METHODS OF DIFFERENTIAL GEOMETRY IN ANALYTIC AND ALGEBRAIC GEOMETRY †.

DANIEL LEHMANN

Département des Sciences Mathématiques, Université de Montpellier II
Place Eugène Bataillon, F-34095 Montpellier Cedex 5, France, e-mail: lehmann@darboux.math.univ-montp2.fr

Notes written by Filippo Bracci[1] and Giulio Minervini[2]

[1] Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, 00133 Roma, Italy, e-mail:fbracci@mat.uniroma2.it

[2] Dipartimento di Matematica, Università di Roma "La Sapienza"
E-mail:minervin@mat.uniroma1.it

These notes are thought of as a second part of [23] and we freely assume all such materials and notations in here.

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1. STATEMENTS OF THE BAUM-BOTT THEOREM AND GENERALIZATIONS

1.1. **The Baum-Bott Theorem(s).** Let V be a compact complex manifold, $\dim_{\mathbb{C}} V = n$, let v be a holomorphic vector field on V with only isolated singularities $m_1, \ldots, m_r$. Let $I = (i_1, \ldots, i_n)$ with $i_j \geq 0$ for $j = 1, \ldots, n$. Let $|I| = i_1 + 2i_2 + \ldots + ni_n$ be the *height* of I.

For a $k \times k$ matrix M and $j = 1, \ldots, k$ let $c_j(M)$ be the j-th symmetric function of the eigenvalues of M, *i.e.*, $c_1(M) = \text{trace}(M)$,...., $c_r(M) = \det(M)$. For a multi-index $I = (i_1, \ldots, i_n)$ set $c_I := (c_1)^{i_1} \cdots (c_n)^{i_n}$.

Let m_λ be a singularity of v, *i.e.*, $v(m_\lambda) = 0$. Let $\{z_1, \ldots, z_n\}$ be a system of local coordinates on V defined on an open set $U_\lambda \subset V$ such that $m_\lambda \in U_\lambda$ and $m_\lambda = (0, \ldots, 0)$. Then

$$v|_{U_\lambda} = \sum_{i=1}^{n} A_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}, \tag{1.1}$$

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for some holomorphic functions A_i. Note that, since m_λ is an isolated singularity, then for any $m \in U_\lambda \setminus \{m_\lambda\}$ there exists (at least one) i such that $A_i(m) \neq 0$. Moreover we let

$$J := \text{Jac}\begin{pmatrix} A_1, \ldots, A_n \\ z_1, \ldots, z_n \end{pmatrix} = \frac{D(A_1, \ldots, A_n)}{D(z_1, \ldots, z_n)}.$$

Let $f : U_\lambda \to \mathbb{C}$ be holomorphic. The *Grothendieck residue* is defined as

$$\begin{bmatrix} f dz_1 \wedge \ldots dz_n \\ A_1, \ldots, A_n \end{bmatrix} := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^n \int_{R_{12\ldots n}} \frac{f(z_1, \ldots, z_n)}{A_1 A_2 \cdots A_n} dz_1 \wedge \ldots dz_n,$$

where $R_{12\ldots n}$ is a (real) n-dimensional manifold given by $R_{12\ldots n} = \{m \in U_\lambda : |A_1(m)| = \ldots = |A_n(m)| = \epsilon\}$ for some small $\epsilon > 0$ (and with some orientation, see [23]).

Definition 1.1.1. We say that v is *non-degenerate* at m_λ if $J(m_\lambda)$ is invertible.

As usual, if ω is an element of $H^{2m}(V, \mathbb{C})$ and C is an element of $H_{2m}(V, \mathbb{C})$ we denote by $\omega \frown C$ the integration of a $2m$-form representative of ω over a smooth representative of C.

