QUANTUM DISTRIBUTIONAL SYMMETRIES I: ROTATABILITY

VITONOFRIO CRISMALE, FRANCESCO FIDALEO, AND STEFANO ROSSI

Contents

1. Introduction

In probability theory, the so-called *distributional symmetries* of stochastic processes play a fundamental role in many situations. As an example, we mention De Finetti theorem(s) and its consequences and generalisations like Hewitt and Savage and Ryll-Nardzewski theorems, and many others. These symmetries concern the invariance properties enjoyed by all finite joint distributions of a stochastic process.

Among the most analysed distributional symmetries, there are the stationarity, exchangeability and spreadability, the first two ones involving the shifts and the permutations on the index-set, respectively, whereas the last one arises by considering the monoid generated by the so called partial shifts, see e.g. [13] for an exhaustive treatment on this point. For the most common distributional symmetries in classical probability, the reader is referred e.g. to the seminal monograph [23] and the references cited therein, for applications and further details.

Recently, the attempt to generalise and study such symmetries in the quantum setting has provided also a huge amount of results and, correspondingly, of the associated literature. For a, perhaps partial, view on the topic, we refer to [10, 11, 12, 13, 14, 18] and the literature cited therein.

The symmetry involving the invariance under the rotations for real random variables is among the most natural ones. The invariance under

Date: December 27, 2021.

unitary transformations for complex random variables can be described in an analogous way, and thus we restrict our preliminary analysis to the former situation.

The rotational invariance for any set $(X_\iota)_{\iota\in I}$ of real, possibly unbounded, random variables is easily described in the following way. For each integer $n = 2, 3, \dots$ and for each choice of n indices $\iota_1, \iota_2, \dots, \iota_n \in I$, consider the joint distribution $\mu_{\iota_1,\iota_2,\dots,\iota_n}$ of $X_{\iota_1}, X_{\iota_2}, \dots, X_{\iota_n}$. which is a probability measure on $\mathbb{R}\times\cdots\times\mathbb{R} \equiv \mathbb{R}^n$.[1] The set $(X_\iota)_{\iota\in I}$ of random variables is said to be rotatable or invariant under rotations if, for each choice as above, and for each rotation $O \in \mathrm{O}(n)$, the corresponding distributions are invariant: $\mu_{\iota_1,\iota_2,\dots,\iota_n} = \mu_{\iota_1,\iota_2,\dots,\iota_n} \circ O$.

We also recall that an extension of such a symmetry to the full quantum case is done in [16], relative to the study of invariance property of noncommutative processes under the "bigger" object consisting of the so-called quantum rotations. Among the most important results, in that paper an operator-valued version of the celebrated Friedman theorem is proven.

As it is not so difficult to recognise, the complete and clear understanding of the distributional symmetry associated to rotations presents some conceptual and technical problems even in the classical case, as we are going to explain with a commutative toy-model whose index-set is made of only two elements. In other words, we consider the case of the toy stochastic process indexed by the two-point set $\{1, 2\}$, by noticing that the analysis can be easily extended to any set of indices.

Indeed, we start with two real-valued random variables X_1, X_2 defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. In such a situation, *the joint distribution* of X_1 and X_2 is the Borel probability measure $\mu = \mu_{X_1,X_2}$ on $\mathbb{R}^2$ determined by the cumulative function F_μ by

$$F_\mu(x_1, x_2) := \mu\big((-\infty, x_1] \times (-\infty, x_2]\big) = \mathrm{P}\big(\{X_j \leq x_j,\ j = 1, 2\}\big)\,,$$

see e.g. [4], Section 12.[2]

Definition 1.1. *The random variables X_1 and X_2 are said to be rotatable if*

$$\mu_{X_1,X_2} \circ O = \mu_{X_1,X_2}\,, \quad O \in \mathrm{O}(2)\,. \tag{1.1}$$

[1] See below for the definition of the joint distribution of a (finite) set of random variables.

[2] Notice that the integrals involving (summable) functions f of X_1 and X_1 can be directly expressed by the Lebesgue-Stieltjes integral against the cumulative functions F_μ: $\int_\Omega f\big(X_1(\omega), X_2(\omega)\mathrm{P}(\mathrm{d}\omega) = \int_{\mathbb{R}^2} f(x_1, x_2)F_\mu(\mathrm{d}x_1, \mathrm{d}x_2)$.

We note that (1.1) is equivalent to

$$\int_{\mathbb{R}^2} f\circ O^{\mathrm{t}}\mathrm{d}\mu=\int_{\mathbb{R}^2} f\mathrm{d}\mu\,,\quad f\in L^1(\mathbb{R}^2,\mu),\ O\in\mathrm{O}(2)\,. \tag{1.2}$$

Suppose now that

$$\int_{\mathbb{R}^2}|x_1|^{n_1}|x_2|^{n_2}\mathrm{d}\mu<+\infty\,,\quad n_1,n_2\in\mathbb{N}\,.$$

In this case, we say that μ *has moments of all orders.* It is then meaningful to define the sequence $(m_{n_1,n_2})_{n_1,n_2}\subset\mathbb{R}$ of moments of μ by $m_{n_1,n_2}:=\int_{\mathbb{R}^2}x_1^{n_1}x_2^{n_2}\mathrm{d}\mu$.

Recall that a probability measure μ on $\mathbb{R}^2$ having moments of all orders is uniquely determined by its moment if, given another probability measure ν having moments of all orders with the same sequence of moments as μ, one has $\nu=\mu$.[3] Note that, if μ is compactly supported, then μ is determined by its moments thanks to the Stone-Weierstrass theorem.

Proposition 1.2. *If the joint distribution μ of the random variables X_1 and X_2 has the moments of all orders and X_1 and X_2 are rotatable, then for each $n_1,n_2\in\mathbb{N}$ and $O\in\mathrm{O}(2)$,*

$$\int_{\mathbb{R}^2}x_1^{n_1}x_2^{n_2}\mathrm{d}\mu=\int_{\mathbb{R}^2}\big(O_{11}x_1+O_{21}x_2\big)^{n_1}\big(O_{12}x_1+O_{22}x_2\big)^{n_2}\mathrm{d}\mu\,. \tag{1.3}$$

Conversely, if μ is uniquely determined by the sequence $(m_{n_1,n_2})_{n_1,n_2\in\mathbb{N}}$, then the condition in (1.3) *implies the rotatablity for X_1 and X_2.*

Proof. We note that all functions $x_1^{n_1}x_2^{n_2}$ are in $L^1(\mathbb{R}^2,\mu)$ by assumption, and then (1.2) yields (1.3).

