vector bundles techniques as in [11, 19]. We consider the Grassmannian $\mathbb{G}(1, n+1)$ of lines $\ell \subset \mathbb{P}^{n+1}$ and we define the locus

$$\tilde{\Delta}_{d-c, X} := \{(p, [\ell]) \in X \times \mathbb{G}(1, n+1) \mid \ell \cdot X \geq (d-c)p\}.$$

The gonality map $\varphi: C \dashrightarrow \mathbb{P}^1$ clearly determines a rational curve in $\tilde{\Delta}_{d-c, X}$. Since the curves C cover X , then $\tilde{\Delta}_{d-c, X}$ contains a uniruled subvariety of dimension at least n . This, and an analysis of positivity properties of the canonical bundle of $\tilde{\Delta}_{d-c, X}$, yield numerical restrictions on $d-c$, leading to the lower bound in Theorem 1.1 (cf. Corollary 3.6 and Theorem 4.1).

In order to conclude the proof of Theorem 1.1, we construct a family of irreducible plane curves covering X and having a singularity of multiplicity $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$. The cones V_p^h are contained in the tangent hyperplane $T_p X \cong \mathbb{P}^n$, and any hyperplane section of V_p^h not containing p is defined by the vanishing of $h-2$ polynomials of degrees $2, 3, \dots, h-1$, respectively. Then we slightly improve (in the case of lines) a classical result about linear spaces in complete intersections in a projective space (cf. [17, 6] and Proposition 2.3), from which we deduce the existence of a subvariety $Z \subset X$ of dimension at least $n-1$ such that for any $p \in Z$ and $h \leq \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$, the cone V_p^h contains a line ℓ_p not passing through p (see Lemma 2.7). Hence the span of p and ℓ_p cuts out on X a plane curve C_p having a singularity at p of multiplicity at least h , so that the projection from $p \in C_p$ is a map $C_p \dashrightarrow \mathbb{P}^1$ of degree at most $d-h$. The family of plane curves obtained by varying $p \in Z$ covers X , and the assertion follows by setting $h = \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$ (see Theorem 2.8).

A couple of questions are in order. First, it would be interesting to characterize the curves computing the covering gonality of X and, in particular, to understand whether, at least if n is sufficiently large and $d \geq 2n+1$, they are only the plane curves presented above. We discuss this question in §5.1.

Concerning the exceptional values $n \in \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 \mid \alpha \in \mathbb{N}\}$, apart from the trivial case $n=1$, we cannot decide if (1.4) holds for other exceptional values. However, especially if one believes that for sufficiently large dimension of X the covering gonality is computed by plane curves, it is natural to make the following:

Conjecture 1.3. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Then*

$$\text{cov. gon}(X) = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor.$$

The paper is organized as follows. In Section 2 we are concerned with the relations between the cones of lines having high tangency order at a point $p \in X$ and the geometry of curves having low gonality and covering X . On one hand, we discuss the existence of lines in V_p^h not belonging to the ruling of the cone, and we achieve the upper bound in Theorem 1.1. On the other hand, we prove that any curve $C \subset X$ through a general point of X having sufficiently small gonality lies on some V_p^h and we describe the gonality map $C \dashrightarrow \mathbb{P}^1$.

In Section 3 we study positivity properties of the loci $\tilde{\Delta}_{d-c, X}$, deducing numerical conditions on the existence of uniruled subvarieties of $\tilde{\Delta}_{d-c, X}$. In Section 4 we finish the proof of Theorem 1.1, and in Section 5 we make some final remarks and discuss some open problems.

Notation. We work over \mathbb{C} . By *variety* we mean a complete reduced algebraic variety X , unless otherwise stated. By *curve* we mean a variety of dimension 1. We say that a property holds for a

general (resp. very general) point $x \in X$ if it holds on a Zariski open nonempty subset of X (resp. on the complement of the countable union of proper subvarieties of X).

2. GEOMETRY OF COVERING FAMILIES OF CURVES

2.1. High tangency cones to hypersurfaces. Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface defined by the vanishing of a non-zero homogeneous polynomial $F \in \mathbb{C}[y_0, \dots, y_{n+1}]$ of degree $d \geq 2$.

Definition 2.1. Given a point $x \in X$ and an integer $h \geq 2$, the cone $V_x^h(X) \subset \mathbb{P}^{n+1}$ of tangent lines of order h at $x \in X$ (denoted by V_x^h if there is no danger of confusion) is the set of all lines $\ell \subset \mathbb{P}^{n+1}$ having intersection multiplicity at least h with X at x .

The variety V_x^h is a cone with vertex containing x , and (see [6, p. 186]) it is defined by the $h-1$ equations

$$G_k(y_0, \dots, y_{n+1}) := \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n+1} y_{i_1} \cdots y_{i_k} \frac{\partial^k F}{\partial y_{i_1} \cdots \partial y_{i_k}}(x) = 0 \quad \text{for } 1 \leq k \leq h-1.$$

In particular, if X is smooth at x , then V_x^2 is the (projective) tangent hyperplane $T_x X \subset \mathbb{P}^{n+1}$. When $h \geq 3$, the variety V_x^h is a cone in $T_x X \cong \mathbb{P}^n$ with vertex at $x \in X$, and any hyperplane section of V_x^h not containing x is a subvariety $\Lambda_x^h \subset \mathbb{P}^{n-1}$, uniquely determined up to isomorphism, defined by the vanishing of $h-2$ polynomials of degrees $2, 3, \dots, h-1$, respectively.

Lemma 2.2. *Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$ and let $x \in X$ be a general point. Then Λ_x^h is a general complete intersection of type $(2, 3, \dots, h-1)$ in \mathbb{P}^{n-1} .*

Proof. We may assume that $x = [0, \dots, 0, 1]$ and that $T_x X = \{y_n = 0\}$. Then X has equation of the form

$$F(y_0, \dots, y_{n+1}) = y_n y_{n+1}^{d-1} + f_2(y_0, \dots, y_n) y_{n+1}^{d-2} + \dots + f_{d-1}(y_0, \dots, y_n) y_{n+1} + f_d(y_0, \dots, y_n) = 0,$$

where the polynomials f_i are homogeneous of degree i . Then Λ_x^h , as a subvariety of the \mathbb{P}^{n-1} with equations $y_n = y_{n+1} = 0$, is defined by the equations

$$f_2(y_0, \dots, y_{n-1}, 0) = \dots = f_{h-1}(y_0, \dots, y_{n-1}, 0) = 0.$$

Let W be the sub-vector space of $\mathbb{C}[y_0, \dots, y_{n+1}]_d$ consisting of all polynomials F as above. Consider the linear map

$$\begin{aligned} \zeta: W &\longrightarrow \prod_{i=2}^{h-1} H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(i)) \\ F &\longmapsto (f_2(y_0, \dots, y_{n-1}, 0), \dots, f_{h-1}(y_0, \dots, y_{n-1}, 0)). \end{aligned}$$

To prove the assertion amounts to show that ζ is surjective. One has

$$\dim(W) = \binom{d+n+1}{d} - n - 2 \quad \text{and} \quad \dim\left(\prod_{i=2}^{h-1} H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(i))\right) = \binom{h+n-1}{h-1} - n - 1.$$

Moreover $\ker(\zeta)$ consists of polynomials divisible by y_n , i.e. polynomials F such that $f_i(y_0, \dots, y_n) = y_n g_{i-1}(y_0, \dots, y_n)$, where $2 \leq i \leq h-1$ and the polynomials g_j have degree j . Then

$$\dim(\ker(\zeta)) = \binom{d+n+1}{d} - \binom{h+n-1}{h-1} - 1 = \dim(W) - \dim\left(\prod_{i=2}^{h-1} H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(i))\right),$$

proving that ζ is surjective. \square

2.2. Lines on complete intersections. Let $1 \leq s \leq m-2$ and $d_1, d_2, \dots, d_s \in \mathbb{N}$ be integers, and set $\underline{d} := (d_1, \dots, d_s)$. We consider the vector space

$$S_{\underline{d}} := \bigoplus_{i=1}^s H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)),$$

and its Zariski open subset

$$S_{\underline{d}}^* := \bigoplus_{i=1}^s (H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)) \setminus \{0\}).$$

For any $u := (F_1, \dots, F_s) \in S_{\underline{d}}^*$, we denote by $Y_u := V(F_1, \dots, F_s) \subset \mathbb{P}^m$ the closed subscheme defined by the vanishing of the s homogeneous polynomials F_1, \dots, F_s . When $u \in S_{\underline{d}}^*$ is a general point, $Y_u \subset \mathbb{P}^m$ is a smooth irreducible complete intersection of dimension $m-s \geq 2$.

In [17], Predonzan gave necessary and sufficient conditions for the existence of a k -dimensional linear subvariety of Y_u , with $u := (F_1, \dots, F_s) \in S_{\underline{d}}^*$ (see also [6, Theorem 2.1] and [5, 9]). The following proposition provides a slight extension of Predonzan's result in the case $k=1$.

Proposition 2.3. *Let $1 \leq s \leq m-2$ and d_1, d_2, \dots, d_s be positive integers such that $\prod_{i=1}^s d_i > 2$. Consider the locus*

$$W_{\underline{d}} := \left\{ u \in S_{\underline{d}}^* \mid Y_u \text{ contains a line} \right\} \subseteq S_{\underline{d}}$$

and set

$$t := \max \left\{ 0, \sum_{i=1}^s d_i + s - 2(m-1) \right\} \quad \text{and} \quad \theta := \max \left\{ 0, 2(m-1) - \sum_{i=1}^s d_i - s \right\}. \quad (2.1)$$

Then $W_{\underline{d}}$ is nonempty, irreducible, and

$$\text{codim}_{S_{\underline{d}}}(W_{\underline{d}}) = t.$$

Furthermore, if $u \in W_{\underline{d}}$ is a general point, then $Y_u \subset \mathbb{P}^m$ is a smooth complete intersection of dimension $m-s$, containing a family of lines of dimension θ . If $t > 0$, and if $u \in W_{\underline{d}}$ is a general point, then Y_u contains a unique line.