With these notations we have:

Theorem 1.1.2 (Baum-Bott 1970)**.** *For any multi-index $I = (i_1, \ldots, i_n)$ such that $|I| = n$ it follows*

$$c_I(TV) \frown [V] = \sum_{\lambda=1}^{r} \begin{bmatrix} c_I(J) dz_1 \wedge \ldots dz_n \\ A_1, \ldots, A_n \end{bmatrix}_{m_\lambda}. \tag{1.2}$$

These theorem is due to Chern and Bott (1966) in case v is non-degenerate at each of its singularity.

In general, instead of working with vector fields on V one might work with one-dimensional foliations; in such a case however Theorem 1.1.2 does not hold. To see how it changes we need some notations.

Definition 1.1.3. A *one dimensional holomorphic foliation* $\mathcal{F}$ is a holomorphic line bundle $\mathcal{L}$ on V together with a morphism of vector bundles $h : \mathcal{L} \to TV$. The *singularities* of $\mathcal{F}$ are the points $p \in V$ such that $h(p) = 0$.

Note that if $p \in V$ is an isolated singularity of a one-dimensional foliation $\mathcal{F}$, then $\mathcal{F}$ is represented in some open neighborhood U of p by a vector field $v : U \to TV$ with an isolated singularity at p, and then one may define the Grothendieck residue of such a vector field v at p. Since any other vector field representing $\mathcal{F}$ near p is given by $u \cdot v$ for some invertible holomorphic function u defined on U and the Grothendieck residue of v at p is equal to the Grothendieck residue of $u \cdot v$ for any invertible holomorphic function u, then one can well-define the Grothendieck residue of $\mathcal{F}$ at p to be the Grothendieck residue of v at p.

For dealing with the case of singular foliations one has to introduce the virtual bundle $TV - \mathcal{L}$ and its Chern classes. We recall briefly how these are defined.

1.1.1. *Virtual bundles and their Chern classes.* Let $\mathcal{T}(V)$ denote the set of isomorphic classes of complex vector bundles on V. $(\mathcal{T}(V), \oplus)$ is a commutative semi-group. We set an equivalence relation on $\mathcal{T}(V) \times \mathcal{T}(V)$ as follows: $(E, F) \sim (E', F')$ if there exists $G \in \mathcal{T}(V)$ such that $E \oplus E' \oplus G = F \oplus F' \oplus G$. We set $K^0(V) = \mathcal{T}(V) \times \mathcal{T}(V)/\sim$. There exists a natural map $\mathcal{T}(V) \to K^0(V)$ given by $E \mapsto [(E, 0)]$, and the class $[(E, 0)]$ is called the *stable class of* E. Note that such a map is not injective.

Example 1.1.4. Let TS^2 be the (real) tangent bundle to the 2-dimensional sphere $S^2 \subset \mathbb{R}^3$. Let N_{S^2} be the normal bundle of S^2 in $\mathbb{R}^3$. Since S^2 is orientable then $N_{S^2} \simeq S^2 \times \mathbb{R}$. Then $TS^2 \oplus N_{S^2} \simeq S^2 \times \mathbb{R}^3 = (S^2 \times \mathbb{R}^2) \oplus N_{S^2}$ and thus $(TS^2, 0) \sim (S^2 \times \mathbb{R}^2, 0)$.

Note that in $K^0(V)$ the (stable class) of E has inverse given by $(0, E)$. However the stable class of a trivial bundle $V \times \mathbb{C}^r$ is not the neutral element of the group unless $r = 0$. To avoid this, one may introduce a new group as follows. Observe first that a C^∞ map $f : V \to V'$, for V' a complex manifold, induces a natural map $f^* : K^0(V') \to K^0(V)$. Let $V' = \{x\}$ for some fixed point $x \in V$. Note that a complex vector bundle over x is nothing but a complex vector space and therefore it is uniquely determined up to isomorphism by its dimension. Thus $\mathcal{T}(\{x\}) = \mathbb{N}$ and $K^0(\{x\}) = \mathbb{Z}$. The map $K^0(V) \to K^0(\{x\})$ induced by $x \to V$ associates to every element of $\mathcal{T}(V)$ its rank, and the map $K^0(\{x\}) \to K^0(V)$ induced by the constant map $V \mapsto x$ associates to any $r \in \mathbb{N}$ the complex trivial bundle of rank r over V. Therefore there is an injective map $0 \to \mathbb{Z} \to K^0(V)$ and one can define $\tilde{K}^0(V) := K^0(V)/\mathbb{Z}$. The exact sequence