Conversely, by an elementary change of variables, we argue that $\mu\circ O$, $O\in\mathrm{O}$, has also moments of all order, and again a change of variable in (1.3) leads to

$$\int_{\mathbb{R}^2}x_1^{n_1}x_2^{n_2}\mathrm{d}\mu=\int_{\mathbb{R}^2}x_1^{n_1}x_2^{n_2}\mathrm{d}(\mu\circ O)\,,\quad n_1,n_2\in\mathbb{N},\ O\in\mathrm{O}(2)\,.$$

But μ is assumed to be determined by its moments, and therefore $\mu=\mu\circ O$ for each $O\in\mathrm{O}(2)$.

□

[3]A useful sufficient condition under which μ is uniquely determined by the sequence of all moments, corresponding to the multivariate generalisation of the Carleman condition (*cf.* [7]), is provided in [32], Theorem 12.1. See also [25] and the literature cited therein.

We conclude this preliminary discussion by noticing that, by density, (1.2) is equivalent to

$$\int_{\mathbb{R}^2} f\circ O^{\mathrm{t}}\mathrm{d}\mu=\int_{\mathbb{R}^2} f\mathrm{d}\mu\,,\quad f\in C_o(\mathbb{R})\otimes C_o(\mathbb{R}),\ O\in\mathrm{O}(2)\,, \tag{1.4}$$

where the C^*-tensor product on $C_o(\mathbb{R})\otimes C_o(\mathbb{R})=C_o(\mathbb{R}^2)$ is uniquely determined, see *e.g.* [26], Theorem 6.4.1.

We are now in a position to rephrase the previous considerations about the simple classical example described above in the setting of (quantum) stochastic processes, described in [10, 11, 13].

Indeed, consider the sample algebra $C_o(\mathbb{R})$ together with its *free abelian product* C^*-algebra

$$\mathbf{ab}_{\{1,2\}}C_o(\mathbb{R})\sim C_o(\mathbb{R})\otimes C_o(\mathbb{R})=C_o(\mathbb{R}^2)\,,$$

together with the $*$-representations $\iota_j:C_o(\mathbb{R})\to\mathcal{B}\big(L^2(\mathbb{R}^2,\mu)\big)$, $j=1,2$ given, for $f\in C_o(\mathbb{R})$, by

$$\begin{aligned}\big(\iota_1(f)g\big)(x_1,x_2)&:=f(x_1)g(x_1,x_2),\quad g\in L^2(\mathbb{R}^2,\mu)\\ \big(\iota_2(f)g\big)(x_1,x_2)&:=f(x_2)g(x_1,x_2),\quad g\in L^2(\mathbb{R}^2,\mu).\end{aligned}$$

According to [11], Definition 2.2, we have a realisation of this simple stochastic process by the quadruple $\big(C_o(\mathbb{R}),L^2(\mathbb{R}^2,\mu),\{\iota_1,\iota_2\},1\big)$, where $1\in L^2(\mathbb{R}^2,\mu)$ is the constant function assuming the value 1, μ-almost everywhere. Such a process is rotatable if, by definition, the measure μ is rotation-invariant. Obviously, such a definition of rotatability cannot be extended to the quantum case without providing some further comments we are going to describe.

For such a purpose, we consider the "single generator" X, given by $X(x)=x$, of the sample algebra, and note that the C^*-algebra $C_0(\mathbb{R}^2)$, concretely acting on the Hilbert space $L^2(\mathbb{R}^2,\mu)$ by multiplication operators, is generated by the embeddings $\iota_1(X)=x_1$ and $\iota_2(X)=x_2$, obtaining the two "coordinate-functions" x_1 and x_2, which provide multiplication operators by the coordinate x_1 and x_2.

Equally well, we can investigate the invariance under actions of unitary operators instead of elements in the orthogonal group, that is for the unitary group $\mathrm{U}(2)$ (or $\mathrm{U}(n)$ when n random variable are involved). In this case, we consider the processes generated by a complex random variable Z which, when the index-set is $\{1,2\}$ as above, leads to a probability measure μ on $\mathbb{C}^2$, and the embeddings $\iota_j(Z)$, $j=1,2$, for which $\iota_j(Z)$ are the multiplication operator for the coordinate function z_j, acting on $L^2(\mathbb{C}^2,\mu)$. Such, possibly unbounded, operators are normal, and thus still commute with each other.

By coming back to the real case, although the, mutually commuting and self-adjoint, multiplication operators by the coordinates x_1 and x_2 will in general be unbounded unless μ is compactly supported, they still generate the images of $C_o(\mathbb{R}^2)$ in $\mathcal{B}(L^2(\mathbb{R}^2, \mu)$ through their continuous functional calculus with functions vanishing at infinity.

In such a situation, if the process is rotatable, that is the measure μ is rotation-invariant, and μ is uniquely determined by its moments, Proposition 1.2 asserts that the rotatability condition can be directly stated in terms of the representations of the "single generator" X of $C_o(\mathbb{R})$, by (1.3). As we remarked, if μ is compactly supported, μ is uniquely determined by its moments, and the $*$-representations $\iota_1(X) = x_1$ and $\iota_2(X) = x_2$ are both bounded and generate a C^*-algebra which is $*$-isomorphic to $C\big(\mathrm{supp}(\mu)\big)$.

The attempt to extend the notion of the rotatability, or equally well the invariance under unitary transformations, to more general situations including the quantum one, meets several conceptual and technical problems, for which we are going to discuss some of them.

The first problem is when the (images of the) involved random variables, selfajoint or equally well normal, are unbounded. We note that it could certainly appear when one approaches to (quantum) probability from a purely algebraic setting by using that is usually denoted in literature as *algebraic probability spaces*.

Indeed, in the framework of the algebraic probability spaces, the starting point will be a pair $(\mathcal{A}, \varphi)$ made merely of an involutive unital algebra, the algebra generated by all random variables, equipped with a normalised positive functional, *i.e.* a state, describing all the joint distributions of the process. Even if this would be the most natural approach, many conceptual and technical problems arise soon. The first one concerns the natural attempts to represent the process on a suitable Hilbert space via the (generalisation of the) so-called *Gel'fand-Naimark-Segal* (GNS for short) *representation*. Unfortunately, the random variables, *i.e.* the elements of $\mathcal{A}$, may be represented by unbounded operators and there is in general no cyclic vector reproducing the stochastic process as the state φ. In addition, there might be obstructions to manage the algebraic operations like strong sums and products. The reader is referred to [8], Theorem 3.2, for details about this situation. We also note that, the commutativity of the random variables might provide only little simplifications in order to overcome such difficulties.

