THE DYNAMICS NEAR QUASI-PARABOLIC FIXED POINTS OF HOLOMORPHIC DIFFEOMORPHISMS IN $C^2$

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Abstract. Let $F$ be a germ of holomorphic diffeomorphism of $C^2$ fixing $O$ and such that $dFO$ has eigenvalues 1 and $e^{i\theta}$ with $|e^{i\theta}| = 1$ and $e^{i\theta} \neq 1$. Introducing suitable normal forms for $F$ we define an invariant, $\nu(F) \geq 2$, and a generic condition, that of being dynamically separating. In the case $F$ is dynamically separating, we prove that there exist $\nu(F) - 1$ parabolic curves for $F$ at $O$ tangent to the eigenspace of 1.

1. Introduction. Let $\text{End}(C^2, O)$ denote the group of germs of holomorphic diffeomorphisms at the origin $O$ of $C^2$ fixing $O$. One of the main open problems is to understand the dynamics near $O$ of an element $F \in \text{End}(C^2, O)$ for which the spectrum of the differential $dFO$ is contained in the unit circle (see Question 2.26 in [9]). The case where $O$ is a parabolic point of $F$, that is $dFO = \text{id}$, and $O$ is an isolated fixed point, has been studied by several authors ([7], [17], [10], [1]). To recall their main result we need first a definition:

Definition 1.1. A parabolic curve for $F \in \text{End}(C^2, O)$ at $O$ tangent to (the space spanned by) $v \in C^2 \setminus \{O\}$ is an injective holomorphic map $\varphi : \Delta \to C^2$ satisfying the following properties:

1. $\Delta$ is a simply connected domain in $C$ with $0 \in \partial\Delta$,
2. $\varphi$ is continuous on $\partial\Delta$, $\varphi(0) = O$ and $[\varphi(\zeta)] \to [v]$ as $\zeta \to 0$ (where $[\cdot]$ denote the projection of $C^2 \setminus \{O\}$ to $P^1$),
3. $F(\varphi(\Delta)) \subset \varphi(\Delta)$, and $F^n(\varphi(\zeta)) \to O$ as $n \to \infty$ for any $\zeta \in \Delta$.

Then the main result is:

Theorem 1.2. (Écalle, Hakim, Abate) Let $F \in \text{End}(C^2, O)$ be tangent to the identity and such that $O$ is an isolated fixed point. Let $t(F) \geq 2$ denote the order of vanishing of $F - \text{id}$ at $O$. Then there exist (at least) $t(F) - 1$ parabolic curves for $F$ at $O$.

Actually, Écalle [7] and Hakim [10] proved such a theorem for any dimension, but only for generic mappings, while Abate [1] using an ingenious index theorem...
makes the result holds for any map, but just in \( \mathbb{C}^2 \). The case where there is a curve of fixed points passing through \( O \) has also been studied ([11], [5], [2]), and actually one can see Theorem 1.2 as a consequence of results on dynamics near curves of fixed points by means of blow-ups of \( O \) in \( \mathbb{C}^2 \) (see [1], [4]). We also wish to mention that for the semi-attractive case in \( \mathbb{C}^n \) (that is one eigenvalue \( 1 \) with some multiplicity and the others of modulus strictly less than 1) the existence of parabolic curves is provided by Rivi [13].

Roughly speaking the underlying idea in all previous results is to find “good invariants” attached to \( F \) which read dynamical properties of \( F \) itself (for instance Hakim’s nondegenerate characteristic directions or Abate’s indices in [1], and residues in [4]).

In this paper we deal with the case of a map \( F \in \text{End}(\mathbb{C}^2, O) \) with \( \text{Sp}(dF_O) = \{ 1, e^{i\theta} \} \) for \( \theta \in \mathbb{R} \) and \( e^{i\theta} \neq 1 \). We call \( O \) a quasi-parabolic fixed point for \( F \).

If \( e^{i\theta} \) satisfies some Brjuno condition then Pöschel proved that there exists a (germ of) complex curve \( \Gamma \) tangent to the eigenspace of \( e^{i\theta} \) which is invariant for \( F \) and on which \( F \) is conjugated to the rotation \( \zeta \mapsto e^{i\theta} \zeta \) (see [12]). However nothing is known about the dynamics in the direction tangent to the eigenspace of 1.

Our starting point is the following trivial observation: the map \( F : (z, w) \mapsto (z + z^3, e^{i\theta} w) \) has \( \{ w = 0 \} \) as invariant curve and thus, by the one-dimensional Fatou theory (see, e.g., [6]) there exist two parabolic curves for \( F \) at \( O \) tangent to the eigenspace of 1, no matter what \( e^{i\theta} \) is. However, conjugating \( F \) with a map \( G \in \text{End}(\mathbb{C}^2, O) \) tangent to \( \text{id} \) at \( O \), it might be very difficult to check that the new map has an invariant curve tangent to the eigenspace of 1 and two parabolic curves in there.

Motivated by the previous results for germs tangent to the identity, we direct our study in searching invariants for \( F \) at a quasi-parabolic point which are related to dynamical properties of \( F \) along the direction tangent to the eigenspace of 1.

The main difference between the parabolic and quasi-parabolic case is that in the first, all terms of \( F \) are resonant in the sense of Poincaré-Dulac (see, e.g., [3]), while in the second case some are not, and this allows us to dispose of those terms with suitable transformations. More precisely, let \( F = (F_1, F_2) \in \text{End}(\mathbb{C}^2, O) \) be given in some system of local coordinates by

\[
\begin{align*}
F_1(z, w) &= z + \sum_{j+k \geq 2} p_{j,k} z^j w^k, \\
F_2(z, w) &= e^{i\theta} w + \sum_{j+k \geq 2} q_{j,k} z^j w^k,
\end{align*}
\]

for \( p_{j,k}, q_{j,k} \in \mathbb{C}, \theta \in \mathbb{R} \) and \( e^{i\theta} \neq 1 \). A monomial \( z^m w^n \) in \( F_1 \) is resonant if \( 1 = 1^m e^{i\theta n} \), while a monomial \( z^m w^n \) in \( F_2 \) is resonant if \( e^{i\theta} = 1^m e^{i\theta n} \), for \( m, n \in \mathbb{N}, m+n \geq 2 \). A germ \( F \) is said to be in Poincaré-Dulac normal form if it is given by (1.1) and \( p_{j,k} = q_{j,k} = 0 \) for all nonresonant monomials \( z^j w^k \). The Poincaré-Dulac Theorem states that it is always possible to formally conjugate \( F \) to a (formal) map \( G \) in normal form by means of a (formal) transformation tangent to the
identity, and actually the method of Poincaré-Dulac is constructive in the sense that given \( k \in \mathbb{N} \) it is possible to analytically conjugate \( F \) to a (convergent) map \( G \) which is in normal form up to order \( k \) (that is, nonresonant monomials of degree less than or equal to \( k \) are all zero) by means of a (convergent) transformation tangent to the identity.