Proof. The case $t=0$, is the one considered by Predonzan, in which the assertion holds $W_{\underline{d}} = S_{\underline{d}}^*$ (cf. [6, Theorem 2.1]).

Hence we may assume $t > 0$. Consider $\mathbb{G} := \mathbb{G}(1, m)$ and the incidence correspondence

$$J := \left\{ ([\ell], u) \in \mathbb{G} \times S_{\underline{d}}^* \mid \ell \subset Y_u \right\}$$

with the projections

$$\mathbb{G} \xleftarrow{\pi_1} J \xrightarrow{\pi_2} S_{\underline{d}}^*.$$

Notice that J is a vector bundle over \mathbb{G} via π_1 . Indeed, for any $[\ell] \in \mathbb{G}$, the fiber $\pi_1^{-1}([\ell])$ equals $\bigoplus_{i=1}^s (H^0(\mathcal{I}_{\ell/\mathbb{P}^m}(d_i)) \setminus \{0\})$, where $\mathcal{I}_{\ell/\mathbb{P}^m}$ is the ideal sheaf of ℓ in \mathbb{P}^m . Thus J is smooth and irreducible and

$$\dim(J) = 2(m-1) + \sum_{i=1}^s h^0(\mathcal{O}_{\mathbb{P}^m}(d_i)) - \sum_{i=1}^s (d_i + 1) = \sum_{i=1}^s \binom{d_i + m}{m} - t > 0.$$

Since $W_{\underline{d}} = \pi_2(J)$, then $W_{\underline{d}}$ is nonempty and irreducible.

We claim that, for any $[\ell] \in \mathbb{G}$, if $([\ell], u) \in \pi_1^{-1}([\ell])$ is general (so that $u \in W_{\underline{d}}$ is general), then Y_u is a smooth complete intersection of dimension $m - s$. This is an immediate consequence of Bertini's theorem applied to the blow-up of \mathbb{P}^m along ℓ , noting that the strict transforms of the linear systems $|H^0(\mathcal{I}_{\ell/\mathbb{P}^m}(d_i))|$, with $1 \leq i \leq s$, are base point free.

If $u \in W_{\underline{d}}$ is general, then

$$\dim(W_{\underline{d}}) = \dim(J) - \dim(\pi_2^{-1}(u)) = \sum_{i=1}^s \binom{d_i + m}{m} - t - \dim(\pi_2^{-1}(u)),$$

thus (2.1) gives

$$\text{codim}_{S_{\underline{d}}}(W_{\underline{d}}) = t + \dim(\pi_2^{-1}(u)). \quad (2.2)$$

Next we show that, for $u \in W_{\underline{d}}$ general, one has $\dim(\pi_2^{-1}(u)) = 0$. To this aim, we argue as in [5, Proposition 2.1] and therefore we will be brief. Let $[\ell] \in \mathbb{G}$ and let $[y_0, y_1, \dots, y_m]$ be coordinates in \mathbb{P}^m such that $I_{\ell} := (y_2, \dots, y_m)$. For

$$([\ell], u) \in \pi_1^{-1}([\ell]) \subset J, \quad \text{with} \quad u = (F_1, \dots, F_s),$$

we can write

$$F_h = \sum_{i=2}^m y_i P_h^{(i)} + R_h, \quad 1 \leq h \leq s,$$

where

$$P_h^{(i)} = \sum_{\mu_0 + \mu_1 = d_h - 1} c_{h, \mu_0, \mu_1}^{(i)} y_0^{\mu_0} y_1^{\mu_1} \in \mathbb{C}[y_0, y_1]_{d_h - 1}, \quad \text{for } 1 \leq h \leq s \text{ and } 2 \leq i \leq m, \quad (2.3)$$

whereas $R_h \in I_{\ell}^2$. We may assume u general, so that Y_u is smooth and the normal sheaf N_{ℓ/Y_u} is a vector bundle on ℓ , fitting in the exact sequence

$$0 \rightarrow N_{\ell/Y_u} \rightarrow N_{\ell/\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(m-1)} \rightarrow N_{Y_u/\mathbb{P}^m}|_{\ell} \cong \bigoplus_{h=1}^s \mathcal{O}_{\mathbb{P}^1}(d_h) \rightarrow 0. \quad (2.4)$$

Any $\xi \in H^0(\ell, N_{\ell/\mathbb{P}^m})$ can be identified with a collection of $m - 1$ linear forms on $\mathbb{P}^1 \cong \ell$

$$\varphi_i^{\xi}(y_0, y_1) := a_{i,0}y_0 + a_{i,1}y_1, \quad \text{with } 2 \leq i \leq m,$$

whose coefficients form the $(m - 1) \times 2$ matrix

$$A_{\xi} := (a_{i,j}), \quad \text{where } 2 \leq i \leq m \text{ and } 0 \leq j \leq 1.$$

By abusing notation, we identify ξ with A_{ξ} . Then the map

$$H^0(\ell, N_{\ell/\mathbb{P}^m}) \xrightarrow{\sigma} H^0\left(\ell, N_{Y_u/\mathbb{P}^m}|_{\ell}\right)$$

arising from (2.4), is given by

$$A_{\xi} \xrightarrow{\sigma} \left(\sum_{0 \leq j \leq 1 < i \leq m} a_{i,j} y_j P_h^{(i)} \right)_{1 \leq h \leq s}. \quad (2.5)$$

Notice that $t > 0$ is equivalent to $h^0(\ell, N_{\ell/\mathbb{P}^m}) < h^0(\ell, N_{Y_u/\mathbb{P}^m}|_{\ell})$.

Claim 2.4. *The map σ is injective. Thus $h^0(N_{\ell/Y_u}) = 0$, i.e. the Fano scheme $F(Y_u)$ parametrizing lines in Y_u contains $[\ell]$ as a zero-dimensional integral component.*

Proof of Claim 2.4. By (2.3), the expression $\sum_{0 \leq j \leq 1 < i \leq m} a_{i,j} y_j P_h^{(i)}$, for $1 \leq h \leq s$, reads

$$\begin{aligned} & a_{2,0} \left(\sum_{\mu_0 + \mu_1 = d_h - 1} c_{h,\mu_0,\mu_1}^{(2)} y_0^{\mu_0+1} y_1^{\mu_1} \right) + a_{2,1} \left(\sum_{\mu_0 + \mu_1 = d_h - 1} c_{h,\mu_0,\mu_1}^{(2)} y_0^{\mu_0} y_1^{\mu_1+1} \right) + \cdots + \\ & + \cdots + a_{m,0} \left(\sum_{\mu_0 + \mu_1 = d_h - 1} c_{h,\mu_0,\mu_1}^{(m)} y_0^{\mu_0+1} y_1^{\mu_1} \right) + a_{m,1} \left(\sum_{\mu_0 + \mu_1 = d_h - 1} c_{h,\mu_0,\mu_1}^{(m)} y_0^{\mu_0} y_1^{\mu_1+1} \right). \end{aligned}$$

By equating to 0 the coefficients of $y_0^{d_h-k} y_1^k$, for $0 \leq k \leq d_h$, we find a homogeneous linear system of $\sum_{h=1}^s d_h + s$ equations in the $2(m-1)$ variables $a_{i,j}$ and coefficients $c_{h,\mu_0,\mu_1}^{(i)}$, where $2 \leq i \leq m$ and $0 \leq j \leq 1$. By (2.5), the map σ is injective if and only if this system admits only the trivial solution. One checks that this is the case for a general choice of the coefficients $c_{h,\mu_0,\mu_1}^{(i)}$ and because of the assumption $t > 0$ equivalent to $\sum_{i=1}^s d_i > 2(m-1) - s$. Thus we deduce from (2.4) that $h^0(\ell, N_{\ell/Y_u}) = 0$. \square

By the irreducibility of J and Claim 2.4, for $u \in W_{\underline{d}}$ general, the Fano scheme $F(Y_u)$ is zero-dimensional, i.e. Y_u contains finitely many lines. In particular, (2.2) yields that $\text{codim}_{S_{\underline{d}}}(W_{\underline{d}}) = t$ as desired.

Finally, to show that Y_u contains only one line for $u \in W_{\underline{d}}$ general, one makes a count of parameters, left to the reader, similar to the one in (2.2), which shows that the codimension in $S_{\underline{d}}^*$ of the locus of u such that Y_u contains at least two lines is strictly larger than t . \square

Remark 2.5. The case of quadrics is not covered by Proposition 2.3. For our purposes, it will suffice to recall that any quadric of dimension at least 2 contains a line, whereas the locus parameterizing conics containing a line coincides with the locus of singular conics and it has codimension 1 in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

2.3. The upper bound for the covering gonality. Given a hypersurface $X \subset \mathbb{P}^{n+1}$, a smooth point $x \in X$ and an integer $h \geq 3$, we are concerned with the existence of lines in V_x^h which do not pass through x . More precisely, we want to consider the Zariski closure X_1^h of the set of smooth points $x \in X$ such that V_x^h contains a line which does not pass through x .

Lemma 2.6. *Let $n, d, h \geq 3$ be integers such that $d \geq 2n$ and $h(h+1) = 4n$. Then there exist a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d and two points $p, q \in X$ such that:*

- (i) X is smooth at p and q ;
- (ii) $p \notin X_1^h$;
- (iii) $q \in X_1^h$ and V_q^h contains a line ℓ such that $q \notin \ell$, $p \in \ell$ and V_q^h is smooth along ℓ .