Concerning instead the quantum case, additional conditions must be added to try to solve all those difficulties. Concerning this point, the reader is referred to [1, 2] where an analogous approach is followed to handle gaussian states on the so-called *Canonical Commutation Relation* (CCR for short) algebra in the early rigorous analysis of quantum field theory.

We now consider the simpler situation when we can reduce the matter to bounded random variables, that is when a stochastic process is described as in [10, 11] by a quadruple $(\mathfrak{A}, \{\iota_j\}_{j\in J}, \mathcal{H}, \xi)$, made of a C^*-algebra, called the *algebra of samples*, a set of $*$-homomorphisms $\{\iota_j\}_{j\in J}$ of $\mathfrak{A}$ whose images act on the Hilbert space $\mathcal{H}$, and finally a unit vector ξ which is cyclic for the $*$-algebra in $\mathcal{B}(\mathcal{H})$ generated by all images $\iota_j(\mathfrak{A})$. In such a situation, the free product C^*-algebra $\star_J\mathfrak{A}$ plays a crucial role, and the stochastic process is described by a state, precisely the vector state induced by $\xi \in \mathcal{H}$, on such a free product $\star_J\mathfrak{A}$.

A distributional symmetry of the process under consideration, might be then handled in the above mentioned simplified, but common, situation by using $\star_J\mathfrak{A}$ and the corresponding state φ. It might be expected that any such a distributional symmetry induces $*$-automorphisms, or merely completely positive (unital) maps of $\star_J\mathfrak{A}$ under which the state φ is invariant.

This is certainly true when the symmetry under consideration can be managed by looking at the index-set J. Among those, we mention stationarity, exchangeability, and finally spreadability studied in [13]. It can also happen that some distributional symmetry cannot be directly established in terms of completely positive maps of the C^*-algebra describing the involved random variables, which is indeed a suitable quotient of the free product C^*-algebra. This is the case of the exchangeability of the so called *monotone stochastic processes* because the permutations of the indices destroy the order of such index-sets. However, yet in this situation, we can reduce the matter to a careful use of the permutations as described in [12]. Unfortunately, when one manages "continuous" symmetries like rotatability, the approaches described above fail also in the commutative cases.

Another natural fact we would like to remark is the following one. Suppose that the algebra $\mathfrak{A}$ of the samples is singularly generated by a unique (bounded) element x as in the simple commutative scheme previously described. Take a (continuous and bounded) vector-valued function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and look at the process which is a "function" of the previous one through $\mathbf{F}$, namely it is generated by the new random variables

$$(Y_1, Y_2) := (\mathbf{F}_1(X_1, X_2), \mathbf{F}_2(X_1, X_2)) .$$

It is easy to convince ourselves that this new process might not enjoy the same invariance properties of the previous one, as we can see with the following simple cases.

The first one involves two classical variables X_1, X_2 as before and concerns exchangeability. For instance, if X_1, X_2 are Bernoulli-distributed with parameter $0 < p < 1$ different from $\frac{1}{2}$, then $1 - X_1$ and X_2 no longer have the same distribution, for $1 - X_1$ is a Bernoulli variable with parameter $1 - p$. Therefore, $1 - X_1$ and X_2 are not exchangeable because exchangeable variables are in fact equally distributed. Note that the transformation of the two variables X_1, X_2 is induced by the function $\mathbf{F}(x_1, x_2) = (1 - x_1, x_2)$ which realises a homeomorphism of the spectrum $\{0, 1\}^2$ of the product $\mathbb{C}^2 \otimes \mathbb{C}^2$ of the sample algebra $\mathbb{C}^2$.

The same can be viewed with rotatability, where we are supposing that two random variables $\iota_1(x) = X_1$ and $\iota_2(x) = X_2$ are rotatable, which simply means that their joint distribution μ_{X_1,X_2} is rotation-invariant as in Definition 1.1. For the function $\mathbf{F}(x_1, x_2) = (1 - x_1, 1 - x_2)$, the new variables $Y_1 = 1 - X_1$ and $Y_2 = 1 - X_2$ are no longer rotatable as can be easily seen as follows. Indeed, applying (1.3) with $k = 2$, $n_1 = 1$ and $n_2 = 0$ (or $n_1 = 0$ and $n_2 = 1$), $O \in \mathrm{O}_2$ the rotation by π, one sees at once that $E[X_1] = E[X_2] = 0$. But then, $(1 - X_1, 1 - X_2)$ cannot be rotatable since their expectations are not zero.

As a final consideration, it is not so difficult to recognise that following fact. Once have established the generator of the algebra of the samples, any reasonable way to manage the invariance by rotations, or equally well that by unitary transformations, by $*$-automorphisms of the free product C^*-algebra is doomed to fail. This is certainly enough for the purpose of the present paper which deals with distributional symmetries.

At the light of the previous considerations explaining all possible troubles encountered when one try to manage more complicated symmetries like rotatability for real random variables, or unitary invariance for general ones, and extend them to the quantum context in a reasonable way, it clearly emerges that we must restrict the matter to the case when the algebra of samples is the involutive algebra generated by a single generator. Therefore, for such a purpose a slightly extended definition of quantum stochastic processes based on an algebra of samples which is merely an involutive algebra is also described below. This picture can be also considered merely in providing the right quantum generalisation of an arbitrary set of "quantum random variables".

2. Preliminaries

We gather here some facts useful for the following sections.

Basic facts.

For any Hilbert space $\mathcal{H}$ (including $\mathbb{C}$), by I we denote the identity $\mathbf{1}_{\mathcal{H}}$ of $\mathcal{B}(\mathcal{H})$, if it causes no confusion.

Dynamical systems.

The triplet $(\mathfrak{A}, M, \Gamma)$ is said, with an abuse of notations, a C^*-dynamical system if $\mathfrak{A}$ is a C^*-algebra, M is a monoid, and finally Γ is a representation $g \in M \mapsto \Gamma_g$ of M by completely positive identity preserving (i.e., unital) maps of $\mathfrak{A}$.

In some cases, M is replaced by a group G, and in the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ α is indeed a representation of G into the group of the $*$-automorphisms $\mathrm{Aut}(\mathfrak{A})$ of $\mathfrak{A}$. In the latter case, one speaks of reversible dynamics, whereas dissipative dynamics appears in absence of bijections, see, e.g., [6]. In this situation, we say that *the monoid M or the group G are acting on* $\mathfrak{A}$

By $\mathcal{S}(\mathfrak{A})$ we denote the convex of the states on $\mathfrak{A}$, that is the positive normalised linear functionals on $\mathfrak{A}$. $\mathcal{S}(\mathfrak{A})$ is weakly $*$-compact, provided that $\mathfrak{A}$ is unital with unit $\mathbf{1}_{\mathfrak{A}}$.

Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be invariant under the action of each element of M, i.e., $\varphi \circ \Gamma_g = \varphi$, $g \in M$, and consider the Gel'fand-Naimark-Segal (GNS for short) representation $\big(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi\big)$. Then there exists a unique contraction $V_{\varphi,g} \in \mathcal{B}(\mathcal{H}_\varphi)$ such that $V_{\varphi,g}\xi_\varphi = \xi_\varphi$ and

$$V_{\varphi,g}\pi_\varphi(a)\xi_\varphi = \pi_\varphi(\Gamma_g(a))\xi_\varphi\,, \quad a \in \mathfrak{A}\,,$$

see, e.g., [28], Lemma 2.1. The quadruple $\big(\mathcal{H}_\varphi, \pi_\varphi, V_{\varphi,g}, \xi_\varphi\big)$ is called the covariant GNS representation associated to the invariant state φ. If the Γ_g are multiplicative, the $V_{\varphi,g}$ are isometries. If in addition the Γ_g are invertible, then the $V_{\varphi,g}$ are unitaries.

The convex, compact in the $*$-weak topology, subset of all invariant states is

$$\mathcal{S}_M(\mathfrak{A}) := \big\{\varphi \in \mathcal{S}(\mathfrak{A}) \mid \varphi \circ \Gamma_g = \varphi\,,\ g \in M\big\}\,.$$

The set of the extremal invariant states (i.e., the extreme boundary) is denoted by $\mathcal{E}_M(\mathfrak{A}) := \partial\mathcal{S}_M(\mathfrak{A})$. Those are, by definition, nothing else than the ergodic states under the action Γ of M.

Direct limit of matrix-groups.

Let us denote by $\mathbb{F}$ the field of real or complex numbers $\mathbb{R}$ or $\mathbb{C}$, and for any natural number n the general linear group $\mathrm{GL}(n, \mathbb{F})$. If J is an arbitrary set of indices, the direct limits of such general linear groups can be viewed in $\mathcal{B}(\ell^2(J))$ as the matrices S whose entries are those of the $I = \mathbf{1}_{\ell^2(J)}$, but k, l that belongs to some finite subset $F \subset J$ and S, when restricted to these entries, defines in a canonical way an element of $\mathrm{GL}(|F|, \mathbb{F})$.[4] We denote by $\mathrm{GL}(J, \mathbb{F})$ such a direct limit of the finite dimensional general linear groups $\varinjlim_{F \uparrow J} \mathrm{GL}(F, \mathbb{F})$.

Since all matrix-groups $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{R})$ are subgroups of $\mathrm{GL}(n, \mathbb{C})$, all direct limits $\mathrm{O}(J)$, $\mathrm{U}(J)$ and $\mathrm{GL}(J, \mathbb{R})$ can be viewed as subgroups of $\mathrm{GL}(J, \mathbb{C})$.

The group $\mathbb{P}_J$ of all permutations of the set leaving fixed all indices, but some finite subset of J, is also viewed as a subgroup of all the above direct limits. In the forthcoming analysis, we restrict our attention to $\mathrm{O}(J)$, $\mathrm{U}(J)$ and $\mathbb{P}_J$.

Algebraic stochastic processes and distributional symmetries. We present a purely algebraic version of a stochastic process which is suitable for the investigation of the distributional symmetries associated to the direct limit groups previously described.

Indeed, a (realisation of a) quantum *stochastic process* labelled by the index set J is a quadruple $\big(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j \in J}, \xi\big)$. Here, $\mathfrak{A}$ is an involutive algebra, referred to as the sample algebra of the process, $\mathcal{H}$ is a Hilbert space whose inner product, denoted by $\langle \cdot\,, \cdot \rangle$, is linear w.r.t. the left argument, the maps ι_j are $*$-morphisms from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$, and $\xi \in \mathcal{H}$ is a unit vector, cyclic for the $*$-algebra generated by all ranges $\{\iota_j(\mathfrak{A}) \mid j \in J\}$. If $\mathfrak{A}$ is unital and $\iota_j(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathcal{H}}$, $j \in J$, the process is said to be *unital*.

The finite dimensional joint distribution of a stochastic process as above are simply defined as follows. For each positive integer n, first consider any any finite subset $\{j_1, \dots, j_n\} \subset J$ where, to simplify, we can suppose that the contiguous indices are all different. Then for each subset $\{a_1, \dots, a_n\} \subset J$,

$$\mathcal{E}_{j_1,\dots,j_n}(a_1, \dots, a_n) := \langle \iota_{j_1}(a_1) \dots \iota_{j_n}(a_n)\xi, \xi \rangle \,. \tag{2.1}$$

The set of multi-linear functionals

$$\{\mathcal{E}_{j_1,\dots,j_n} \mid j_1, \dots, j_n \in J, \;\; n = 1, 2, \dots\}$$

on $\underbrace{\mathfrak{A} \times \cdots \times \mathfrak{A}}_{\text{n-times}}$ constitutes all such finite dimensional joint distributions, which determine the process (up the unitary equivalences).

It should be noticed that, by universality, all maps ι_j determine a universal (unital) $*$-morphism $\boldsymbol{\iota} : (*_J\mathfrak{A})^{(o)} \to \mathcal{B}(\mathcal{H})$ of the (*unital*) *algebraic free product* $(*_J\mathfrak{A})^{(o)}$ into $\mathcal{B}(\mathcal{H})$.

[4]We consider $\ell^2(J)$ over the complex field, and equipped with the canonical basis made of $e_i(j) = \delta_{i,j}$, $i, j \in J$, $\delta_{i,j}$ being the Kronecker symbol.

In the case when $\mathfrak{A}$ is a C^*-algebra, a quadruple as above can equivalently be assigned through a state φ on the free product C^*-algebra $*_J\mathfrak{A}$, the last being the C^*-completion of $(*_J\mathfrak{A})^{(o)}$, see *e.g.* [35], Appendix L.

Indeed, if one starts with a stochastic process, then a state φ on $*_J\mathfrak{A}$ is uniquely determined by its values on the words in $(*_J\mathfrak{A})^{(o)}$ given by the $\mathcal{E}_{j_1,\dots,j_n}$, such that the GNS representation is $(\pi,\mathcal{H},\xi)$ and $\pi : *_J\mathfrak{A}\to\mathcal{B}(\mathcal{H})$ is uniquely determined from the set of $*$-homomorphisms $\{\iota_j\}_{j\in J}$ by universality.