Therefore if there exist invariants for \( F \) at a quasi-parabolic fixed point they have to be found in normal forms. Unfortunately normal forms are not unique and also they do not reflect the character of \( e^{i\theta} \), while our leading example does not make differences. Also, normal forms are not stable under blow-ups, which are one of the basic ingredients of parabolic theory. Indeed the only invariant terms are those we call ultra-resonant monomials, that is, for \( F \) given by (1.1), of type \( z^m \) in \( F_1 \) and \( z^m w \) in \( F_2 \), \( m \in \mathbb{N} \). And we say that \( F \) is an asymptotic ultra-resonant normal form if \( q_{0,0} = 0 \) for any \( j \). Note that Poincaré-Dulac normal forms are in fact examples of asymptotic ultra-resonant normal forms but the converse is not true in general, and indeed there are convergent asymptotic ultra-resonant normal forms which have no convergent Poincaré-Dulac normal forms. With a simplified Poincaré-Dulac method we prove that given \( F \in \text{End}(\mathbb{C}^2, O) \) with \( O \) as quasi-parabolic fixed point, there always exist (possibly formal) asymptotic ultra-resonant normal forms conjugated to \( F \) by means of transformations tangent to the identity. Again asymptotic ultra-resonant normal forms are not unique, but we show that the first \( j \in \mathbb{N} \) such that \( p_{j,0} \neq 0 \) is an invariant for (even formal) conjugated ultra-resonant normal forms. Therefore we find the first invariant \( \nu(F) \in \mathbb{N} \cap [2, \infty] \) associated to \( F \). Of course this invariant could also have been defined from Poincaré-Dulac normal forms. However, the following result justifies the usage of ultra-resonant normal forms instead of Poincaré-Dulac normal forms:

**Proposition 1.3.** Let \( F \in \text{End}(\mathbb{C}^2, O) \) and assume \( O \) is a quasi-parabolic fixed point of \( F \). Then there exists an invariant nonsingular complex curve \( \Gamma \) for \( F \) passing through \( O \) and tangent to the eigenspace of 1 if and only if \( F \) is analytically conjugated to a convergent asymptotic ultra-resonant normal form. Moreover in this case, if \( \nu(F) = \infty \) then \( F \) pointwise fixes \( \Gamma \), while if \( \nu(F) < \infty \) there exist \( \nu(F) - 1 \) parabolic curves for \( F \) at \( O \) contained in \( \Gamma \).

For the practical purpose of calculating \( \nu(F) \) one does not need to find an asymptotic ultra-resonant normal form. Indeed it is enough to find what we call a ultra-resonant normal form, that is, \( F \) given by (1.1) for which the first pure non-zero term in \( z \) of \( F_2 \) has degree greater than or equal to the first non-zero pure term in \( z \) of \( F_1 \) (see Section 2).

In the generic case \( \nu(F) < \infty \), we can associate to \( F \) a second invariant, essentially the sign of \( \Theta(F) \). The latter, for \( F \) in ultra-resonant normal form given by (1.1), is defined as \( \Theta(F) = \nu(F) - j - 1 \) where \( j \) is the first integer for which \( q_{j,1} \neq 0 \) and, roughly speaking, measures the “degree of mixing” of the dynamics along the eigenspace associated to 1 and \( e^{i\theta} \). Therefore, given any \( F \in \text{End}(\mathbb{C}^2, O) \) for which \( O \) is quasi-parabolic for \( F \), we say that \( F \) is dynamically separating
if $\nu(F) < \infty$ and $\Theta(F) \leq 0$ for some ultra-resonant normal form $\tilde{F}$ of $F$ (see Definition 2.7). Our main result can now be stated as follows:

**Theorem 1.4.** Let $F \in \operatorname{End}(\mathbb{C}^2, O)$ and assume $O$ is a quasi-parabolic point of $F$. If $F$ is dynamically separating then there exist $\nu(F) - 1$ parabolic curves for $F$ at $O$ tangent to the eigenspace of 1.

One remarkable consequence of this theorem is that if $F$ is given by (1.1) and $p_{2,0} \neq 0$ then there always exists a parabolic curve for $F$ at $O$ tangent to the eigenspace of 1. This is similar to a result in the quasi-hyperbolic case—one eigenvalue 1, the other of modulus $< 1$—where, under similar hypothesis, the existence of a basin of attraction for $F$ is proved (cf. [8], [14], [15]).

The plan of the paper is the following: In Section 2 we introduce ultra-resonant normal forms, the invariant $\nu(F)$ and dynamically separating maps and give the proof of Proposition 1.3. In Section 3 we prove Theorem 1.4. Finally, in Section 4 we conclude with some remarks and discuss the case $\Theta(F) = 1$ for some $s \geq 2$, especially relating parabolic curves provided by Theorem 1.4 with the ones given by Hakim’s and Abate’s theory for $F^s$.

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### 2. Ultra-resonant normal forms.

**Definition 2.1.** Let $F \in \operatorname{End}(\mathbb{C}^2, O)$ be given by (1.1). We call ultra-resonant the monomials of type $z^n$ in $F_1$ and of type $z^m w$ in $F_2$, $m \in \mathbb{N}$.

In case there exists $j \in \mathbb{N}$ such that $p_{j,0} \neq 0$ we let

$$
\mu(F, z) := \min\{j \in \mathbb{N} : p_{j,0} \neq 0\},
$$

and let $\mu(F, z) = +\infty$ if $p_{j,0} = 0$ for all $j$'s. Similarly if there exists $j \in \mathbb{N}$ such that $q_{j,1} \neq 0$, we let

$$
\mu(F, w) := \min\{j \in \mathbb{N} : q_{j,1} \neq 0\},
$$

setting $\mu(F, w) = +\infty$ if $q_{j,1} = 0$ for all $j$'s.

Finally, if $\mu(F, z) < +\infty$ we let $\Theta(F) := \mu(F, z) - \mu(F, w) - 1$ (with the convention that $\Theta(F) = -\infty$ if $\mu(F, w) = +\infty$).

In general $\mu(F, z)$ and $\mu(F, w)$ are not invariant under change of coordinates. However $\mu(F, z)$ and the sign of $\Theta(F)$ are invariant under a suitable normalization which we are going to describe.

**Definition 2.2.** We say that a (possibly formal) germ of diffeomorphism $F \in \operatorname{End}(\mathbb{C}^2, O)$ is in ultra-resonant normal form if $F$ is given by (1.1) and $q_{j,0} = 0$ for $j = 2, \ldots, \mu(F, z) - 1$. If $q_{j,0} = 0$ for any $j$ we call $F$ an asymptotic ultra-resonant normal form.
The first result we prove is the existence of (possibly formal) asymptotic ultra-resonant normal form.