$$0 \to \mathbb{Z} \to K^0(V) \to \tilde{K}^0(V) \to 0$$

splits and one can regard $\tilde{K}^0(V)$ as a subgroup of $K^0(V)$ as well. In $\tilde{K}^0(V)$ the class of a trivial bundle of any rank over V is the neutral element of the group. More details can be found, *e.g.*, in [1].

For a complex vector bundle E over V let $c(E) = 1 + c_1(E) + \ldots + c_n(E)$ be the *total Chern class of* E; the Whitney formula for the sum of vector bundles states that

$$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2).$$

Let $H_0^*(V)$ be the group (with respect to the product) of invertible elements in $H^*(V) := \oplus_i H^i(V)$ with 1 as term of degree 0. Thus $c : \mathcal{T}(V) \to H_0^*(V)$ is a semigroup morphism and naturally extends to a group morphism $K^0(V) \to H_0^*(V)$ (and to $\tilde{K}^0(V) \to H_0^*(V)$) in such a way that

$$c(E_1 - E_2) = \frac{c(E_1)}{c(E_2)}.$$

One can then also define the single Chern classes $c_j(E_1 - E_2)$ as the $2j^{th}$ degree term in the total Chern class $c(E_1 - E_2)$.

We remark that for a complex line bundle $\mathcal{L}$ there is only one *a priori* non trivial Chern class, *i.e.*, $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$. Therefore in our case

$$c(TV - \mathcal{L}) = \frac{1 + c_1(V) + \ldots + c_n(V)}{1 + c_1(\mathcal{L})}.$$

Note that $\mathcal{L}$ is trivial if and only if there exists a (global) non-zero section $v : V \to \mathcal{L}$; in such a case $h \circ v$ is a vector field on V with zeros at the singularities of $\mathcal{F}$ and $c_1(\mathcal{L}) = 0$ (in terms of K-theory, $TV - \mathcal{L} = TV$ in $K^0(V)$ and thus $c(TV - \mathcal{L}) = c(TV)$).

Theorem 1.1.5 (Baum-Bott 1970/72). *If $\mathcal{F}$ is a one-dimensional foliation with only isolated singularities $\{m_\lambda\}$ then*

$$c_I\,(TV - \mathcal{L}) \cap [V] = \sum_\lambda \begin{bmatrix} c_I\,(J)\,dz_1 \wedge \dots \wedge dz_n \\ A_1\; A_2 \dots\; A_n \end{bmatrix}_{m_\lambda} \tag{1.3}$$

Observe that one retrieves Theorem 1.1.2 for vector fields in the case $\mathcal{L}$ is trivial, since $c\,(\mathcal{L}) = 1$ and so $c\,(TV - \mathcal{L}) = c\,(TV)$.

1.2. **Generalizations.** We now give a more general point of view of the previous statements.

Definition 1.2.1. Let E be a holomorphic, rank r vector bundle over V and $v \in \Gamma(TV)$ a holomorphic vector field on V. An *action* of v on E is an operator on the C^∞-sections $\Gamma(E)$ of E into itself,

$$\theta_v : \Gamma\,(E) \to \Gamma\,(E)\,,$$

such that:

1. θ_v is $\mathbb{C}$-linear.
2. $\sigma \in \Gamma\,(E)\,,\ \sigma$ holomorphic $\Longrightarrow \theta_v\,(\sigma)$ is holomorphic.
3. $\theta_v\,(f\sigma) = f\theta_v\,(\sigma) + v\,(f)\,\sigma$ for any $f \in C^\infty\,(V)$.

Example 1.2.2. (1) Any holomorphic vector field $v \in \Gamma(TV)$ defines the *Lie derivative action* $\mathfrak{L}_v = [v, \cdot] : \Gamma\,(TV) \to \Gamma\,(TV)$ on TV.