Rather interestingly, all states on the free product $*_J\mathfrak{A}$ arise in this way, see [10], Theorem 3.4. Phrased differently, starting now with a state $\varphi\in\mathcal{S}\big(*_J\mathfrak{A}\big)$, it is possible to recover a stochastic process by looking at the GNS representation $(\pi_\varphi,\mathcal{H}_\varphi,\xi_\varphi)$ of φ, and at the natural maps $i_j:\mathfrak{A}\to *_J\mathfrak{A}$, $j\in J$. Indeed, for any $j\in J$ we can set $\iota_j(a):=\pi_\varphi(i_j(a))$, $a\in\mathfrak{A}$, so as to get the quadruple $\big(\mathfrak{A},\mathcal{H}_\varphi,\{\iota_j\}_{j\in J},\xi_\varphi\big)$ which satisfies all the properties that define a quantum stochastic process. In addition, the state associated with this quadruple, and uniquely determined by all finite dimensional joint distributions, is nothing but the state φ we started with.

It is of certain relevance that some distributional symmetry can be handled directly looking at the corresponding properties of the invariance of the corresponding state, provided such a distributional symmetry is described by the action of the involved group or monoid. This is certainly the case of stationarity, exchangeability and spreadability as explained in the sequel of papers [10, 11, 12, 13, 14, 18]. As we have previously explained, this seems not to be the case for the invariance under the orthogonal and unitary symmetries.

Quantum random variables.

We provide the definition of an arbitrary sequences of *quantum random variables*. Namely, it is a triple $V:=\big(\mathcal{H},\{T_j\}_{j\in J},\xi\big)$, where $\mathcal{H}$ is an Hilbert space $\{T_j\}_{j\in I}\subset\mathcal{B}(\mathcal{H})$, J being an arbitrary set of indices, is a subset of bounded operators acting on the Hilbert space $\mathcal{H}$, and finally $\xi\in\mathcal{H}$ is a unit vector.

The set of their finite dimensional joint distributions can be computed as in (2.1), by replacing the $\iota_j(a_j)$ with the T_j, respectively. Therefore, we can suppose without loosing generality, that ξ is cyclic for the T_j. In fact, the orthogonal projection P_V onto the cyclic subspace $\big[\{T_j\}_{j\in J}\xi\big]$ commutes with all the T_j, and so reduces simultaneously all of them. We then argue that $V:=\big(\mathcal{H},\{T_j\}_{j\in J},\xi\big)$ is weakly equivalent (in the sense that all the finite dimensional joint distributions coincide) to $V_{P_V}=\big(P_V\mathcal{H},\{T_jP_V\}_{j\in J},\xi\big)$.

We say that the sequences of random variables $V := \big(\mathcal{H}, \{T_j\}_{j\in J}, \xi\big)$ is real if all the T_j are selfadjoint, and in this situation we speak of a set of real random variables.

The bridge with the algebraic stochastic processes previously defined is, first to consider the algebra of the samples as the free involutive algebra $\mathfrak{A}$ generated by a single generator a_o, second to define the ι_j on the generator $\iota_j(a_o) := T_j$, and finally extend in the obvious manner the ι_j to $*$-morphisms on $\mathfrak{A}$.

3. Invariance under the rotations and unitary transformations

We first manage the invariance under the group of rotations. For such a purpose, we fix once for all an abstract selfadjoint generator $a_o = a_o^*$, and consider the involutive complex free abelian algebra $\mathfrak{A} := \operatorname{span}\{a_o^n \mid n \in \mathbb{N}\}$ obtained by linear combinations of monomials of arbitrary degrees, where $a_o^0 := 1 = \mathbf{1}_{\mathfrak{A}}$. We should have denoted such an involutive algebra by $\mathfrak{A}_{a_o}$, but we drop the subscript when it causes no confusion. We also consider unital stochastic processes if it is not otherwise specified.

Let us consider the stochastic process $P := \big(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \xi\big)$ with $\mathfrak{A} = \mathfrak{A}_{a_o}$ as above. For each choice of finite subsets $F \subset J$ and $O \in \mathrm{O}(|F|)$, define the new process $P^{(F,O)} = \Big(\mathfrak{A}, \mathcal{H}, \big\{\iota_j^{(F,O)}\big\}_{j\in J}, \xi\Big)$, where $\iota^{(F,O)}$ are the same as before, but $j \in F$ where, for the generator a_o,

$$\Big(\iota_{j_1}^{(F,O)}(a_o), \dots, \iota_{j_n}^{(F,O)}(a_o)\Big) := \big(\iota_{j_1}(a_o), \dots, \iota_{j_n}(a_o)\big) O \,, \tag{3.1}$$

where the product is understood as the row-column one between the row-vector $\big(\iota_{j_1}(a_o), \dots, \iota_{j_n}(a_o)\big)$ and the matrix O.

For monomials a_o^k, which linearly generate $\mathfrak{A}$, it is elementary to recognise that, with an abuse the notations,

$$\iota_j^{(F,O)}(a_o^k) := \iota_j^{(F,O)}(a_o)^k \,, \quad j \in J,\ k \in \mathbb{N} \,, \tag{3.2}$$

by using (3.1), $*$-morphisms $\iota_j^{(F,O)}$ of $\mathfrak{A}$ are uniquely determined.

Definition 3.1. *We say that the process $P = \big(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j\in J}, \xi\big)$, with $\mathfrak{A}$ generated by a selfadjoint element a_o, is rotatable if the finite dimensional joint distributions in* (2.1) *of the process P and those of the transformed processes $P^{(F,O)}$ under the action of the rotations coincide, for each choice of n, finite subsets $F \subset J$, orthogonal matrices $O \in \mathrm{O}(|F|)$, and finally $a_1, \dots, a_n \in \mathfrak{A}$:*

$$\langle \iota_{j_1}^{(F,O)}(a_1) \dots \iota_{j_n}^{(F,O)}(a_n)\xi, \xi\rangle = \langle \iota_{j_1}(a_1) \dots \iota_{j_n}(a_n)\xi, \xi\rangle \,.$$

In order to manage the invariance under the the unitary group, we proceed as follows. Fix an abstract generator a_o, and consider all noncommutative monomials, or reduced words of arbitrary length n, $w=(a_o^{\#_1})^{k_1}(a_o^{\#_2})^{k_2}\cdots(a_o^{\#_n})^{k_n}$. Obviously, for the empty word $(a_o^*)^0=a_o^0=:\mathbf{I}_{\mathfrak{A}}$, $\#_i\in\{1,*\}$, and with

$$\#_i^*=\begin{cases} *\,\text{if}\,\#=1\,,\\ 1\,\text{if}\,\#=*\,,\end{cases}$$

$w^*:=\big(a_o^{\#_n^*}\big)^{k_n}\big(a_o^{\#_{n-1}^*}\big)^{k_{n-1}}\cdots\big(a_o^{\#_1^*}\big)^{k_1}$, all words as above linearly generate the free involutive unital algebra $\mathfrak{A}$, which in this case is not commutative. The case when the generator is supposed to be normal (*i.e.* $a_oa_o^*=a_o^*a_o$) is achieved simply considering all commutative monomials $\{a_o^m(a_o^*)^n\mid m,n\in\mathbb{N}\}$.