**Proposition 2.3.** Let $F \in \text{End}(\mathbb{C}^2, O)$ and assume $O$ is a quasi-parabolic fixed point for $F$. Then there exists a formal transformation $\hat{K} \in \text{End}(\mathbb{C}^2, O)$ tangent to $\text{id}$ such that $\hat{K}^{-1} \circ F \circ \hat{K} = \hat{F}$, with $\hat{F}$ a formal asymptotic ultra-resonant normal form.

**Proof.** We may assume $F$ in the form (1.1). Let $q_{s,0} \neq 0$ be the first nonzero coefficient of a pure term in $z$ in $F$. Consider the transformation

$$K_s = \begin{cases} z = Z \\
 w = W + aZ^s
\end{cases}$$

with $a = -q_{s,0}/(e^{i\theta} - 1)$. Then $K_s^{-1} \circ F \circ K_s$ has pure term in $Z$ in the second component of degree $\geq s + 1$. Proceeding this way we can get rid of all pure terms in $z$ in the second component, and $\hat{K}$ is given by composition of the $K_s$'s. \hfill \Box

Ultra-resonant normal forms are by no means unique as the following example shows.

**Example 2.4.** The germs $F(z,w) = (z + z^2, e^{i\theta}w)$ and $G(z,w) = (z + z^2, e^{i\theta}w - e^{i\theta}wz^2/(1 + z + z^2))$ are both in normal forms and conjugated by the transformation $(z,w) \mapsto (z, w + zw)$. Moreover $\mu(F,z) = \mu(G,z) = 2$, $\Theta(F) = -\infty$ while $\Theta(G) = -1$.

Using ultra-resonant normal forms we can define some invariants associated to $F$. Before doing that, we need the following basic lemma.

**Lemma 2.5.** Let $F, G \in \text{End}(\mathbb{C}^2, O)$ be (possibly formal) germs of diffeomorphisms in ultra-resonant normal form. If $F$ is conjugated to $G$ then $\mu(F,z) = \mu(G,z)$. Moreover if $\mu(F,z) = \mu(G,z) < \infty$ then $\Theta(F) \leq 0$ if and only if $\Theta(G) \leq 0$, while if $\mu(F,z) = \mu(G,z) = \infty$ then $\mu(F,w) = \mu(G,w)$.

**Proof.** Let $F$ be given by (1.1), and let

$$G(z,w) = (z + \sum_{j+k \geq 2} \tilde{p}_{j,k}z^j w^k, e^{i\theta}w + \sum_{j+k \geq 2} \tilde{q}_{j,k}z^j w^k).$$

If $T$ is the transformation which conjugates $F$ to $G$, then its differential at the origin must be a diagonal matrix, which we can assume to be the identity. Thus let $T : (z,w) \mapsto (z + \varphi_1(z,w), w + \varphi_2(z,w))$ be the transformation conjugating $F$ to $G$.

We introduce the following notation: we denote by $H_m$ any term which has order greater than or equal to $m$. Also, for $m, n \in \mathbb{N}$, $m \leq n$, we write $B_{m,n}$ for...
indicating terms of order greater than or equal to \( m \) but less than or equal to \( n \); we also set \( B_{n,n} = 0 \) for \( m > n \). Moreover we let \( S_k \) denote any term of order strictly smaller than \( k \). We also set \( a := \mu(F,z) \), \( b = \mu(F,w) \) and \( \tilde{a} = \mu(G,z) \), \( \tilde{b} = \mu(G,w) \). In case \( a = \infty \) we agree that terms of type \( p_{a,0}z^a \) and symbols like \( O(z^a) \) should be understood as zeros (similarly if \( \tilde{a} = \infty \)). With this convention we can deal with all cases at the same time. Since \( F = (F_1, F_2) \) and \( G = (G_1, G_2) \) are both in normal form, we can write

\[
F(z, w) = \begin{cases} 
F_1(z, w) = z + p_{a,0}z^a + wB_{1,a-1} + H_{a+1}, \\
F_2(z, w) = e^{i\theta}w + q_{b,1}z^b w + w^2 S_b + O(z^a, z^{b+1}w, w^2H_b), 
\end{cases}
\]

and

\[
G(z, w) = \begin{cases} 
G_1(z, w) = z + \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + wB_{1,\tilde{a}-1} + H_{\tilde{a}+1}, \\
G_2(z, w) = e^{i\theta}w + \tilde{q}_{\tilde{b},1}z^{\tilde{b}} w + w^2 S_{\tilde{b}} + O(z^{\tilde{a}}, z^{\tilde{b}+1}w, w^2H_{\tilde{b}}). 
\end{cases}
\]

Let \( c_h \geq 2 \) be the order of vanishing of \( \varphi_h(z, 0) \) at \( 0, h = 1, 2 \). Since \( F \circ T = T \circ G \), using (2.2) and (2.3) and equating components we obtain

\[
\varphi_1(z, w) + p_{a,0}z^a + \varphi_2(z, w)B_{1,a-1} + H_{a+1} + O(w) = \varphi_1(G(z, w)) + \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + H_{\tilde{a}+1},
\]

and

\[
e^{i\theta} \varphi_2(z, w) + q_{b,1}(z + \varphi_1(z, w))b^b w + \varphi_2(z, w) + [2w\varphi_2(z, w) + \varphi_2(z, w)^2]S_b + O(z^a, z^{b+1+c}, z^{b+1}w) + O(w^2) = \varphi_2(G(z, w)) + \tilde{q}_{\tilde{b},1}z^{\tilde{b}} w + O(z^{\tilde{a}}, z^{\tilde{b}+1}w).
\]

Write \( \varphi_h(z, w) = \sum_{j+k \geq 2} \varphi_h^{jk} z^j w^k \), for \( \varphi_h^{jk} \in \mathbb{C} \) and \( h = 1, 2 \). Then

\[
q_{b,1}(z + \varphi_1(z, w))b^b (w + \varphi_2(z, w)) = q_{b,1}z^b w + O(w^2, z^{b+1}w, z^{b+c_2}),
\]

\[
\varphi_2(G(z, w)) - e^{i\theta} \varphi_2(z, w) = (1 - e^{i\theta})\varphi_2^{c_2, 0}z^{c_2} + O(z^{a}, z^{c_2+1}, w),
\]

and putting (2.6), (2.7) into (2.5) we get that

\[
c_2 \geq \min\{ a, \tilde{a} \},
\]

where we understood \( c_2 = \infty \) (that is \( \varphi_2^{c_2, 0} = 0 \) for any \( j \)) in case \( a = \tilde{a} = \infty \). In particular equation (2.4) reads now as

\[
\varphi_1(G(z, w)) - \varphi_1(z, w) = p_{a,0}z^a - \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + O(w, z^{a+1}, z^{\tilde{a}+1}).
\]
We examine the left-hand side of (2.9). Using (2.3) we have

\[(2.10) \quad \varphi_1(G(z, w)) = \sum_{j+k \geq 2} \varphi_{j,k}^1 [z + O(z^\alpha, w)]^j [e^{i\theta} w + O(z^\alpha, w^2)]^k = \varphi_1(z, w) + O(w, z^{\hat{a}+1}).\]

Therefore from (2.9) and (2.10) we get \(a = \hat{a}\), that is \(\mu(F, z) = \mu(G, z)\).