Proof. Let $[y_0, \dots, y_{n+1}]$ be the coordinates in \mathbb{P}^{n+1} and consider a homogeneous polynomial

$$F(y_0, \dots, y_{n+1}) = \sum_{|I|=d} a_I y^I$$

of degree d , where for a multi-index $I = (i_0, \dots, i_{n+1})$, we denote by $|I|$ its length and we set $y^I := y_0^{i_0} \cdots y_{n+1}^{i_{n+1}}$. Note that I varies among the points with integral coordinates in the n -simplex

$\Delta_d \subset \mathbb{R}_{\geq 0}^{n+1}$ with vertices $Q_0 := (d, 0, \dots, 0), \dots, Q_{n+1} := (0, \dots, 0, d)$. For $0 \leq \alpha \leq d$, we denote by $\Delta_\alpha^{(0)} \subset \Delta_d$ the n -subsimplex with vertices

$$Q_0 = (d, 0, \dots, 0), (d - \alpha, \alpha, 0, \dots, 0), \dots, (d - \alpha, 0, \dots, 0, \alpha).$$

Let $q := [1, 0, \dots, 0], p := [0, 1, \dots, 0] \in \mathbb{P}^{n+1}$ and assume that the hypersurface $X := V(F) \subset \mathbb{P}^{n+1}$ passes through q and p , so that $a_{(d,0,\dots,0)} = a_{(0,d,\dots,0)} = 0$. Then V_q^h depends on the multi-indices I belonging to $\Delta_{h-1}^{(0)}$. In fact, V_q^h has equations

$$G_k(y_0, \dots, y_{n+1}) := \sum_{|J|=k} \frac{\partial^k F}{\partial y^J}(q) y^J = 0 \quad \text{for } 1 \leq k \leq h-1 \quad \text{where} \quad \frac{\partial^k}{\partial y^J} := \frac{\partial^k}{\partial y_0^{j_0} \dots \partial y_{n+1}^{j_{n+1}}}.$$

In particular

$$\frac{\partial^k F}{\partial y^J} = \sum_{|I|=d, |J|=k, I \geq J} \frac{I!}{(I-J)!} a_I y^{I-J},$$

where, as usual, $I! := i_0! i_1! \dots i_{n+1}!$ and $I - J$ and $I \geq J$ are defined componentwise. For any $I \notin \Delta_k^{(0)}$, J such that $|J| = k$ and $I \geq J$, the value of y^{I-J} at q is zero. Therefore the non-zero coefficients a_I in G_k are such that $I \in \Delta_k^{(0)}$, for all $1 \leq k \leq h-1$.

Similarly, the non-zero coefficients a_I in the equations of V_p^h are such that I is an integral point in the n -subsimplex $\Delta_{h-1}^{(1)}$ with vertices

$$Q_1 = (0, d, 0, \dots, 0), (h-1, d-h+1, 0, \dots, 0), \dots, (0, d-h+1, 0, \dots, 0, h-1).$$

Choose the coefficients of F so that $G_1 = y_{n+1}$, hence q is a smooth point for X . The cone V_q^h lies in $T_q X = V(y_{n+1}) \cong \mathbb{P}^n$. Moreover, setting $H := V(y_0)$, the section of V_q^h with H is $\Lambda_q^h = V(y_0, y_{n+1}, G_2, \dots, G_{h-1}) \subset V(y_0, y_{n+1}) \cong \mathbb{P}^{n-1}$. Next we apply Proposition 2.3 with $m = n-1$, $s = h-2$ and $d_i = i+1$. By (2.1), we have

$$t = \sum_{i=1}^s d_i + s - 2(m-1) = \frac{h(h+1)}{2} - 2n + 1 = 1.$$

If $h \geq 4$, Proposition 2.3 ensures the existence of a locus $W_{(2,\dots,h-1)}$ of codimension 1 in $S_{(2,\dots,h-1)}$, whose general point is a complete intersection of type $(2, \dots, h-1)$ in \mathbb{P}^{n-1} containing a line. The same holds for $h = 3 = n$ by Remark 2.5.

Since $W_{(2,\dots,h-1)}$ has codimension 1 in $S_{(2,\dots,h-1)}$ we can choose the coefficients a_I , with $I \in \Delta_{h-1}^{(1)}$ and $I \neq (0, d, 0, \dots, 0)$, general enough so that X is smooth at p and Λ_p^h contains no line. Moreover, we can choose the coefficients a_I , with $I \in \Delta_{h-1}^{(0)}$ which we did not fix yet, so that $p \in V_q^h$ and V_q^h contains a line through p different from the line $\langle p, q \rangle$ and it is smooth along this line. This can be done without altering the coefficients a_I , with $I \in \Delta_{h-1}^{(1)}$, we already chose, since $\Delta_{h-1}^{(0)}$ and $\Delta_{h-1}^{(1)}$ have no integral point in common. Then p and q satisfy (i)–(iii). \square

Lemma 2.7. *Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2n$ and let $h \geq 3$ be an integer such that $h(h+1) \leq 4n$.*

- (i) *If $h(h+1) < 4n$, then $X_1^h = X$ and if $x \in X$ is a general point, then V_x^h is the cone over a general complete intersection in $S_{\underline{\delta}}^*$, with $\underline{\delta} = (1, 2, \dots, h-1)$, which contains a family of lines of dimension θ given by (2.1);*

- (ii) If $h(h+1) = 4n$, then all components of X_1^h have dimension $n-1$ and there is some irreducible component Z of X_1^h such that, if $x \in Z$ is a general point, then V_x^h is the cone over a general complete intersection in $W_{\underline{\delta}} \subset S_{\underline{\delta}}^*$, with $\underline{\delta} := (1, 2, \dots, h-1)$, which is smooth and contains only one line.

Proof. Let

$$\mathcal{X} := \{(x, F) \mid F \in S_d^* \text{ and } x \in X = V(F)\} \subset \mathbb{P}^{n+1} \times S_d^*$$

be the universal hypersurface of degree d , and let

$$\mathbb{P}^{n+1} \xleftarrow{\tau} \mathcal{X} \xrightarrow{\sigma} S_d^*$$

be the projections onto the two factors. Fix a hyperplane $H \subset \mathbb{P}^{n+1}$ and let

$$U := \mathcal{X} \cap ((\mathbb{P}^{n+1} \setminus H) \times S_d^*).$$

Let $\underline{\delta} := (1, 2, \dots, h-1)$ and consider the rational map

$$\begin{aligned} \varphi: \quad U &\dashrightarrow S_{\underline{\delta}}^* \\ (x, F) &\longrightarrow V_x^h \cap H, \end{aligned}$$

Thanks to Lemma 2.2, the map φ is dominant. As in Proposition 2.3, we consider $W_{\underline{\delta}} \subseteq S_{\underline{\delta}}^*$.

If $h(h+1) < 4n$, then $W_{\underline{\delta}} = S_{\underline{\delta}}^*$. Indeed, if $h \geq 4$, Proposition 2.3 applied to $H \cong \mathbb{P}^n$, with $s = h-1$, $d_i = i$ and $1 \leq i \leq h-1$, gives $t = 0$. If $h = 3$, then $n \geq 4$ and V_x^3 is a cone over a quadric of dimension at least 2, which always contains lines (cf. Remark 2.5). Thus, in both cases the general $(x, F) \in U$, and hence the general $(x, F) \in \mathcal{X}$, is such that $V_x^h \cap H$ is general in $S_{\underline{\delta}}^*$ and (i) follows by Proposition 2.3.

If $h(h+1) = 4n$, Proposition 2.3 (and Remark 2.5 if $h = n = 3$) yields $\text{codim}_{S_{\underline{\delta}}^*}(W_{\underline{\delta}}) = 1$. Therefore $\varphi^{-1}(W_{\underline{\delta}})$ has codimension 1 in U , Lemma 2.6 implies that $\sigma(\varphi^{-1}(W_{\underline{\delta}}))$ is dense in S_d and (ii) follows. \square

We can now prove the upper bound in Theorem 1.1.

Theorem 2.8. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Given a general point $x \in X$, there exists a plane curve $C \subset X$ passing through x , and having a singular point of multiplicity at least $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$ and gonality $\text{gon}(C) \leq d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$. In particular,*

$$\text{cov. gon}(X) \leq d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor.$$

Proof. The cases $n = 1$ and $n = 2$ are covered by [7, Teorema 3.14] and [13, Corollary 1.8], respectively. So we assume $n \geq 3$. We set

$$h := \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor.$$

and note that this is the maximal integer such that $h(h+1) \leq 4n$. We will prove the existence of a family of plane curves covering X and having gonality at most $d-h$.

Thanks to Lemma 2.7, the hypersurface X contains an irreducible component Z of X_1^h such that for the general $z \in Z$, the cone V_z^h contains some line ℓ not passing through z , and for each such line ℓ we can consider the plane $\pi_{z,\ell} := \langle z, \ell \rangle \subset V_z^h$. To ease the notation we set $\pi := \pi_{z,\ell}$. Note that π is not contained in X , since X contains no rational curve (see [19, 20]), hence π cuts

out on X a curve $C := C_{z,\ell}$ passing through z (C is possibly reducible, but none of its component is rational, in particular it is not a line). Since $\pi \subset V_q^h$, any line $L \subset \pi$ passing through z is such that the intersection cycle $L \cdot X$ is of the form $L \cdot X = hz + x_1 + \cdots + x_{d-h}$, for some $x_1, \dots, x_{d-h} \in C$. Therefore, if \tilde{C} is the normalization of C , the projection from z induces a morphism $\tilde{C} \rightarrow \ell \cong \mathbb{P}^1$, which is non-constant on any component of C and has degree at most $d-h$. In particular, the gonality of any irreducible component Γ of \tilde{C} satisfies $\text{gon}(\Gamma) \leq d-h$.

Actually, it shall follow from Theorem 4.1 and Lemma 4.2 that for general $z \in Z$, any such a component Γ satisfies $\text{gon}(\Gamma) \geq d-h-1$, and hence the general $C := C_{z,\ell} \subset \pi$ shall turn out to be an irreducible plane curve with gonality $\text{gon}(C) \leq d-h$. Moreover, any line $L \subset \pi$ passing through z meets C at z with multiplicity at least h , so that C has a singular point of multiplicity at least h at z .