For any process $P:=\big(\mathfrak{A},\mathcal{H},\{\iota_j\}_{j\in J},\xi\big)$, where now $\mathfrak{A}$ is generated by a non selfadjoint (possibly normal) element a_o, it is possible to define its transformed process $P^{(F,U)}$, where $U\in\mathrm{U}(|F|)$ as above, by putting

$$\begin{aligned}&\Big(\iota_{j_1}^{(F,U)}(a_o),\dots,\iota_{j_n}^{(F,O)}(a_o)\Big):=\big(\iota_{j_1}(a_o),\dots,\iota_{j_n}(a_o)\big)\overline{U}\,,\\ &\iota_j^{(F,U)}\big((a_o^{\#})^k\big):=\big(\iota_j^{(F,U)}(a_o)^{\#}\big)^k\,,\quad j\in J,\ \#=1,*,\ k\in\mathbb{N}\,.\end{aligned}\tag{3.3}$$

Here, $\overline{U}_{\iota,\kappa}:=\overline{U_{\iota\kappa}}$ for all $\iota,\kappa\in J$.[5]

Equally well, it is clear as before that the $\iota_j^{(F,U)}$ in (3.3) uniquely extend by linearity to $*$-morphisms of $\mathfrak{A}$, and therefore determine the transformed process $P^{(F,U)}:=\Big(\mathfrak{A},\mathcal{H},\big\{\iota_j^{(F,U)}\big\}_{j\in J},\xi\Big)$,.

Definition 3.2. *We say that the process $P=\big(\mathfrak{A},\mathcal{H},\{\iota_j\}_{j\in J},\xi\big)$, with $\mathfrak{A}$ generated by a, possibly non selfadjoint element, a_o, is invariant under unitary transformations if the finite dimensional joint distributions in* (2.1) *of the process P and those of the transformed processes $P^{(F,U)}$ under the action of the rotations coincide, for each choice of n, finite subsets $F\subset J$, unitary matrices $U\in\mathrm{U}(|F|)$, and finally $a_1,\dots,a_n\in\mathfrak{A}$:*

$$\langle\iota_{j_1}^{(F,U)}(a_1)\dots\iota_{j_n}^{(F,U)}(a_n)\xi,\xi\rangle=\langle\iota_{j_1}(a_1)\dots\iota_{j_n}(a_n)\xi,\xi\rangle\,.$$

We would like to point out that the invariance under $\mathrm{O}(J)$ or $\mathrm{U}(J)$ can be equally well managed by considering only the corresponding invariance for the set V of random variables given by $V=\big(\mathcal{H},\{\iota_j(a_o)\}_{j\in J},\xi\big)$.

[5]The reason to choosing the conjugate matrix is clarified in the proof of Proposition 4.1.

We can also see in simple but pivotal examples like the so-called boolean stochastic processes, that the invariance under orthogonal or unitary matrices can be managed in a more direct way.

4. Invariant boolean processes

Let $\mathcal{H}$ be a complex Hilbert space. Recall that the boolean Fock space over $\mathcal{H}$ is given by $\Gamma_{\text{Boole}}(\mathcal{H}) := \Gamma(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H}$, where the vacuum vector Ω is $(1,0)$. On $\Gamma(\mathcal{H})$, we define the creation and annihilation operators, respectively given for $f \in \mathcal{H}$, by

$$b^\dagger(f)(\alpha \oplus g) := 0 \oplus \alpha f, \quad b(f)(\alpha \oplus g) := \langle g, f\rangle_{\mathcal{H}} \oplus 0, \quad \alpha \in \mathbb{C},\, g \in \mathcal{H}.$$

They are mutually adjoint, and satisfy the following relations for $f, g \in \mathcal{H}$,

$$b(f)b^\dagger(g) = \langle g, f\rangle_{\mathcal{H}} \langle\,\cdot\,,\Omega\rangle\Omega\,, \quad b^\dagger(f)b(g) = \langle\,\cdot\,, 0 \oplus g\rangle 0 \oplus f\,.$$

As shown in [10], Section 7, the unital C^*-algebra $\mathfrak{b}$ acting on $\Gamma(\ell^2(\mathbb{N}))$ generated by the annihilators $\{b(f) \mid f \in \ell^2(\mathbb{N})\}$ coincides with $\mathcal{K}(\Gamma(\mathcal{H}))+\mathbb{C}I$.

It is then natural to view any state on $\mathfrak{b} = \mathcal{K}(\Gamma(\mathcal{H})) + \mathbb{C}I$ as a boolean stochastic process (*cf.* [18]), and thus some invariance properties of the distributional symmetries can be managed directly in terms of invariance properties of the states under consideration. We will see that it is possible for rotatability and unitary invariance. We start with the latter.

For $U \in \mathcal{U}(\mathcal{H})$, define $\Gamma(U) := I \bigoplus U \in \mathcal{U}(\Gamma(\mathcal{H}))$, and put $G := \mathrm{ad}_{\Gamma(\mathcal{U}(\mathcal{H}))} \subset \mathrm{Aut}(\mathfrak{b})$. As we are going to see, the states $\omega \in \mathcal{S}(\mathfrak{b})$, or equally well the boolean stochastic processes, which are invariant under the transposed action of (a subgroup of) G play a crucial role in the definition of invariance under the unitary action.

For such a purpose, we specialise the situation to $\mathcal{H} = \ell^2(J)$ for the index-set J of arbitrary cardinality. We also note that the algebra of samples $\mathfrak{A}$ is isomorphic to $\mathbb{M}_2(\mathbb{C})$ and is generated by the single annihilator $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In such a situation,

$$\mathfrak{b} = \mathcal{K}(\ell^2(\{\#\} \bigsqcup J)) + \mathbb{C}I\,,$$

Under such an identification, annihilators and creators are expressed by the system of matrix-units as follows:

$$b_\iota = \varepsilon_{\#\iota},\ b_\iota^\dagger = \varepsilon_{\iota\#},\ b_\iota b_\kappa^\dagger = \delta_{\iota,\kappa}\varepsilon_{\#\#},\ b_\iota^\dagger b_\kappa = \varepsilon_{\iota\kappa}, \quad \iota, \kappa \in J\,, \tag{4.1}$$

where the additional symbol "#" takes into account of the vacuum vector Ω. We also note that

$$\varepsilon_{l_1 l_2} = \langle\,\cdot\,,\delta_{l_2}\rangle\delta_{l_1}\,,\quad l_1,l_2\in\{\#\}\bigsqcup J\,,$$

is the rank-one operator mapping the element δ_{l_2} in the canonical basis of $\ell^2(\{\#\}\bigsqcup J)$, to the other one δ_{l_1}. In such a situation, $\mathbb{U}(J)$ in Section 2 is viewed as a subgroup of $\mathcal{U}(\ell^2(\{\#\}\bigsqcup J))$ in a natural way.