Let \(a < \infty\). We assume \(\Theta(F) \leq 0\) and want to show that \(\Theta(G) \leq 0\) (the other implication will follow reversing the role of \(F\) and \(G\)). We have already proved that \(\hat{a} = a\) and now we are assuming \(b \geq a - 1\). Seeking for a contradiction we suppose that \(\hat{b} < a - 1\). Taking into account (2.6) and (2.8), equation (2.5) becomes

\[(2.11) \quad \varphi_2(G(z, w)) - e^{i\theta} \varphi_2(z, w) = -\tilde{q}_{b,1} \zeta^b w + O(wz^{\hat{b}+1}, z^\alpha, w^2).\]

We examine the left-hand side of (2.11). Since \(\varphi_{j,0}^2 = 0\) for \(j < c\) and \(c \geq a\) by (2.8), using (2.3) we have

\[(2.12) \quad \varphi_2(G(z, w)) = \sum_{j \geq 0} \varphi_{j,0}^2 [z + O(z^\alpha, w)]^j + \sum_{j+k \geq 1} \varphi_{j,k}^2 [z + O(z^\alpha, w)]^j [e^{i\theta} w + O(wz^\hat{b}, z^\alpha, w^2)]^k = \varphi_2(z, e^{i\theta} w) + O(w^2, z^\alpha, wz^{\hat{b}+1}).\]

Put (2.12) into (2.11) and noting that \(e^{i\theta} \varphi_2(z, w) - \varphi_2(z, e^{i\theta} w)\) does not contain terms in \(z^m w\) for any \(m \in \mathbb{N}\), we reach a contradiction. Therefore \(\hat{b} \geq a - 1\) and \(\Theta(G) \leq 0\) as wanted.

Finally suppose \(a = \hat{a} = \infty\). Then by hypothesis and by (2.8) the maps \(G(z, w), F(z, w)\) and \(\varphi_2(z, w)\) do not contain pure terms in \(z\). Therefore, using (2.6), equation (2.5) becomes

\[\varphi_2(G(z, w)) - e^{i\theta} \varphi_2(z, w) = -\tilde{q}_{b,1} \zeta^b w + q_{b,1} \zeta^b w + O(wz^{b+1}, wz^{\hat{b}+1}, w^2),\]

where, as usual, we set all the terms containing \(z^b\) or \(\zeta^b\) equal to zero if \(b = \infty\) or \(\hat{b} = \infty\). From this and from (2.12) it follows that \(b = \hat{b}\).

Remark 2.6. If \(F\) and \(G\) are conjugated and in ultra-resonant normal form (and \(\mu(F, z) = \mu(G, z) < \infty\), \(\mu(F, w)\) might be different from \(\mu(G, w)\), as one can see in the Example 2.4.

Now we are in the position to define our invariants:

**Definition 2.7.** Let \(F \in \text{End}(\mathbb{C}^2, O)\) and assume \(O\) is a quasi-parabolic fixed point for \(F\). Let \(\hat{F}\) be a (possibly formal) asymptotic ultra-resonant normal form of...
F. We let $\nu(F) := \mu(\hat{F}, z)$. In case $\mu(\hat{F}, z) < \infty$ we call $F$ dynamically separating if $\Theta(\hat{F}) \leq 0$.

**Remark 2.8.** By Lemma 2.5 the previous definition is well posed. Moreover, if $\nu(F) < \infty$ one can find a (convergent) ultra-resonant normal form conjugated to $F$ after a finite number of transformations of type (2.1).

Let $F \in \text{End}(\mathbb{C}^2, O)$. The Poincaré-Dulac normal form theorem states that it is always possible to find a resonant formal normal form for $F$. Namely there exists a formal transformation $T : (z, w) \mapsto (z + \ldots, w + \ldots)$ such that $T^{-1} \circ F \circ T(z, w) = (z + R_1(z, w), e^{\theta w} w + R_2(z, w))$, where $R_1, R_2$ are series of resonant monomials, that is $R_1(z, w)$ is a combination of terms of type $z^n$, $z^n w^n$, while $R_2(z, w)$ is a combination of terms of type $z^n w, z^n w^{m+1}$ for $m, n \in \mathbb{N}$, where $s \in \mathbb{N}$ is such that $e^{i\theta} = 1$ (thus $s = 0$ if $e^{i\theta}$ is not a root of unity).

Due to Lemma 2.5 our (formal) asymptotic ultra-resonant form is equivalent to the Poincaré-Dulac normal form for the purpose of calculating $\mu(F, z)$ and $\Theta(F)$. However, asymptotic ultra-resonant normal forms reflect better the dynamics of $F$, as claimed in Proposition 1.3. Here is its proof.

**Proof of Proposition 1.3.** If $F$ has a convergent asymptotic ultra-resonant normal form then $F$ is conjugated to a germ of biholomorphism $G = (G_1, G_2)$ such that $G_2(z, w) = w A(z, w)$ for some holomorphic function $A(z, w)$. In particular $w = 0$ is invariant by $G$. For the converse, if there exists an invariant curve tangent to the eigenspace of $1$ we can choose coordinates in such a way that $\Gamma = \{(z, w) : w = 0\}$ and $F(z, w) = (z + \ldots, e^{i\theta w} + w A(z, w))$ for some holomorphic function $A(z, w)$. In particular $F$ has a (convergent) asymptotic ultra-resonant form. By Lemma 2.5, if $F$ has a convergent asymptotic ultra-resonant normal form $G$ then $\mu(G, z) = \nu(F)$. Thus if $\nu(F) = \infty$ then $G_1(z, w) = z + w A_1(z, w)$ and $\{w = 0\}$ is a curve of fixed points for $G$. If $\nu(F) < \infty$ then the classical one-dimensional Fatou theory gives the result. \qed

**3. Dynamics.** In this section we give the proof of Theorem 1.4. The idea is that starting from an ultra-resonant normal form, if $\Theta(F) \leq 0$, it is possible to blow up $O$ a certain number of times in order to find some simpler expression for $F$, where one can apply a modified Hakim’s argument to produce parabolic curves.