To finish the proof we need to show that the curves $C_{z,\ell}$ cover an open subset of X . If $h(h+1) < 4n$, then $Z = X$ by Lemma 2.7, and the curves $C_{z,\ell}$ cover X .

If $h(h+1) = 4n$, then Z has codimension 1 in X by Lemma 2.7. In this case, proving that the curves $C_{z,\ell}$ cover X is equivalent to prove that the (closure of the) $(n-1)$ -dimensional family of planes of the form $\pi_{z,\ell}$ as above is *non-degenerate*, i.e. it sweeps out the whole projective space \mathbb{P}^{n+1} . To prove this fact, it suffices to prove it for some special hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d such that $\dim(X_1^h) = n-1$.

By Lemma 2.6, for any $d \geq 2n$, there exist a hypersurface $X \subset \mathbb{P}^{n+1}$ and two smooth points $p, q \in X$ such that $p \notin X_1^h$, $q \in Z$, $\dim(Z) = n-1$, and $p \in \pi_{q,L}$, with L a line through p not containing q and V_q^h smooth along L . Assume by contradiction that the family \mathcal{P} of planes $\pi_{z,\ell}$ is degenerate, and let Π be the proper subvariety of \mathbb{P}^{n+1} which is the union of the planes of \mathcal{P} . To simplify the argument, we assume that \mathcal{P} , and hence Π , is irreducible (the general case can be treated similarly by replacing \mathcal{P} with each of its irreducible components).

Since $Z \subset \Pi$, then

$$n \geq \dim(\Pi) \geq \dim(\Pi \cap X) \geq \dim(Z) = n-1.$$

Note that Π is not contained in X because X contains no plane. Therefore $\dim(\Pi) > \dim(Z)$, hence $\dim(\Pi) = n$ and Π and X intersect along a pure $(n-1)$ -dimensional variety containing Z as a component, and also along some other irreducible component Y passing through $p \in X \setminus Z$.

By (1.1) one has

$$\text{cov. gon}(Z) + \text{cov. gon}(Y) \geq 2(d-n-1). \quad (2.6)$$

On the other hand, the intersection $\Pi \cap X$ is covered by the curves $C_{z,\ell}$, and the sum of the gonalities of their irreducible components is at most $d-h$. Hence

$$\text{cov. gon}(Z) + \text{cov. gon}(Y) \leq d-h. \quad (2.7)$$

By combining (2.6) and (2.7), we deduce $d-2n-2+h \leq 0$, which is impossible for $d \geq 2n$ and $n \geq 3$. Thus we reach a contradiction and we conclude that Π does coincide with \mathbb{P}^{n+1} , as wanted. \square

2.4. Covering families of curves with low gonality and high tangency cones. Let X be an irreducible complex projective variety of dimension n .

Definition 2.9. A *covering family of c -gonal curves* consists of a smooth family $\mathcal{C} \xrightarrow{\pi} T$ of irreducible curves endowed with a dominant morphism $f: \mathcal{C} \rightarrow X$ such that for general $t \in T$, the

fibre $C_t := \pi^{-1}(t)$ is a smooth curve with gonality $\text{gon}(C_t) = c$ and the restriction $f_t: C_t \rightarrow X$ is birational onto its image.

The *covering gonality* of X is the least integer $c > 0$ such that there exists a covering family of c -gonal curves. According to [4, Remark 1.5], we may assume that both T and \mathcal{C} are smooth, with $\dim(T) = n - 1$. Furthermore, up to base change, we may consider a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & T \times \mathbb{P}^1 \\ & \searrow \pi & \downarrow \text{pr}_1 \\ & & T, \end{array}$$

where the restriction $\varphi_t: C_t \rightarrow \{t\} \times \mathbb{P}^1$ is a c -gonal map. If $y \in \mathbb{P}^1$ is a general point, we set $\varphi_t^{-1}(y) = \{q_1, \dots, q_c\} \subset C_t$. We can argue as in [1, Example 4.7] to construct a correspondence $\Gamma \subset X \times (T \times \mathbb{P}^1)$ of degree c with null trace (cf. [1, Section 4]). Then [3, Theorem 2.5] implies the following.

Proposition 2.10. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n + 3$, and let $\mathcal{C} \xrightarrow{\pi} T$ be a covering family of c -gonal curves, as above. If $c \leq 2d - 2n - 3$, then $f(q_1), \dots, f(q_c) \in X$ are contained on a line $\ell_{(t,y)} \subset \mathbb{P}^{n+1}$.*

Set $\mathbb{G} := \mathbb{G}(1, n + 1)$. For a general point $x \in X$, there exists some line $\ell_{(t,y)} \subset \mathbb{P}^{n+1}$ passing through x . Moreover, since X is smooth of degree $d \geq n + 3$, then X is of general type, hence it is not covered by lines, so that $\ell_{(t,y)}$ meets X along a 0-dimensional scheme of length $d = \deg X$. As we vary $(t, y) \in T \times \mathbb{P}^1$, the line $\ell_{(t,y)}$ describes a subvariety $B_0 \subset \mathbb{G}$ of dimension n . By taking a desingularization $B \rightarrow B_0$, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{\mu} & \mathfrak{P}_0 \longrightarrow \mathbb{P}^{n+1} \\ \downarrow \phi & & \downarrow \\ B & \longrightarrow & B_0 \end{array} \quad (2.8)$$

where $\mathfrak{P} \xrightarrow{\phi} B$ is the \mathbb{P}^1 -bundle obtained as the pullback of the universal \mathbb{P}^1 -bundle on \mathbb{G} , and $\mu: \mathfrak{P} \rightarrow \mathbb{P}^{n+1}$ is the obvious morphism, which is clearly dominant.

Finally, by arguing similarly to [4, Theorem C], we prove the following result.

Proposition 2.11. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Consider a covering family of c -gonal curves as above with $c \leq d - 3$. Then:*

- (i) *there exists a point $x_t \in f(C_t)$ such that $f(C_t) \subset V_{x_t}^{d-c} \cap X$;*
- (ii) *the c -gonal map $\varphi_t: C_t \rightarrow \mathbb{P}^1$ is the composition of f_t with the projection from x_t .*

In particular, the image of $f(C_t)$ under the projection from x_t is a rational curve $R_t \subset \Lambda_{x_t}^{d-c}$.

Proof. Since $d \geq 2n + 2$, we deduce that $c \leq d - 3 \leq 2d - 2n - 3$. Hence by Proposition 2.10 there exists a line $\ell_{(t,y)}$ containing $f(\varphi_t^{-1}(y)) = \{f(q_1), \dots, f(q_c)\} \subset f(C_t)$. We will prove that there exists a point $x_t \in f(C_t)$ depending only on t such that $\ell_{(t,y)} \cdot X = (d - c)x_t + f(q_1) + \dots + f(q_c)$.

We argue by contradiction and we assume that, as $(t, y) \in \{t\} \times \mathbb{P}^1$ varies, the 0-cycle $(\ell_{(t,y)} \cdot X) - f(q_1) - \dots - f(q_c)$ moves describing a (possibly reducible) curve $D_t \subset X$. We note

that D_t is dominated by the curve $E_t := \overline{\{(x, y) \in D_t \times \mathbb{P}^1 \mid x \in \ell_{(t,y)}\}}$ under the first projection, whereas the second projection has degree $d - c$. Thus any irreducible component of E_t has gonality at most $d - c$, and hence any irreducible component D'_t of D_t satisfies $\text{gon}(D'_t) \leq d - c$. Moreover, since $c \geq \text{cov.gon}(X) \geq d - n$ by (1.2) and $d \geq 2n + 2$, we deduce $\text{cov.gon}(X) > d - c$, so that $\text{cov.gon}(X) > \text{gon}(D'_t)$. Thus the closure of the locus swept out by the curves D_t is a proper subvariety of X .

Let S be an irreducible component of such a subvariety and let $1 \leq s \leq n - 1$ be its dimension. By (1.1), one has

$$\text{cov.gon}(S) \geq d - 2n + s. \quad (2.9)$$

On the other hand, let us consider the family $\mathfrak{P} \xrightarrow{\phi} B$ in (2.8). By construction, the general line $\ell_{(t,y)}$ intersects S . Moreover, since $s \leq n - 1$ and $\dim(B) = n$, if $x \in S$ is a general point, there is a family of dimension $n - s > 0$ of lines of the original family passing through x . Let us denote by $R \subset \mathfrak{P}$ the ramification divisor of the generically finite morphism $\mu: \mathfrak{P} \rightarrow \mathbb{P}^{n+1}$ in (2.8). Thus there exists an irreducible component Z of R such that $\mu(Z) = S$ and the restriction $\phi|_Z: Z \rightarrow B$ is dominant.

Setting $e := \deg \phi|_Z$, we claim that $\text{cov.gon}(S) \leq e$. Indeed, if we vary $(t, y) \in \{t\} \times \mathbb{P}^1$, the lines $\ell_{(t,y)}$ describe a rational curve $Q_t \subset B$ and the inverse image $\phi^{-1}(Q_t)$ intersects Z along a curve G_t which dominates D_t by construction. Since Q_t is rational, we deduce

$$\text{cov.gon}(S) \leq \text{gon}(D'_t) \leq \deg(\phi|_{G_t}: G_t \rightarrow Q_t) \leq e. \quad (2.10)$$

Now, we recall that for general $[\ell] \in B$, the fibre $L := \phi^{-1}([\ell])$ satisfies $(L \cdot R) = n$ (see e.g. [10, Proposition 1]), and the contribution of Z to this intersection product is $e \cdot \text{ord}_Z(R)$, where $\text{ord}_Z(R)$ is the multiplicity of Z in R . By [4, Corollary A.6], one has $\text{ord}_Z(R) \geq n - s$. Therefore $e(n - s) \leq n$, and (2.10) yields

$$\text{cov.gon}(S) \leq \frac{n}{n - s}. \quad (2.11)$$

Finally, by pooling (2.9), (2.11) and the assumption $d \geq 2n + 2$, we obtain

$$s + 2 \leq \frac{n}{n - s} = \frac{s}{n - s} + 1,$$

which fails for $1 \leq s \leq n - 1$. Hence we get a contradiction, so that for general $(t, y) \in \{t\} \times \mathbb{P}^1$, the 0-cycle $(\ell_{(t,y)} \cdot X) - f(q_1) - \cdots - f(q_c)$ is supported at a point $x_t \in X$, which depends only on t .