The following fact will be crucial in defining the invariance under unitary matrices of boolean processes.

Proposition 4.1. *For the set* $\{\varepsilon_{kl}\mid k,l\in\{\#\}\bigsqcup J\}$ *of matrix-units in* (4.1) *and* $U\in\mathbb{U}(J)$,

$$\Gamma(U)\{\varepsilon_{kl}\mid k,l\in\{\#\}\bigsqcup J\}\Gamma(U)^* =: \{e_{kl}\mid k,l\in\{\#\}\bigsqcup J\}$$

provides a set $\{e_{kl}\mid k,l\in\{\#\}\bigsqcup J\}$ *of matrix-units, denoted by* e_{kl}*, satisfying the same relations as those of the original one in* (4.1)*, and generating together the identity* I*, the whole algebra* $\mathfrak{b}$*.*

In particular $e_{\#\#}=\varepsilon_{\#\#}$*, and for the generators* $\{b_\iota\mid\iota\in J\}$ *and the transformed generator*

$$B_\iota := e_{\#\iota} = \Gamma(U)b_\iota\Gamma(U)^*\,,$$

we get

$$B_\iota = \sum_{\kappa\in J} b_\kappa\overline{U_{\kappa\iota}}\,. \tag{4.2}$$

Proof. We start by noticing that, since $V=\Gamma(U)$ is unitary, $\{e_{kl}\mid k,l\in\{\#\}\bigsqcup J\}$ still generates $\mathcal{K}(\ell^2(\{\#\}\bigsqcup J))$, and thus the whole $\mathfrak{b}$ after adding the identity. Since V is the second-quantised of U, $\varepsilon_{\#,\iota}$ and correspondingly its adjoint $\varepsilon_{\iota,\#}$, are sent in $e_{\#,\iota}$ and $e_{\iota,\#}$, respectively. It is also immediate to check that $\Gamma(U)\varepsilon_{\#\#}\Gamma(U)^*=\varepsilon_{\#\#}$.

Concerning the remaining part, we first note that the sum in (4.2) is finite because $U\in\mathbb{U}(J)$. Therefore, we easily compute

$$\begin{aligned}B_\iota =&\Gamma(U)b_\iota\Gamma(U)^* = \langle\,\cdot\,,U\delta_\iota\rangle\delta_\# = \Big\langle\,\cdot\,,\sum_\kappa U_{\kappa\iota}\delta_\kappa\Big\rangle\delta_\#\\ =&\sum_\kappa\langle\,\cdot\,,U_{\kappa\iota}\delta_\kappa\rangle\delta_\# = \sum_\kappa\overline{U_{\kappa\iota}}\langle\,\cdot\,,\delta_\kappa\rangle\delta_\# = \sum_\kappa b_\kappa\overline{U_{\kappa\iota}}\,.\end{aligned}$$

□

As before, we can easily compute

$$\begin{aligned}B^\dagger_\iota = (B_\iota)^* =&\sum_\kappa b^\dagger_\kappa U_{\kappa\iota} = \langle\,\cdot\,,\delta_\#\rangle\sum_\kappa U_{\kappa\iota}\delta_\kappa\\ =&\langle\,\cdot\,,\delta_\#\rangle U\delta_\iota = \Gamma(U)b^\dagger_\iota\Gamma(U)^*\,,\end{aligned}$$

which implies the analogous formulas for the remaining matrix-units.

To pass to the invariance under the orthogonal group, we consider the selfadjoint part of the annihilators $s_\iota := b_\iota + b^\dagger$, usually called the *position variable* according to the standard terminology of quantum field theory. It is shown that, differently from that happens for systems satisfying the q-relations but in accordance with monotone commutation relations (*cf.* CFL), the concrete C^*-algebra generated by $\{s_\iota \mid \iota \in J\}$ and the identity coincides with the whole boolean algebra.

For such a purpose, we first note that $\mathrm{O}(J) \subset \mathrm{U}(J)$ with real entries, and for $O \in \mathrm{O}(J)$, we compute as before,

$$\begin{aligned}\Gamma(O)s_\iota\Gamma(O)^* =&\Gamma(O)(b_\iota + b_\iota^\dagger)\Gamma(O)^* = \sum_{\kappa\in J} b_\kappa\overline{O_{\kappa\iota}} + \sum_{\kappa\in J} b_\kappa^\dagger O_{\kappa\iota}\\ =&\sum_{\kappa\in J}(b_\kappa + b_\kappa^\dagger)O_{\kappa\iota} = \sum_{\kappa\in J} s_\kappa O_{\kappa\iota}\,.\end{aligned} \tag{4.3}$$

As the previous symmetries, like stationarity, exchangeability and spreadability studied for boolean stochastic processes, and also for other kind of quantum stochastic processes, the idea is to provide the definition of the orthogonal and unitary invariance directly on the involved algebras, the boolean one $\mathfrak{b}$ in this case. In such a way, the problems arising with relations and quotients to pass from the universal free (C^*-)algebra to the particular ones is automatically overcame. In this case, as well as for the monotone stochastic processes, the problem involving relations and quotients directly in the sample $*$-algebra is also automatically overcame.

Therefore, by taking into account (3.1) and (3.3), Definitions 3.1 and 3.2, (4.3) and (4.2), and identifying boolean stochastic processes with states on the boolean algebra, it is meaningful to provide the following

Definition 4.2. *A boolean stochastic process with arbitrary index-set J, is invariant under the action of the orthogonal (resp. unitary) matrices if the corresponding state $\omega \in \mathcal{S}(\mathfrak{b})$ is invariant under the (transpose of the) adjoint action $\mathrm{ad}_{\Gamma(O)}$ (resp. $\mathrm{ad}_{\Gamma(U)}$), for each $O \in \mathrm{O}(J)$ (resp $U \in \mathrm{U}(J)$).*

We recall the definition of the *state at infinity*

$$\omega_\infty(a + \alpha I) := \alpha\,, \quad a \in \mathcal{K}(\ell^2(\{\#\}\bigsqcup J)),\ \alpha \in \mathbb{C}\,.$$

The vacuum state is simply given by $\omega_\# = \langle\,\cdot\,,\delta_\#,\delta_\#\rangle$.