We divide the proof into several steps, which might be of some interest on their own.

Recall that if $F \in \text{End}(\mathbb{C}^2, O)$ and $\pi : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$ is the blow-up (quadratic transformation) of $\mathbb{C}^2$ at $O$, then there exists a holomorphic map $\hat{F}$ defined near the exceptional divisor $D := \pi^{-1}(O)$ such that $\pi \circ \hat{F} = F \circ \pi$, $\hat{F}(D) = D$ and the action of $\hat{F}$ on $D$ is given by $D \ni [\nu] \mapsto [dF_O(\nu)] \in D$ (see for instance [1], [17]). We call such a $\hat{F}$ the blow-up of $F$. 
LEMMA 3.1. Suppose $F$ is given by (1.1). If

(1) $q_{j,0} = 0$ for $j < \mu(F, z)$ and

(2) $q_{j,1} = 0$ for $j < \mu(F, z) - 1,$

then one can perform a finite number of blow-ups and changes of coordinates in such a way that the blow-up map $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ is given by

\begin{equation}
\begin{aligned}
\tilde{F}_1(z, w) &= z - z^{\nu(F)} + O(z^{\nu(F)+1}, z^{\nu(F)}w), \\
\tilde{F}_2(z, w) &= e^{\theta}w - \lambda w z^{\nu(F)-1} + O(w z^{\nu(F)} - z^{\nu(F)-1} w^2, z^{\nu(F)+2}),
\end{aligned}
\end{equation}

with $\text{Re}(\lambda e^{-i\theta}) < 0.$

Proof. Note that by hypothesis $F$ is an ultra-resonant normal form, thus $\nu(F) = \mu(F, z).$ First of all, we can use transformations of type (2.1), for $s = \nu(F),$ as in the proof of Proposition 2.3, to dispose of $q_{\nu(F),0}.$ Note that $K_j$ does not decrease the order of vanishing of $F_1(z, w) - z$ and $F_2(z, w) - e^{\theta}w,$ nor it effects the ultra-resonant monomials of order $\leq \nu(F).$ Now we blow-up the point $O$ in $\mathbb{C}^2.$ Recalling that $1/(1+\xi) = \sum_{k \geq 0} (-1)^k \xi^k$ for $|\xi| < 1,$ in coordinates $(z = u, w = u\nu)$ we have that the blow-up map $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ is given by

\begin{equation}
\begin{aligned}
\tilde{F}_1(u, v) &= u + \sum_{j+k \geq 2} p_{j,k} u^{j+k} v^k = u + \sum_{j+k \geq 2} \tilde{p}_{j,k} u^{j+k} v^k, \\
\tilde{F}_2(u, v) &= e^{\theta} v + \sum_{j+k \geq 2} q_{j,k} u^{j+k-1} v^k \left[ 1 - \sum_{j+k \geq 2} p_{j,k} u^{j+k-1} v^k \right. \\
&\quad + \left. \left( \sum_{j+k \geq 2} p_{j,k} u^{j+k-1} v^k \right)^2 + \cdots \right] = e^{\theta} v + \sum_{j+k \geq 2} \tilde{q}_{j,k} u^{j+k} v^k.
\end{aligned}
\end{equation}

Thus, setting $p_{j,k} = 0$ for $j + k < 2,$ it follows that $\tilde{p}_{j,k} = p_{j-k,k}. In particular $\mu(F, z) = \mu(\tilde{F}, u)$ and $p_{\mu(F), 0} = \tilde{p}_{\mu(F), 0}.$ Moreover, if $m_1$ was the order of vanishing of $F_1(z, w) - z$ (that is $p_{j,k} = 0$ for $j + k < m_1$), then the order of vanishing of $\tilde{F}_1(u, v) - u$ is at least $m_1 + 1$ if $m_1 < \nu(F)$ or it is equal to $m_1$ if $m_1 = \nu(F).$ Also, the lowest nonzero non ultra-resonant term in $\tilde{F}_1,$ i.e., the one of type $w^a z^b,$ $a \geq 1,$ $b \geq 0,$ has degree strictly greater than the lowest one in $F_1.$

The terms $\tilde{q}_{j,k}$ in the second component of $\tilde{F}$ are more difficult to write explicitly. We use the notations $H_m$ and $B_{m,n}$ introduced in the proof of Lemma 2.5. Denote by $m_2$ the order of vanishing of $F_2(z, w) - e^{\theta}w.$ Note that, since we assumed that $q_{j,0} = 0$ for $j < \nu(F) + 1$ and by hypothesis (2), then for every $q_{j,k} \neq 0$ with $j + k < \nu(F)$ it follows that $k \geq 2.$ Thus, using hypothesis (1) and (2)
we have
\[
\tilde{F}_2(u, v) = [e^{i\theta} v + q_{\nu(F)-1,1} u^\nu v + v^2 B_{m_2-1,\nu(F)-2} + H_{\nu(F)+1}]1
\]
\[+ \sum_{k=1}^{\infty} (-1)^k (p_{\nu(F),0} u^\nu v + v_j^2 B_{m_1-1,\nu(F)-2}) + H_{\nu(F)+1}k\]
\[+ p_{\nu(F),0} u^2 v + v \sum_{j=m_1-1}^\nu \tilde{F}_m \tilde{F}_n + v^2 H_{m_1-1} + v^2 H_{m_2-1} + H_{\nu(F)+1}.\]

In particular note that the ultra-resonant terms in \(\tilde{F}_2\) are vanishing up to order \(\nu(F) - 1\). Also \(\tilde{q}_{\nu(F)-1,1} = (q_{\nu(F)-1,1} - e^{i\theta} p_{\nu(F),0})\) and then
\[\Re(e^{-i\theta} \tilde{q}_{\nu(F)-1,1}/\tilde{p}_{\nu(F),0}) = \Re(e^{-i\theta} q_{\nu(F)-1,1}/p_{\nu(F),0}) - 1.\]

Finally note that the order of vanishing of \(\tilde{F}_2(u, v) - e^{i\theta} v\) is at least \(\min\{\nu(F), m_1 + 1, m_2 + 1\}\). This time the lowest nonzero non ultra-resonant term in \(\tilde{F}_2\) might be of degree strictly smaller than the one in \(F_2\). However, its degree is at least \(\min\{\nu(F)+1, m_1 + 1, m_2 + 1\}\). In particular the map \(\tilde{F}\) has properties (1), (2) in the hypothesis and its lowest nonzero non ultra-resonant term (in both components) has degree at least \(\min\{\nu(F)+1, m_1 + 1, m_2 + 1\}\). Moreover \(\Re(e^{-i\theta} \tilde{q}_{\nu(F)-1,1}/\tilde{p}_{\nu(F),0})\) is one less than \(\Re(e^{-i\theta} q_{\nu(F)-1,1}/p_{\nu(F),0})\).