Moreover, denoting by Σ_t the closure of the surface swept out by the lines $\ell_{(t,y)}$ with $(t, y) \in \{t\} \times \mathbb{P}^1$, the argument above assures that the intersection $X \cap \Sigma_t$ is supported on $f(C_t)$, so that $x_t \in f(C_t)$. In particular, we have that $\ell_{(t,y)} \cdot X = (d - c)x_t + f(q_1) + \cdots + f(q_c)$, i.e. $\ell_{(t,y)}$ is a line of the ruling of the cone $V_{x_t}^{d-c}$. Then (ii) and the final assertion of the proposition follow. \square

3. THE VECTOR BUNDLES APPROACH

Let $X_F \subset \mathbb{P}^{n+1}$ be a smooth hypersurface defined by the vanishing of a non-zero polynomial F of degree d , and set $\mathbb{G} := \mathbb{G}(1, n + 1)$. For a positive integer r , we consider the variety

$$\tilde{\Delta}_{r,F} := \{(x, [\ell]) \in X_F \times \mathbb{G} \mid \ell \cdot X_F \geq rx\} \quad (3.1)$$

which, for $r \leq \min\{d, 2n + 1\}$, turns out to be nonempty, smooth, irreducible of dimension $2n + 1 - r$ (cf. Lemma 3.2). The main goal of this section is to find necessary conditions on the integers r, d, n, l

for the existence of an uniruled subvariety $Y \subset \tilde{\Delta}_{r,F}$ of dimension $l \geq n$, for X_F very general (see Corollary 3.6). To this aim, we will follow the argument of [15], relying on the approach of [11, 19].

3.1. Hypersurfaces and lines of high contact order. Given two integers $n, d \geq 2$, we set

$$\mathbb{P} := \mathbb{P}^{n+1}, \quad N + 1 := \dim_{\mathbb{C}}(S_d).$$

The universal hypersurface $\mathcal{X} \subset \mathbb{P} \times S_d^*$ over S_d^* has dimension $N + 1 + n$, and let

$$S_d^* \xleftarrow{\sigma} \mathcal{X} \xrightarrow{\tau} \mathbb{P}$$

be the two projections. Besides, we denote by $\mathcal{P} \subset \mathbb{P} \times \mathbb{G}$ the universal family of lines over \mathbb{G} , endowed with the two projections

$$\mathbb{G} \xleftarrow{\pi_2} \mathcal{P} \xrightarrow{\pi_1} \mathbb{P}$$

The Picard group $\text{Pic}(\mathcal{P})$ is generated by the line bundles

$$L := \pi_2^*(\mathcal{O}_{\mathbb{G}}(1)) \quad \text{and} \quad H := \pi_1^*(\mathcal{O}_{\mathbb{P}}(1))$$

where $\mathcal{O}_{\mathbb{G}}(1)$ is the *Plücker line bundle* on \mathbb{G} (cf. e.g. [20, p. 609]).

Let $1 \leq r \leq \min\{d, n + 1\}$ be an integer, and consider the variety

$$\tilde{\Delta}_r := \{(x, [\ell], F) \in \mathcal{P} \times S_d^* \mid \ell \cdot X_F \geq rx\}$$

endowed with the two projections

$$\mathcal{P} \xleftarrow{\psi} \tilde{\Delta}_r \xrightarrow{\phi} S_d^*.$$

Since $r \leq d$, the map ψ is surjective; indeed, for any $(x, [\ell]) \in \mathcal{P}$, there is the triple $(x, [\ell], F) \in \tilde{\Delta}_r$, where X_F is the hypersurface consisting of d general hyperplanes through $x \in \mathbb{P}$. For any $(x, [\ell]) \in \mathcal{P}$, one has

$$\psi^{-1}((x, [\ell])) \cong \{F \in S_d^* \mid X_F \cdot \ell \geq rx\} \cong H^0(\mathbb{P}, \mathcal{I}_{rx/\mathbb{P}}(d)) \setminus \{0\}.$$

Moreover, the line bundle $\mathcal{O}_{\mathbb{P}}(d)$ is $(r - 1)$ -very ample as $d \geq r$. Thus from the exact sequence

$$0 \rightarrow \mathcal{I}_{rx/\mathbb{P}}(d) \rightarrow \mathcal{O}_{\mathbb{P}}(d) \rightarrow \mathcal{O}_{rx} \rightarrow 0$$

we deduce that $h^1(\mathcal{I}_{rx/\mathbb{P}}(d)) = 0$. Hence ψ is smooth of relative dimension $N + 1 - r$, and each fiber is irreducible. Then $\tilde{\Delta}_r$ is smooth, irreducible, of dimension

$$\dim(\tilde{\Delta}_r) = \dim(\mathcal{P}) + h^0(\mathcal{I}_{rx/\mathbb{P}}(d)) = 2n + 2 + N - r.$$

Remark 3.1. It follows from the definition that $\tilde{\Delta}_r$ is invariant under the action of $\text{GL}(n + 2)$ on $\mathcal{P} \times S_d^*$ defined as follows. Given any $g \in \text{GL}(n + 2)$ and any triple $(x, [\ell], F) \in \mathcal{P} \times S_d^*$, then

$$g \cdot (x, [\ell], F) := \left(g(x), g(\ell), (g^{-1})^*(F) \right) \in \mathcal{P} \times S_d^*,$$

where $g(\ell)$ denotes the line which is (projectively) equivalent to ℓ under g .

We note that $\tilde{\Delta}_r$ is endowed with the natural map $\rho: \tilde{\Delta}_r \rightarrow \mathcal{X}$ given by $(x, [\ell], F) \mapsto (x, F)$, which fits in the following commutative diagram

$$\begin{array}{ccccc}
 & & \phi & & \\
 & & \curvearrowright & & \\
 \tilde{\Delta}_r & \xrightarrow{\rho} & \mathcal{X} & \xrightarrow{\sigma} & S_d^* \\
 \psi \downarrow & & \downarrow \tau & & \\
 \mathcal{P} & \xrightarrow{\pi_1} & \mathbb{P} & & \\
 \pi_2 \downarrow & & & & \\
 \mathbb{G} & & & &
 \end{array}$$

Since $r \leq n + 1$, [15, Lemma 4.1] may be rephrased as follows.

Lemma 3.2. *The map ρ is surjective. Furthermore, if $F \in S_d^*$ is a general polynomial, then*

(a) *the subvariety*

$$\tilde{\Delta}_{r,F} := \phi^{-1}(F) \subset \tilde{\Delta}_r$$

is smooth, irreducible, of dimension $2n + 1 - r$;

(b) *the restriction $\tau \circ \rho|_{\tilde{\Delta}_{r,F}}: \tilde{\Delta}_{r,F} \rightarrow \mathbb{P}$ maps onto*

$$\Delta_{r,F} := \{p \in X_F \mid \exists [\ell] \in \mathbb{G} \text{ s.t. } \ell \cdot X_F \geq rp\} \subseteq X_F.$$

3.2. The canonical bundle of $\tilde{\Delta}_{r,F}$. Let $F \in S_d^*$ be general. The restriction to $\tilde{\Delta}_{r,F}$ of the map $\psi: \tilde{\Delta}_r \rightarrow \mathcal{P}$ is an isomorphism onto its image, realizing $\tilde{\Delta}_{r,F}$ as in (3.1).

Let \mathcal{L} be the universal rank 2 quotient bundle on \mathbb{G} . For any positive integer m , we set $\mathcal{E}_m := \text{Sym}^m(\mathcal{L})$. Since its fiber over $[\ell] \in \mathbb{G}$ identifies with $H^0(\ell, \mathcal{O}_\ell(m))$, the rank of \mathcal{E}_m is $m + 1$ and $c_1(\mathcal{E}_m) = \mathcal{O}_{\mathbb{G}}(\frac{m(m+1)}{2})$. For any $1 \leq r \leq d$, the vector bundle $\pi_2^*(\mathcal{E}_d)$ on \mathcal{P} contains a rank $d + 1 - r$ sub-vector bundle $\mathcal{A}_{d,r} \hookrightarrow \pi_2^*(\mathcal{E}_d)$, whose fiber over $(x, [\ell]) \in \mathcal{P}$ identifies with $H^0(\ell, \mathcal{O}_\ell(d - rx))$.