Remark 4.3. *Since $\mathbb{P}_J \subset \mathrm{O}(J) \subset \mathrm{U}(J)$, and the state at infinity ω_∞ and the vacuum state $\omega_\#$ are invariant under the action of the unitary* *group* $\mathrm{U}(J)$*, by* [14]*, Section 7, we have the expected result*

$$\begin{aligned}\mathcal{E}_{\mathrm{U}(J)}(\mathfrak{b}) =&\mathcal{E}_{\mathrm{O}(J)}(\mathfrak{b}) = \mathcal{E}_{\mathbb{P}_J}(\mathfrak{b})\\ =&\{\gamma\omega_{\#} + (1-\gamma)\omega_{\infty} \mid \gamma \in [0,1]\}\,.\end{aligned}$$

We end by point out that the quite interesting emerging fact is not just to discover that the states invariant under rotation and unitary groups coincide with the symmetric ones (the last corresponding to exchangeable boolean stochastic processes), but the way to achieve the action of the orthogonal and unitary symmetries for the Boole algebra.

References

[1] Araki H. *On quasifree states of the canonical commutation relations. II*, Publ. Res. Inst. Math. Sci. **7** (1971/72), 121-152.

[2] Araki H., Shiraishi M. *On quasifree states of the canonical commutation relations. I*, Publ. Res. Inst. Math. Sci. **7** (1971/72), 105-120.

[3] Avitzour D. *Free products of* C^**-algebras*, Trans. Am. Math. Soc. **271** (1982), 423–435.

[4] Billingsley P. *Probability and measure*, John Wiley & Sons, 1985.

[5] Bożejko M., Speicher R. *Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann.* **300** (1994), 97–120.

[6] Bratteli O., Robinson D.W. *Operator algebras and quantum statistical mechanics. 1*, Springer-Verlag: New York, 1987.

[7] Carleman T. *Les functions quasi analytiques*, Gauthier-Villars 1926.

[8] Crismale V., Duvenhage R., Fidaleo F. C^**-fermi systems and detailed balance*, Anal. Math. Phys. **11** (2021), Paper No. 11.

[9] Crismale V., Fidaleo F. *De Finetti theorem on the CAR algebra*, Commun. Math. Phys. **315** (2012), 135–152.

[10] Crismale V., Fidaleo F. *Exchangeable stochastic processes and symmetric states in quantum probability*, Ann. Mat. Pura Appl., **194** (2015), 969–993.

[11] Crismale V., Fidaleo F. *Symmetries and ergodic properties in quantum probability*, Colloq. Math. **149** (2017), 1–20.

[12] Crismale V., Fidaleo F., Griseta M.E. *Wick order, spreadability and exchangeability for monotone commutation relations*, Ann. Henri Poincaré **19** (2018), 3179–3196.

[13] Crismale V., Fidaleo F., Griseta M.E. *Spreadability for quantum stochastic processes, with an application to boolean commutation relations*, Entropy **22** no.5 (2020) Article number 532, 17 pp.

[14] Crismale V., Fidaleo F., Lu Y.G. *Ergodic theorems in quantum probability: an application to monotone stochastic processes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **XVII** (2017), 113–141.

[15] Crismale V., Fidaleo F., Lu Y. G. *From discrete to continuous monotone* C^**-algebras via quantum central limit theorems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **20** No.2 (2017), 1750013 (18 pages).

[16] Curran S. *Quantum Rotatability*, Trans. Amer. Math. Soc. **362** (2010), 4831–4851.

[17] Dykema K., Fidaleo F. *Unique mixing of the shift on the C^*–algebras generated by the q–canonical commutation relations*, Houston J. Math. **36** (2010), 275–281.
[18] Fidaleo F. *A note on Boolean stochastic processes*, Open Syst. Inf. Dyn. **22** no. 1 (2015), 1550004, 10 pp.
[19] Graham R., Knuth D.E., Patashnik O. *Concrete Mathematics*, Addison-Wesley, Reading, 1989.
[20] Gray R.D., Kambites M. *Amenability and geometry of semigroups*, Trans. Amer. Math. Soc. **369** (2017), 8087–8103.
[21] Kadison R., Ringrose J.R. *Fundamentals of the Theory of Operator Algebras*, Volume I, Academic Press, Orlando, Florida (1986).
[22] Kadison R., Ringrose J.R. *Fundamentals of the Theory of Operator Algebras*, Volume II, Academic Press, Orlando, Florida (1986).
[23] Kallenberg, O. *Probabilistic Symmetries and Invariance Principles*, Springer, Berlin (2005).
[24] Klawe M. *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions*, Pacific J. Math. **73** (1977), 91–106.
[25] Kleibert C., Stoyanov J. *Multivariate distributions and the moment problem*, J. Mult. Anal. **113** (2013), 7-18.
[26] Murphy G. *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.
[27] Namioka I. *Følner's conditions for amenable semigroups*, Math. Scand. **15** (1964), 18–28.
[28] Niculescu C.P., Ströh A., Zsidó L. *Noncommutative estension of classical and multiple recurrence theorems*, J. Operator Theory **50** (2003), 3–52.
[29] Olsen C.L., Zame W. *Some C^*-algebras with a single generator*, Trans. Amer. Math. Soc. **215** (1976), 205–217.
[30] Olshen R.A. *The coincidence of measure algebras under an exchangeable probability*, Probab. Theory Related Fields **18** (1971), 153–158.
[31] Ryll-Nardzewski V. *On stationary sequences of random variables and the de Finetti's equivalence*, Colloq. Math. **4** (1957), 149–156.
[32] Shohat J. A. *The problem of moments*, American Mathematical Society 1970.
[33] Takesaki M. *Theory of operator algebras III*, Springer, Berlin-Heidelberg-New York 2001.
[34] Voiculescu D.V., Dykema K.J., Nica A. *Free random variables*, CRM Monograpy Series, **1**, American Mathematical Society, Providence, 1992.
[35] Wegge-Olsen *K-Theory and C^*- algebras*, Oxford university press Oxford-New York-Tokyo, 1993.

Vitonofrio Crismale, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: vitonofrio.crismale@uniba.it

Francesco Fidaleo, Dipartimento di Matematica, Università degli studi di Roma Tor Vergata, Via della Ricerca Scientifica 1, Roma 00133, Italy
Email address: fidaleo@mat.uniroma2.it

Stefano Rossi, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: stefano.rossi@uniba.it