Repeating the previous arguments (conjugation with \(K_s\) followed by blow-up) we will eventually find a map in ultra-resonant normal form given by (1.1) with

(i) \(q_{j,k} = 0\) for \(j + k < \nu(F)\),

(ii) \(p_{j,k} = 0\) for \(j + k < \nu(F)\),

(iii) \(\Re(e^{-i\theta} q_{\nu(F)-1,1}/p_{\nu(F),0}) < 1\).

Note that \(\nu(F)\) is the same as for the starting map. Eventually performing some more transformations \(K_s\) as in (2.1), with \(s = \nu(F), \nu(F) + 1, \nu(F) + 2\), we can assume \(q_{j,0} = 0\) for \(j < \nu(F) + 3\).

Let \(\alpha^{\nu(F)-1} = -p_{\nu(F),0}\) and let \(T\) be the transformation given by \(Z = \alpha z, W = w\). The map \(\tilde{F} = T \circ F \circ T^{-1}\) satisfies (i), (ii) and \(\nu(\tilde{F}) = \nu(F)\). Moreover, denoting with \(\tilde{q}\) the coefficients of \(\tilde{F}\), we have \(\tilde{p}_{\nu(F),0} = -1, \tilde{q}_{j,0} = 0\) for \(j < \nu(F) + 3\) and \(\tilde{q}_{\nu(F)-1,1} = -q_{\nu(F)-1,1}/p_{\nu(F),0}\). In particular property (iii) becomes \(\Re(e^{-i\theta} \tilde{q}_{\nu(F)-1,1}) > -1\).

Now we perform a final blow-up of \(O\). Let \(\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2\) be the blow-up and \(\tilde{F}\) the blow-up map. In the coordinates \((z, w)\) such that the projection \(\pi(z, w) = (z, zw)\), we have that \(\tilde{F} = (\tilde{F}_1, \tilde{F}_2)\) is given by (3.1), with \(\lambda = -(e^{i\theta} + \tilde{q}_{\nu(F)-1,1})\).

Now we prove that form (3.1) is actually useful.
**Lemma 3.2.** Let $F \in \text{End}(\mathbb{C}^2, O)$ be given by (3.1), with $\nu(F) \geq 2$ and $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda e^{-i\theta}) < 0$. Then there exist $\nu(F) - 1$ parabolic curves for $F$ at $O$ tangent to $[1 : 0]$.

**Proof.** The proof is a modification of that of Theorem 3.1 of [1]. Let $r = \nu(F) - 1$. Let $D_{\delta,r} := \{ \zeta \in \mathbb{C} : |\zeta^r - \delta| < \delta \}$ and let $\mathcal{E}(\delta) := \{ u \in \text{Hol}(D_{\delta,r}, \mathbb{C}) : u(\zeta) = \zeta^r u^0(\zeta), \|u^0\|_{\infty} < \infty \}$. The set $\mathcal{E}(\delta)$ is a Banach space with norm $\|u\|_{\mathcal{E}(\delta)} = \|u^0\|_{\infty}$. For $u \in \mathcal{E}(\delta)$ we let $F^u(\zeta) = F_1(\zeta, u(\zeta))$. The classical Fatou theory for mappings of the form $\zeta - \zeta^{r+1} + O(\zeta^{r+2})$ implies that there exists $\delta_0 = \delta_0(\|u^0\|_{\infty})$ such that if $0 < \delta < \delta_0$ then $F^u$ maps each component of $D_{\delta,r}$ into itself and moreover

$$|(F^u)^n| = O(\frac{1}{n^{1/r}}).$$

Suppose we find $u \in \mathcal{E}(\delta)$ such that $u(F_1(\zeta, u(\zeta)) = F_2(\zeta, u(\zeta))$ for any $\zeta \in D_{\delta,r}$. Thus the map $\varphi^u(\zeta) := (\zeta, u(\zeta))$ restricted to each connected component of $D_{\delta,r}$ is a parabolic curve for $F$.

For $(z, w) \in \mathbb{C}^2$ let $z_1 := F_1(z, w)$ and $w_1 := F_2(z, w)$. Suppose $z, z_1$ belong to the same connected component of $D_{\delta,r}$. Let $\mu := \lambda e^{-i\theta}$ and define

$$H(z, w) := w - e^{-i\theta} \frac{z^\mu}{z_1^\mu} w_1.$$ 

Thus a direct computation shows that

$$H(z, w) = w - z^\mu w - \mu z^r w + O(wz^{r+1}, w^2 z^r, z^{r+3})$$

$$= w - [w - \mu z^r w + O(wz^{r+1}, w^2 z^r, z^{r+3})][1 + \mu z^r + O(z^{r+1}, z^r w)]$$

$$= O(z^{r+1} w, z^r w^2, z^{r+3}).$$

Now $F_2(z, w) = w_1 = e^{i\theta} \frac{z^\mu}{z_1^\mu} (w - H(z, w))$ and therefore we are left to solve the following functional equation:

$$u(z_1(\zeta, u(\zeta)) = e^{i\theta} \frac{z^\mu}{\zeta^\mu} u(\zeta) - H(z, u(\zeta)).$$

For $\zeta_0 \in D_{\delta,r}$ let $\zeta_n := (F^n)^u(\zeta_0)$. For $u \in \mathcal{E}(\delta)$ let

$$Tu(\zeta_0) := \zeta^\mu_0 \sum_{n=0}^{\infty} e^{-in\theta} \frac{z^\mu}{\zeta^\mu_n} H(z_n, u(z_n)).$$

If $n$ is such that $\|u^n\| < c_0$ and $\delta \leq \delta_0(c_0)$, then $H(z_n, u(z_n))$ is defined for any $\zeta_0 \in D_{\delta,r}$. Moreover one can show exactly as in [1] and [10] that the series
converges normally and $Tu \in \mathcal{E}(\delta)$ (essentially because $|e^{i\theta}| = 1$ and thus all the estimates for the parabolic case in [1] go through in this case as well).

Now suppose $u$ is a fixed point for $T$. Then $\varphi^\mu$ is a parabolic curve for $F$. indeed if

$$u(\zeta_0) = Tu(\zeta_0) = \zeta_0^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)),$$

then

$$u(\zeta_1) = \zeta_1^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_{n+1}^{-\mu} H(\zeta_{n+1}, u(\zeta_{n+1})) = e^{i\theta} \zeta_1^\mu \sum_{n=1}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n))$$

$$= \frac{\zeta_1^\mu}{\zeta_0} e^{i\theta} \left( \zeta_0^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)) - H(\zeta_0, u(\zeta_0)) \right)$$

$$= \frac{\zeta_1^\mu}{\zeta_0} e^{i\theta} (u(\zeta_0) - H(\zeta_0, u(\zeta_0))),$$

solving thus (3.4).