Consider the exact sequence

$$0 \rightarrow \mathcal{A}_{d,r} \rightarrow \pi_2^*(\mathcal{E}_d) \rightarrow \mathcal{B}_{d,r} \rightarrow 0 \quad (3.2)$$

defining the quotient $\mathcal{B}_{d,r}$ as a rank r vector bundle. Arguing as in [15, p. 263-264] one sees that, if $F \in S_d^*$ is a general polynomial, then $\tilde{\Delta}_{r,F} \subset \mathcal{P}$ is the vanishing locus of a global section of $\mathcal{B}_{d,r}$. Thus the normal bundle $N_{\tilde{\Delta}_{r,F}/\mathcal{P}}$ of $\tilde{\Delta}_{r,F}$ in \mathcal{P} satisfies

$$N_{\tilde{\Delta}_{r,F}/\mathcal{P}} \cong \mathcal{B}_{d,r}|_{\tilde{\Delta}_{r,F}}. \quad (3.3)$$

Moreover, the smoothness of $\tilde{\Delta}_{r,F}$ and adjunction formula yield

$$\omega_{\tilde{\Delta}_{r,F}} = \mathcal{O}_{\tilde{\Delta}_{r,F}}(K_{\mathcal{P}} \otimes c_1(N_{\tilde{\Delta}_{r,F}/\mathcal{P}})) \cong \mathcal{O}_{\tilde{\Delta}_{r,F}}(K_{\mathcal{P}} \otimes c_1(\mathcal{B}_{d,r})), \quad (3.4)$$

with $K_{\mathcal{P}} = -2H - (n + 1)L$ (cf. [20, p. 609]). In order to compute $c_1(\mathcal{B}_{d,r})$, we note that

$$\mathcal{A}_{d,r} \cong \pi_2^*(\mathcal{E}_{d-r}) \otimes \mathcal{A}_{r,r},$$

where $\mathcal{A}_{r,r} = r(L - H) \in \text{Pic}(\mathcal{P})$. Whence we deduce

$$c_1(\mathcal{A}_{d,r}) = \frac{(d+r)(d-r+1)}{2}L - r(d-r+1)H,$$

so that (3.2) implies

$$c_1(\mathcal{B}_{d,r}) = \frac{r(r-1)}{2}L + r(d-r+1)H$$

and by (3.4), we conclude that

$$\omega_{\tilde{\Delta}_{r,F}} = \mathcal{O}_{\tilde{\Delta}_{r,F}} \left((r(d-r+1) - 2)H + \left(\frac{r(r-1)}{2} - n - 1 \right) L \right). \quad (3.5)$$

3.3. Global generation lemmas. Let $F \in S_d^*$ be a general point, and consider the inclusion $\tilde{\Delta}_{r,F} \subset \tilde{\Delta}_r$. For any integer $1 \leq l \leq \dim(\tilde{\Delta}_{r,F}) = 2n+1-r$, one has

$$\bigwedge^{2n+1-r-l} T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \cong \Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{\vee 2n+1-r-l} \otimes \Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{2n+2+N-r} \cong \Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{N+1+l}. \quad (3.6)$$

Consider the exact sequence

$$0 \longrightarrow M_d \longrightarrow S_d \otimes \mathcal{O}_{\mathbb{G}} \xrightarrow{\text{ev}} \mathcal{E}_d \longrightarrow 0 \quad (3.7)$$

where

$$\begin{aligned} \text{ev}_{[\ell]}: S_d \otimes \mathcal{O}_{\mathbb{G},[\ell]} &\longrightarrow \mathcal{E}_d|_{\ell} \cong H^0(\ell, \mathcal{O}_{\ell}(d)) \\ (F, [\ell]) &\longmapsto F|_{\ell} \end{aligned}$$

and $M_d := \ker(\text{ev})$. The fiber of M_d at $[\ell] \in \mathbb{G}$ identifies with $H^0(\ell, \mathcal{I}_{\ell/\mathbb{P}}(d))$.

Next we consider the exact commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathcal{M}_d & & \\ & & & & \downarrow & & \\ 0 & \rightarrow & \mathcal{N}_{d,r} & \rightarrow & S_d \otimes \mathcal{O}_{\mathcal{P}} & \rightarrow & \mathcal{B}_{d,r} \rightarrow 0 \\ & & & & \downarrow & & \parallel \\ 0 & \rightarrow & \mathcal{A}_{d,r} & \rightarrow & \pi_2^*(\mathcal{E}_d) & \rightarrow & \mathcal{B}_{d,r} \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where the central vertical column is obtained by pulling ev back to \mathcal{P} via π_2 , the bottom row is (3.2), $\mathcal{N}_{d,r}$ and \mathcal{M}_d are defined as the appropriate kernels, and

$$\text{rk}(\mathcal{M}_d) = N - d, \quad \text{rk}(\mathcal{N}_{d,r}) = N + 1 - r.$$

By the Snake Lemma we have the exact commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \rightarrow & \mathcal{M}_d & \rightarrow & \mathcal{N}_{d,r} & \rightarrow & \mathcal{A}_{d,r} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & S_d \otimes \mathcal{O}_{\mathcal{P}} & = & S_d \otimes \mathcal{O}_{\mathcal{P}} & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{A}_{d,r} & \rightarrow & \pi_2^*(\mathcal{E}_d) & \rightarrow & \mathcal{B}_{d,r} \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array} \tag{3.8}$$

Lemma 3.3. *With notation as above, one has:*

- (i) $\mathcal{M}_d = \pi_2^*(M_d)$;
- (ii) $\bigwedge^t \mathcal{M}_d \otimes tL$ is globally generated, for any integer $1 \leq t \leq \text{rk}(\mathcal{M}_d) = N - d$;
- (iii) $\mathcal{M}_{d|\tilde{\Delta}_{r,F}} \hookrightarrow \mathcal{N}_{d,r|\tilde{\Delta}_{r,F}} \hookrightarrow T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}$.

Proof. Assertion (i) follows from (3.7) and the left-most exact column in (3.8).

As for (ii), by [15, Proposition 2.2(ii)], the vector bundle $M_d \otimes \mathcal{O}_{\mathbb{G}}(1)$ is globally generated, i.e. $H^0(\mathbb{G}, M_d \otimes \mathcal{O}_{\mathbb{G}}(1)) \otimes \mathcal{O}_{\mathbb{G}} \twoheadrightarrow M_d \otimes \mathcal{O}_{\mathbb{G}}(1)$. Applying π_2^* to this surjection and using (i), we obtain an induced surjection

$$H^0(\mathbb{G}, M_d \otimes \mathcal{O}_{\mathbb{G}}(1)) \otimes \mathcal{O}_{\mathcal{P}} \twoheadrightarrow \mathcal{M}_d \otimes L. \tag{3.9}$$

Similarly, one has $H^0(\mathcal{P}, \mathcal{M}_d \otimes L) \cong H^0(\mathcal{P}, \pi_2^*(M_d \otimes \mathcal{O}_{\mathbb{G}}(1)))$. We have

$$\pi_{2*}(\pi_2^*(M_d \otimes \mathcal{O}_{\mathbb{G}}(1)) \otimes \mathcal{O}_{\mathcal{P}}) \cong M_d \otimes \mathcal{O}_{\mathbb{G}}(1) \otimes \pi_{2*}(\mathcal{O}_{\mathcal{P}}) \cong M_d \otimes \mathcal{O}_{\mathbb{G}}(1)$$

because $\pi_{2*}(\mathcal{O}_{\mathcal{P}}) \cong \mathcal{O}_{\mathbb{G}}$ since the π_2 -fibers are lines. Thus

$$H^0(\mathcal{P}, \mathcal{M}_d \otimes L) \cong H^0(\mathbb{G}, M_d \otimes \mathcal{O}_{\mathbb{G}}(1)).$$

This isomorphism and (3.9) yield that $\mathcal{M}_d \otimes L$ is globally generated. Then we can conclude the proof of (ii) as in [19, Corollary 1.2].

Finally we prove (iii). Recall that, for any $F \in S_d^*$, $\tilde{\Delta}_{r,F}$ identifies with its image in \mathcal{P} under ψ . Accordingly, we identify $\psi^*(T_{\mathcal{P}})|_{\tilde{\Delta}_{r,F}}$ with $T_{\mathcal{P}|\pi(\tilde{\Delta}_{r,F})}$ and, by abusing notation, we denote it by $T_{\mathcal{P}|\tilde{\Delta}_{r,F}}$. From the inclusions of schemes

$$\tilde{\Delta}_{r,F} \subset \tilde{\Delta}_r \subset \mathcal{P} \times S_d^*$$

and the fact that $\tilde{\Delta}_{r,F}$ is a ϕ -fiber, we obtain the following exact sequence

$$0 \rightarrow T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} \rightarrow T_{\mathcal{P}|\tilde{\Delta}_{r,F}} \oplus (S_d \otimes \mathcal{O}_{\tilde{\Delta}_{r,F}}) \rightarrow N_{\tilde{\Delta}_{r,F}/\mathcal{P}} \rightarrow 0. \tag{3.10}$$

Restricting the right-most exact column in (3.8) to $\tilde{\Delta}_{r,F}$, we get

$$0 \rightarrow \mathcal{N}_{d,r|\tilde{\Delta}_{r,F}} \rightarrow S_d \otimes \mathcal{O}_{\tilde{\Delta}_{r,F}} \rightarrow \mathcal{B}_{d,r|\tilde{\Delta}_{r,F}} \cong N_{\tilde{\Delta}_{r,F}/\mathcal{P}} \rightarrow 0, \tag{3.11}$$

where the isomorphism on the right is (3.3), whereas the injectivity on the left follows from $\mathcal{T}or^1(\mathcal{B}_{d,r}; \mathcal{O}_{\tilde{\Delta}_{r,F}}) = 0$, since $\mathcal{B}_{d,r}$ is locally free. The sequences (3.11) and (3.10) fit in the exact commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \mathcal{N}_{d,r|\tilde{\Delta}_{r,F}} & \rightarrow & S_d \otimes \mathcal{O}_{\tilde{\Delta}_{r,F}} & \rightarrow & N_{\tilde{\Delta}_{r,F}/\mathcal{P}} & \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} & \rightarrow & T_{\mathcal{P}|\tilde{\Delta}_{r,F}} \oplus (S_d \otimes \mathcal{O}_{\tilde{\Delta}_{r,F}}) & \rightarrow & N_{\tilde{\Delta}_{r,F}/\mathcal{P}} & \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & T_{\mathcal{P}|\tilde{\Delta}_{r,F}} & = & T_{\mathcal{P}|\tilde{\Delta}_{r,F}} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & .
\end{array} \tag{3.12}$$

Then the left-most column in (3.12) gives $\mathcal{N}_{d,r|\tilde{\Delta}_{r,F}} \hookrightarrow T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}$. Finally, by restricting the upper sequence in (3.8) to $\tilde{\Delta}_{r,F}$, we get the inclusion $\mathcal{M}_{d|\tilde{\Delta}_{r,F}} \hookrightarrow \mathcal{N}_{d,r|\tilde{\Delta}_{r,F}}$, proving (iii). \square

By (3.6), Lemma 3.3(iii) ensures that for any integer $1 \leq l \leq 2n+1-r$, there is an injection

$$\bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \hookrightarrow \bigwedge^{2n+1-r-l} T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \cong \Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{N+1+l}. \tag{3.13}$$

Moreover, (3.5) implies that

$$\bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \cong \bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \left(\frac{r(r-1)}{2} - n - 1 \right) L_{|\tilde{\Delta}_{r,F}} \otimes (r(d-r+1)-2) H_{|\tilde{\Delta}_{r,F}}. \tag{3.14}$$

Since H is globally generated, $1 \leq r \leq \min\{d, n+1\}$ and $d \geq 2$, then $(r(d-r+1)-2)H_{|\tilde{\Delta}_{r,F}}$ is globally generated, as well. Similarly, by Lemma 3.3(ii), $\bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \left(\frac{r(r-1)}{2} - n - 1 \right) L_{|\tilde{\Delta}_{r,F}}$ is globally generated if $\frac{r(r-1)}{2} - n - 1 \geq 2n+1-r-l$, that is if

$$\frac{r(r+1)}{2} - 3n - 2 + l \geq 0. \tag{3.15}$$

Taking into account (3.14), we deduce the following

Lemma 3.4. *If (3.15) holds, then $\bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}}$ is globally generated.*

3.4. Subvarieties of $\tilde{\Delta}_{r,F}$ of positive geometric genus. Let $U \subset S_d^*$ be the open dense subset parametrizing polynomials $F \in S_d^*$ such that X_F is smooth. To ease notation, we still denote by $\tilde{\Delta}_r$ the restriction of $\tilde{\Delta}_r \subset \mathcal{P} \times S_d$ to U , and let

$$U \xleftarrow{\phi} \tilde{\Delta}_r \subset \mathcal{P} \times U$$

be the projection.

Next we suppose that, up to possibly shrinking U and replacing it with a suitable étale cover, there is a diagram

$$\begin{array}{ccc}
\mathcal{Y} & \hookrightarrow & \tilde{\Delta}_r \\
& \searrow & \downarrow \phi \\
& & U,
\end{array}$$

where $\mathcal{Y} \subseteq \tilde{\Delta}_r$ is an integral scheme and $\phi|_{\mathcal{Y}}$ is flat of relative dimension $l \leq 2n+1-r = \dim(\tilde{\Delta}_{r,F})$. Then

$$\dim(\mathcal{Y}) = N+1+l$$

and, for a general $F \in U$, $Y_F := \phi|_{\mathcal{Y}}^{-1}(F) \subset \tilde{\Delta}_{r,F}$ is irreducible of dimension l .

Then we consider the diagram

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\nu} & \mathcal{Y} \\ & \searrow \tilde{\phi} & \downarrow \phi|_{\mathcal{Y}} \\ & & U \end{array}$$

where ν is a desingularization of \mathcal{Y} and $\tilde{\phi}$ is the induced map. It follows from the smoothness of $\tilde{\mathcal{Y}}$ that the restriction of ν to a general fiber $\tilde{Y}_F := \tilde{\phi}^{-1}(F)$ is a desingularization of Y_F . Following [15, Section 2], we shall also assume \mathcal{Y} to be invariant under the action of $\mathrm{GL}(n+2)$ on $\mathcal{P} \times S_d$ as in Remark 3.1.

Let us consider the map $\iota: \tilde{\mathcal{Y}} \rightarrow \tilde{\Delta}_r$, obtained by composing ν with the inclusion $\mathcal{Y} \hookrightarrow \tilde{\Delta}_r$. For any integer $l \leq 2n+1-r$, the map $\iota^*: \Omega_{\tilde{\Delta}_r}^1 \rightarrow \Omega_{\tilde{\mathcal{Y}}}^1$ induces the map

$$\Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{N+1+l} \xrightarrow{\beta} \Omega_{\tilde{\mathcal{Y}}|\tilde{Y}_F}^{N+1+l} \cong \omega_{\tilde{Y}_F},$$

(cf. [15, Section 2.2(3)]). Composing β with the injection (3.13), we obtain a map

$$\bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \xrightarrow{\alpha} \omega_{\tilde{Y}_F}. \quad (3.16)$$

Lemma 3.5. *Let r, l and d be positive integers such that (3.15) and $d \geq \max\{2, r\}$ hold. Let $\mathcal{Y} \subset \tilde{\Delta}_r$ be any integral subscheme as above (in particular it is invariant under the $\mathrm{GL}(n+2, \mathbb{C})$ -action on $\mathcal{P} \times S_d$). Then, for $F \in U$ general, the map*

$$H^0(\alpha): H^0\left(\tilde{\Delta}_{r,F}, \bigwedge^{2n+1-r-l} \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}}\right) \rightarrow H^0(\tilde{Y}_F, \omega_{\tilde{Y}_F})$$

induced by (3.16) is non-zero. In particular, the geometric genus of \tilde{Y}_F satisfies $p_g(\tilde{Y}_F) := h^0(\tilde{Y}_F, \omega_{\tilde{Y}_F}) > 0$.

Proof. The proof uses the same approach as in [15, Proofs of Lemmas 2.1(i) and 2.3], which in turn follows [11, 12, 19, 20]. For the reader's convenience we recall the argument but we will be brief.

Consider the exact sequence

$$0 \rightarrow T_{\tilde{\Delta}_r}^{\mathrm{vert}} \rightarrow T_{\tilde{\Delta}_r} \xrightarrow{d\psi} \psi^*(T_{\mathcal{P}}) \rightarrow 0. \quad (3.17)$$

which defines $T_{\tilde{\Delta}_r}^{\mathrm{vert}} := \mathrm{Ker}(d\psi)$. With the usual identification of $\tilde{\Delta}_{r,F}$ with its projection to \mathcal{P} via ψ (cf. §3.2), (3.17) yields the exact sequence

$$0 \rightarrow T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{\mathrm{vert}} \rightarrow T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} \rightarrow T_{\mathcal{P}|\tilde{\Delta}_{r,F}} \rightarrow 0, \quad (3.18)$$

where the injectivity on the left follows from $\mathcal{T}\mathrm{or}^1(\psi^*(T_{\mathcal{P}}); \mathcal{O}_{\tilde{\Delta}_{r,F}}) = 0$. By comparing (3.18) with the left-most column in (3.12), we deduce that $\mathcal{N}_{d,r|\tilde{\Delta}_{r,F}} \cong T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{\mathrm{vert}}$. This, (3.6) and Lemma 3.3(iii) yield

$$\bigwedge^c \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \hookrightarrow \bigwedge^c T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{\mathrm{vert}} \otimes \omega_{\tilde{\Delta}_{r,F}} \hookrightarrow \bigwedge^c T_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}} \cong \Omega_{\tilde{\Delta}_r|\tilde{\Delta}_{r,F}}^{N+1+l} \quad \text{where } c := 2n+1-r-l.$$

By the $\mathrm{GL}(n+2)$ -invariance of \mathcal{Y} we have also the exact sequence

$$0 \rightarrow T_{\mathcal{Y}}^{\mathrm{vert}} \rightarrow T_{\mathcal{Y}} \xrightarrow{d\psi} \psi^*(T_{\mathcal{P}}) \rightarrow 0$$

(cf. [12, Remark 2.3(a)] and [15, Lemma 2.1(i)]), hence given a smooth point $\xi := (x, [\ell], F) \in \mathcal{Y}$ we have

$$\mathrm{codim}_{T_{\tilde{\Delta}_r, \xi}^{\mathrm{vert}}}(T_{\mathcal{Y}, \xi}^{\mathrm{vert}}) = \mathrm{codim}_{\tilde{\Delta}_r}(\mathcal{Y}) = c.$$

Now $H^0(\tilde{\Delta}_{r,F}, \bigwedge^c \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}})$ can be considered as a space of global sections of a line bundle over the relative Grassmannian of c -codimensional subspaces of $T_{\tilde{\Delta}_{r,F}}^{\text{vert}} \otimes \omega_{\tilde{\Delta}_{r,F}}$. Then the global generation of $\bigwedge^c \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}}$, which holds by Lemma 3.4, implies that there exists a section

$$s \in H^0\left(\tilde{\Delta}_{r,F}, \bigwedge^c \mathcal{M}_{d|\tilde{\Delta}_{r,F}} \otimes \omega_{\tilde{\Delta}_{r,F}}\right) \subset H^0\left(\tilde{\Delta}_{r,F}, \Omega_{\tilde{\Delta}_{r,F}}^{N+1+l}\right)$$

such that

$$s(x, [\ell]) \notin \text{Ann}(T_{\mathcal{Y}, \xi}^{\text{vert}}).$$

Since $\iota: \tilde{\mathcal{Y}} \rightarrow \tilde{\Delta}_r$ is generically an immersion, we obtain a non-zero element in $H^0(\tilde{Y}_F, \omega_{\tilde{Y}_F})$. \square

If $l \geq n$, then $\frac{r(r+1)}{2} \geq 2n + 2$ implies (3.15), and we have:

Corollary 3.6. *Let $r, l \geq n$ and d be positive integers such that $d \geq \max\{2, r\}$. Suppose there is a \mathcal{Y} as in Lemma 3.5, such that for $F \in U$ general one has $p_g(\tilde{Y}_F) = 0$ (e.g. when \tilde{Y}_F is uniruled), then*

$$\frac{r(r+1)}{2} \leq 2n + 1. \quad (3.19)$$

Moreover, if $F \in U$ is very general and if $\tilde{\Delta}_{r,F}$ has a l -dimensional irreducible subvariety Y_F fitting in a $\text{GL}(n+2, \mathbb{C})$ -invariant family, whose desingularization \tilde{Y}_F has $p_g(\tilde{Y}_F) = 0$, then (3.19) holds.

Proof. The first part follows by Lemma 3.5. As for the final assertion, suppose (3.19) does not hold. Then Lemma 3.5 yields that the set of polynomials $F \in U$ as in the statement is the union of countably many closed subsets of U . Hence the assertion holds. \square

4. COVERING GONALITY OF VERY GENERAL HYPERSURFACES

In this section we conclude the proof of Theorem 1.1. To start, we prove the following.