It remains to show that $T$ does have a fixed point. For doing this we only need to show that $T$ is a contraction on a suitable closed convex subset of $\mathcal{E}(\delta)$. This can be done arguing exactly as in Theorem 3.1 of [1], for all the estimates holding in there actually hold in this case, and we are done.

Now we are in a good shape to prove our main theorem.

**Proof of Theorem 1.4** Since having parabolic curves is obviously a property invariant under changes of coordinates and by Remark 2.8, we can assume $F$ to be in ultra-resonant normal form. By definition of dynamically separating map, $\Theta(F) \leq 0$ and we can thus apply Lemma 3.1 to $F$ and Lemma 3.2 to its blow-up $\tilde{F}$ in order to produce $\nu(F) - 1$ parabolic curves for $\tilde{F}$ at some point of the exceptional divisor. These parabolic curves blow down to $\nu(F) - 1$ parabolic curves for $F$ tangent to the eigenspace of 1 and we are done.

4. Final remarks.

1. Let $F \in \text{End}(\mathbb{C}^2, O)$ and suppose $O$ is a quasi-parabolic fixed point for $F$. In case $e^{i\theta_s} = 1$ for some $s \geq 2$ one can try to apply Hakim and Abate’s theory to produce parabolic curves for $F^s$. If $F$ is dynamically separating one always obtains $\nu(F) - 1$ parabolic curves for $F$ by Theorem 1.4 and these are obviously parabolic curves for $F^s$ as well. The question is whether these parabolic curves are the ones predicted by Hakim’s and Abate’s theory for $F^s$ (if such a theory applies). To give an appropriate answer we need some tools from [10] and [1]. For the reader’s convenience we quickly recall them here.
Let $G \in \text{End}(\mathbb{C}^2, O)$ be such that $dG_O = \text{id}$. Let $G = \text{id} + \sum_{m \geq 2} G_m$ be the homogeneous expansion of $G$. Then the order of $G$, which we denote by $\nu(G)$, is the first $m$ such that $G_m \neq 0$. A vector $v \in \mathbb{C}^2 \setminus \{O\}$ is called a characteristic direction for $G$ if $G_t(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Moreover if $\lambda \neq 0$ the vector $v$ is called a nondegenerate characteristic direction while it is called degenerate in case $\lambda = 0$. Hakim’s theory [10] predicts the existence of at least $\nu(G) - 1$ parabolic curves tangent to each nondegenerate characteristic direction.

We have the following relations:

**Proposition 4.1.** Let $F \in \text{End}(\mathbb{C}^2, O)$ and assume $O$ is a quasi-parabolic fixed point for $F$. Suppose $F$ is given by (1.1) and $e^{i0s} = 1$ for some $s \geq 2$. Let $G := F^s$ and assume $F$ is dynamically separating. Then:

1. $G \neq \text{id}$ and $\nu(G) \leq \nu(F)$.
2. $[1 : 0]$ is a characteristic direction for $G$. Moreover $[1 : 0]$ is a nondegenerate characteristic direction for $G$ if and only if $\nu(F) = t(G)$.
3. The $\nu(F) - 1$ parabolic curves tangent to $[1 : 0]$ at $O$ given by Theorem 1.4 for $G$ can be found applying Hakim’s and Abate’s theory to $G$ after a finite number of blow-ups.

**Proof.** Since $F$ is dynamically separating then there exist parabolic curves for $F$ by Theorem 1.4 which are obviously parabolic curves for $G$. Thus $G \neq \text{id}$. It is then clear that $\nu(F) \geq t(G)$. To prove the other statements we notice that everything involved is invariant under conjugation and thus, using transformations as (2.1) we can assume that $q_{j,0} = 0$ for $j \leq \nu(F)$. Therefore for $F = (F_1, F_2)$ we can write

$$F(z, w) = \begin{cases} F_1(z, w) = z + p_{\nu(F),0}z^{\nu(F)} + O(z^{\nu(F)+1}, zw, w^2) \\ F_2(z, w) = e^{i\theta}w + O(z^{\nu(F)-1}w, w^2, z^{\nu(F)+1}) \end{cases}.$$

Iterating we find that $F^s = G = (G_1, G_2)$ is given by

$$G(z, w) = \begin{cases} G_1(z, w) = z + sp_{\nu(F),0}z^{\nu(F)} + O(z^{\nu(F)+1}, zw, w^2) \\ G_2(z, w) = w + O(z^{\nu(F)-1}w, w^2, z^{\nu(F)+1}) \end{cases}. \tag{4.1}$$

From this it follows that $[1 : 0]$ is a characteristic direction for $G$. Moreover it is nondegenerate if and only if $t(G) = \nu(F)$ for in that case $G_{t(G)} = (p_{\nu(F),0}z^{\nu(F)} + wQ(z, w), wQ'(z, w))$ with $Q, Q'$ suitable homogeneous polynomials of degree $t(G) - 1$.

To prove part (3), we make some preliminary observations. If $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is a blow-up at $O$ and $\tilde{F}$ is the blow-up of $F$, since $\pi \circ \tilde{F}^s = F^s \circ \pi$ and $\pi$ is a biholomorphism outside the exceptional divisor then $\tilde{G} = \tilde{F}^s$. Notice that while $\nu(F) = \nu(\tilde{F})$, in general $t(G) \leq t(\tilde{G})$ (see Lemma 2.1(ii) and (2.1) in [1]). We may assume that after finitely many blow-ups and changes of coordinates $F$ is
that in such a case, if \( p_j \) and we are done.

in such a class then they must be the ones given by Hakim’s and Abate’s theory, (see p. 201–203 in [1]). Since the parabolic curves produced in Lemma 3.2 are \( G \) characteristic direction for \( F \) case \( \in End \ O \) has therefore 4 parabolic curves tangent to \([1 : 0]\) at \( f \) curves are provided by the following construction. Let \( \nu \) point for \( F \) then

\[
\begin{align*}
\nu(F) &= 5 \text{ and thus it has 4 parabolic curves tangent to } [1 : 0] \text{ at } O \text{ by Theorem 1.4. The map } G(z, w) = F^2(z, w) = (z^2 + 2z^5 + O(z^6), w - 2w^3 + O(w^4, z^7, w^2z^5)) \text{ has therefore 4 parabolic curves tangent to } [1 : 0] \text{ at } O. \text{ Moreover } \nu(G) = 3 \text{ and the vector } [1 : 0] \text{ is a degenerate characteristic direction for } G. \text{ However } \tilde{G} \text{ has order 5 at } [1 : 0] \text{ and has } [1 : 0] \text{ as a nondegenerate characteristic direction as a simple computation shows. Notice that } [0 : 1] \text{ is a nondegenerate characteristic direction for } G \text{ and Hakim’s results give 2 parabolic curves for } G \text{ tangent to } [0 : 1] \text{ at } O. \text{ These are contained into } \{z = 0\} \text{ and are exchanged into each other by } F.
\end{align*}
\]

\[\text{Example 4.2. The map } F(z, w) = (z + z^5, -w + w^3 + z^5) \text{ is dynamically separating, } \nu(F) = 5 \text{ and thus it has 4 parabolic curves tangent to } [1 : 0] \text{ at } O \text{ by Theorem 1.4. The map } G(z, w) = F^2(z, w) = (z + 2z^5 + O(z^6), w - 2w^3 + O(w^4, z^7, w^2z^5)) \text{ has therefore 4 parabolic curves tangent to } [1 : 0] \text{ at } O. \text{ Moreover } \nu(G) = 3 \text{ and the vector } [1 : 0] \text{ is a degenerate characteristic direction for } G. \text{ However } \tilde{G} \text{ has order 5 at } [1 : 0] \text{ and has } [1 : 0] \text{ as a nondegenerate characteristic direction as a simple computation shows. Notice that } [0 : 1] \text{ is a nondegenerate characteristic direction for } G \text{ and Hakim’s results give 2 parabolic curves for } G \text{ tangent to } [0 : 1] \text{ at } O. \text{ These are contained into } \{z = 0\} \text{ and are exchanged into each other by } F.\]

\[\text{Remark 4.3. Let } F \in End(\mathbb{C}^2, O), \text{ and assume } O \text{ is a quasi-parabolic fixed point for } F \text{ and } e^{i\theta_s} = 1 \text{ for some } s \geq 2. \text{ Suppose } F \text{ is not dynamically separating. A calculation similar to the one performed in the proof of Proposition 4.1 shows that } [1 : 0] \text{ is always a degenerate characteristic direction for } F^s, \text{ providing } F^s \neq \text{id.}\]

2. Let \( F \in End(\mathbb{C}^2, O) \) and assume \( O \) is a quasi-parabolic fixed point. In case \( F \) is not dynamically separating, there might be no parabolic curves tangent to the eigenspace of 1. A first simple example is when \( F^s = \text{id} \). However note that in such a case, if \( p_j : \mathbb{C}^2 \rightarrow \mathbb{C} \) is the projection on the \( j \)th component, setting

\[
\sigma(z, w) = \left( \sum_{m=0}^{s-1} p_1 \circ F^m(z, w), \sum_{m=0}^{s-1} e^{-i\theta m} p_2 \circ F^m(z, w) \right)
\]

then \( \sigma \circ F \circ \sigma^{-1}(z, w) = (z, e^{i\theta} w) \), thus \( F_1(z, w) = z \), and in particular \( \nu(F) = \infty. \)

Less trivial examples of nondynamically separating map without parabolic curves are provided by the following construction. Let \( f(u, v) = (f_1(u, v), f_2(u, v)) \in End(\mathbb{C}^2, O) \) be given by

\[
\begin{align*}
f_1(u, v) &= e^{i\theta} u + (a_{20} u^2 + a_{11} u v + a_{02} v^2) + \cdots \\
f_2(u, v) &= e^{i\theta} v + (b_{20} u^2 + b_{11} u v + b_{02} v^2) + \cdots
\end{align*}
\]

(4.2)
with $e^{i\theta}$ satisfying the Bryuno condition

$$|e^{im\theta} - 1| \geq cm^{-N}, \ m \in \mathbb{N}$$

for some $c > 0$ and some large $N$. Note that the set of points on the circle satisfying such a condition has full measure. It is a classical result (see, e.g., [3] and [12]) that such a germ $f$ is linearizable, and in particular there cannot exist parabolic curves for $f$. Now suppose that $a_{02} = 0$ in (4.2). Blow up the point $O$ in $\mathbb{C}^2$ and consider the blow up map $F$ of $f$ at the point $[0 : 1]$ of the exceptional divisor. If the projection $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $(u, v) = \pi(z, w) = (zw, w)$ then

$$F = (F_1, F_2)$$

is given by

$$\begin{align*}
F_1(z, w) &= z + e^{-i\theta}w^{(a_{11}-b_{02})z+b_{03}w+\ldots} \\
F_2(z, w) &= e^{i\theta}w + w[b_{02}w + (b_{11}zw + b_{03}w^2 + \ldots)].
\end{align*}$$

Then $[0 : 1]$ is a quasi-parabolic point for $F$ but there cannot exist parabolic curves tangent to the eigenspace of 1 for otherwise these would be parabolic curves for $f$ at $O$. Note that even in this case $\nu(F) = \infty$.

We have to say that at the present we do not have any example of a nondynamically separating mapping $F$ with $\nu(F) < \infty$ and without parabolic curves, even if we believe such a map should exist.

We conclude this work by mentioning a simple family of nondynamically separating maps for which nothing is known, but the understanding of which might unlock the general theory. Let $F_a = (F_{1,a}, F_{2,a})$ be given by

$$F_a(z, w) = \begin{cases} 
F_{1,a}(z, w) = z + z^3 + aw^2 \\
F_{2,a}(z, w) = e^{i\theta}w + zw + z^3,
\end{cases}$$

with $a \in \mathbb{C}$. If $a = 0$, then $\{z = 0\}$ is invariant by $F_0$. Moreover, once fixed $w \in \mathbb{C}$, by the classical Leau-Fatou theory there exist two petals $P_1, P_2 \subset \mathbb{C}$ for $z \mapsto F_{1,0}(z, w)$ at $z = 0$. Then the two open sets $D_j = P_j \times \mathbb{C}$, $j = 1, 2$ are invariant by $F_0$. However we do not know whether there exist parabolic curves contained in $D_1$ or $D_2$.

If $a \neq 0$ and $e^{i\theta}$ is not a root of unity we do not even know whether there exists $P \in \mathbb{C}^2$ such that $F_{1,a}^n(P) \neq O$ for any $n$ but $F_{1,a}^n(P) \to O$ as $n \to \infty$.

Notice that in case $e^{i\theta s} = 1$ for some $s \geq 2$ then Theorem 1.2 provides some parabolic curves for $F^s$. A direct computation shows that these curves are not tangent to $[1 : 0]$. In fact the known techniques for the parabolic case are not applicable to $F^s$ along the direction $[1 : 0]$, not even after blow-ups.

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