Theorem 4.1. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{cov. gon}(X) \geq d - \left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor.$$

Proof. For $n = 1$ and $n = 2$, the assertion holds as the covering gonality of X is $d - 1$ and $d - 2$, respectively (cf. [7, Teorema 3.14] and [13, Corollary 1.8]). So we assume hereafter that $n \geq 3$.

Let $F \in S_d^*$ be very general, set $c := \text{cov. gon}(X_F)$ and consider a covering family $\mathcal{C} \xrightarrow{\pi} T$ of c -gonal curves as in §2.4 from which we keep the notation. We will assume that the curves of \mathcal{C} have minimal degree among all c -gonal curves covering X_F .

Since $n \geq 3$, Theorem 2.8 yields $c \leq d - 3$. Thus Proposition 2.11 applies; for general $(t, y) \in T \times \mathbb{P}^1$, there exists a line $\ell_{(t,y)} \subset V_{x_t}^{d-c}$ such that

$$\ell_{(t,y)} \cdot X = (d-c)x_t + f(q_1) + \cdots + f(q_c)$$

where $q_1 + \cdots + q_c$ is a divisor of a g_c^1 on C_t . Then we define the map

$$\Psi_{\mathcal{C}}: \begin{array}{ccc} T \times \mathbb{P}^1 & \dashrightarrow & \tilde{\Delta}_{d-c,F} \\ (t, y) & \longmapsto & (x_t, [\ell_{t,y}], F) \end{array}.$$

The variety $Y_{F,\mathcal{C}} := \overline{\Psi_{\mathcal{C}}(T \times \mathbb{P}^1)} \subset \tilde{\Delta}_{d-c,F}$ is covered by the rational curves $\overline{\Psi_{\mathcal{C}}(\{t\} \times \mathbb{P}^1)}$, with $t \in T$. As $(t, y) \in T \times \mathbb{P}^1$ vary, the lines $\ell_{t,y}$ describe a n -dimensional subvariety of the Grassmannian $\mathbb{G}(1, n+1)$ (cf. §2.4), so that $\dim(Y_{F,\mathcal{C}}) = n$.

Define

$$Y_F := \bigcup_c Y_{F,\mathcal{C}} \subset \tilde{\Delta}_{d-c,F}$$

with \mathcal{C} varying among the (finitely many maximal) families of c -gonal curves of minimal degree covering X_F . We may pretend Y_F to be irreducible, otherwise we replace it with one of its irreducible components. One has $\dim(Y_F) \geq n$ and Y_F is covered by rational curves. Clearly the last assertion of Corollary 3.6 can be applied to Y_F , with $r = d - c$, so

$$\frac{(d-c)(d-c+1)}{2} \leq 2n+1 \quad \text{hence} \quad d-c \leq \frac{\sqrt{16n+9}-1}{2}.$$

Being c an integer, we find the lower bound in the statement. \square

The conclusion of the proof of Theorem 1.1 is given by the following elementary computational Lemma, whose proof can be left to the reader.

Lemma 4.2. *Let $n \in \mathbb{N}$. Then*

$$\left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor = \begin{cases} \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor + 1 & \text{if } n \in \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 \mid \alpha \in \mathbb{N}\} \\ \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

5. FINAL REMARKS, OPEN PROBLEMS AND SPECULATIONS

5.1. Are all curves computing the covering gonality planar? As we mentioned in the Introduction, an interesting problem is to characterize the curves computing the covering gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n+1$, in particular, one may ask the following:

Question 5.1. If $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n+1$, are the curves computing the covering gonality plane curves?

The proof of Theorem 1.1 shows that this question is related to the existence of rational curves on certain complete intersections. Specifically, if $c = \text{cov.gon}(X)$, $h = d - c$ and $x \in X_1^h$ (see §2.3) is a very general point, one is led to ask the following:

Question 5.2. Does $\Lambda_x^h \subset \mathbb{P}^{n-1}$ contain rational curves other than lines?

Recall that when $Y \subset \mathbb{P}^m$ is a very general hypersurface of degree $2m-3$, then all rational curves on Y are lines, whereas there are no rational curves on very general hypersurfaces of degree $2m-2$ (cf. [15, 19]). The integer $h = \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$ is the largest such that the locus of complete intersections $Y \subset \mathbb{P}^{n-1}$ of type $(2, \dots, h-1)$ containing a line has codimension at most 1 in the parameter space. Thus it is natural to investigate whether the general Y containing a line may contain other rational curves, i.e. Question 5.2 arises very naturally in this context.

A negative answer to Question 5.2 is a necessary condition for an affirmative answer to Question 5.1. However this condition is not sufficient, as the cases $n \leq 7$ show.

If $n = 3$, then $h = 3$ and X_1^3 is the locus of points $x \in X$ such that the conic Λ_x^3 is reducible, so that the answer to Question 5.2 is negative. However, if $x \in X$ is general, then Λ_x^h is an irreducible conic and the intersection of the irreducible cone V_x^h with X is not a plane curve but it still has gonality $d-3$. Similar considerations for $n = 4$, for which again $h = 3$.

If $n = 5$, then $h = 4$, and X_1^4 is the locus of points $x \in X$ such that Λ_x^3 , a $K3$ surface of degree 6 in \mathbb{P}^4 , contains a line. The general such a surface contains no other rational curve, so again the answer to Question 5.2 is negative. However, if $x \in X$ is general, then Λ_x^h is a general $K3$ surface of degree 6 in \mathbb{P}^4 , which contains infinitely many singular rational curves, so that again there are (infinitely many) covering families of X consisting of $(d-4)$ -gonal curves other than plane curves. Similar considerations for $n = 6, 7$, for which still $h = 4$.

Hence, to hope for an affirmative answer to Question 5.1 it is necessary to assume n sufficiently large.

5.2. The connecting gonality. Another birational invariant introduced in [4] is the *connecting gonality*, defined as

$$\text{conn. gon}(Y) := \min \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given two general points } y_1, y_2 \in Y, \exists \text{ an irreducible} \\ \text{curve } C \subset Y \text{ such that } y_1, y_2 \in C \text{ and } \text{gon}(C) = c \end{array} \right. \right\}.$$

Since having $\text{conn. gon}(Y) = 1$ is equivalent to being rationally connected, the connecting gonality can be thought as a measure of the failure of Y to satisfy such a property.

When $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n + 2$, our approach may determine restrictions also to $\text{conn. gon}(X)$. Indeed, one could argue as in §2.2 and §2.3, and look for complete intersections in \mathbb{P}^{n-1} of type $(2, 3, \dots, h-1)$ containing large dimensional families of lines (see e.g. [5, Corollary 2.2]). Since the dimension of a family of curves computing $\text{conn. gon}(X)$ must be at least $2n - 2$ (see e.g. [2, Section 2.1]), a naive computation suggests the following upper bound

$$\text{conn. gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+9} - 1}{2} \right\rfloor. \quad (5.1)$$

On the other hand, the curves computing the connecting gonality of X are still governed by Proposition 2.11. Thus a lower bound on $\text{conn. gon}(X)$ could be obtained by improving the argument of Section 3.

In analogy with the covering gonality, one may naively conjecture that inequality (5.1) is actually an equality. Then (1.3) would imply that the difference between the covering and connecting gonality diverges as n grows. This would answer to a question raised in [4, Section 4].

5.3. The irrationality degree. Given an irreducible projective variety X of dimension n , for any positive integer $k \leq n$ one may define the *k-irrationality degree* of X to be the birational invariant

$$\text{irr}_k(X) := \min \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given a general point } x \in X, \exists \text{ an irreducible} \\ \text{subvariety } Z \subseteq X \text{ of dimension } k \text{ such that } x \in Z \text{ and} \\ \text{there is a rational dominant map } Z \dashrightarrow \mathbb{P}^k \text{ of degree } c \end{array} \right. \right\}.$$

If $k = n$, this is the *irrationality degree* $\text{irr}(X)$ of X (see [1] for references), whereas $\text{irr}_1(X) = \text{cov. gon}(X)$. Of course one has

$$\text{irr}(X) \geq \text{irr}_{n-1}(X) \geq \dots \geq \text{irr}_2(X) \geq \text{cov. gon}(X).$$

It would be interesting to study this string of inequalities for $X \subset \mathbb{P}^{n+1}$ a very general hypersurface as above. It is likely that our methods could be useful for that. We hope to come back to this in the future.

Interestingly enough, a relevant amount of the inequalities above must consist of equalities. Indeed, [4, Theorem A] ensures that $\text{irr}(X) = d - 1$, so that Theorem 1.1 yields that $\text{irr}(X) - \text{cov. gon}(X) \leq \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor$, which asymptotically equals $2\sqrt{n}$.

5.4. Gaps. Finally, given an irreducible projective variety X of dimension n , for any positive integer $k \leq n$ one can consider the following numerical set

$$N_k(X) := \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given a general point } x \in X, \exists \text{ an irreducible} \\ \text{subvariety } Z \subseteq X \text{ of dimension } k \text{ such that } x \in Z \text{ and} \\ \text{there is a rational dominant map } Z \dashrightarrow \mathbb{P}^k \text{ of degree } c \end{array} \right. \right\}.$$

Of course

$$N_n(X) \subseteq N_{n-1}(X) \subseteq \dots \subseteq N_2(X) \subseteq N_1(X).$$

A *gap* for $N_k(X)$ is an integer $c \in \mathbb{N}$ such that $c \notin N_k(X)$. It is not difficult to see that the set of gaps of $N_n(X)$ is bounded, so is the set of gaps of $N_k(X)$ for all $k \leq n$ (see e.g. [4, Problem 4.6]). It would be interesting to study the sets of gaps of $N_k(X)$, for X a very general hypersurface in \mathbb{P}^{n+1} as above. In this direction, we note that [13, Theorem 1.3] leads to relevant results in the case of surfaces in \mathbb{P}^3 .

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