(2) Suppose $V \subset M$ for some complex manifold M. Then there exists the exact sequence of vector bundles

$$0 \to TV \to TM|_V \xrightarrow{\pi} N_V \to 0, \tag{1.4}$$

where $N_V := TM|_V / TV$ is the *normal bundle* to V in M. Assume $v \in \Gamma(TV)$ is a holomorphic vector field which extends to a holomorphic vector field $\tilde{v}$ near V in M. Then we have the action $\theta_v : \Gamma(TM|_V) \to \Gamma(TM|_V)$ defined by $\theta_v(Y) := [\tilde{v}, \tilde{Y}]|_V$, where $\tilde{Y}$ is *any* C^∞ extension of Y in TM. One can easily show that this action is well-defined, that is it is independent of the extension $\tilde{Y}$ chosen to define it. Note however that θ_v depends on the first jet of the extension $\tilde{v}$.

(3) As in (2). Suppose furthermore that $v \in \Gamma(TV)$. Then we can define the action $\theta_v : \Gamma(N_V) \to \Gamma(N_V)$ as follows. If $\sigma \in \Gamma(N_V)$ then there exists $\tilde{Y} \in \Gamma(TM)$ so that $\pi(\tilde{Y}|_V) = \sigma$. Then $\theta_v(\sigma) := \pi([\tilde{v}, \tilde{Y}]|_V)$. Since $v \in \Gamma(TV)$ one can show that the action is well-defined once given $\tilde{v}$.

Suppose now $v \in \Gamma(TV)$ is a holomorphic vector field with isolated zeros $m_1, \dots, m_\lambda, \dots$. Suppose moreover that an action θ_v of v on a holomorphic, rank r vector bundle E is given. For each point m_λ we choose an open neighborhood $U_\lambda \subset V$ such that $E|_{U_\lambda}$ is holomorphically trivial, and we let $\sigma_1, ..., \sigma_r$ be a holomorphic frame of $|_{U_\lambda}$. Moreover we may suppose that U_λ is the domain of a holomorphic chart with coordinates $z_1, ..., z_n$ centered in m_λ. On U_λ one can write v as $v = A_i \frac{\partial}{\partial z_i}$ for some holomorphic functions A_i, such that $A_i(0) = 0$ for any i and for any $z = (z_1, ..., z_n) \neq 0$ there exists j s.t. $A_j(z) \neq 0$.

Using the local basis $\{\sigma_1, ..., \sigma_r\}$ one can locally describe the action θ_v in terms of a matrix of functions $\mathcal{M} = (\mathcal{M}^i_j)$ given by

$$\theta_v(\sigma_i) = \sum_j \mathcal{M}^j_i \sigma_j,$$

the functions $\mathcal{M}^i_j$ being holomorphic because of the axioms of action.

Theorem 1.2.3. *Suppose v is a holomorphic vector field with isolated zeros which defines an action on the holomorphic vector bundle E. With the previous notation, for any multi-index I of height n, the following formula holds:*

$$c_I(E) \cap [V] = \sum\nolimits_\lambda (-1)^{\left[\frac{n}{2}\right]} \begin{bmatrix} c_I(\mathcal{M})\, dz_1 \wedge ... \wedge dz_n \\ A_1\, A_2 \dots A_n \end{bmatrix}_{m_\lambda}. \tag{1.5}$$

In particular

Proposition 1.2.4. *If $v \in \Gamma(TV)$ is a nowhere zero holomorphic vector field defining an action on a holomorphic vector bundle E then $c_I(E) = 0$ for any $|I| = n$.*

Going back to Examples 1.2.2.(2) and (3), one has

Proposition 1.2.5. *Let $V \subset M$ and $v \in \Gamma(TV)$ a nowhere zero holomorphic vector field. If there exists a holomorphic extension $\tilde{v}$ of v to a neighborhood of V in M then $c_I(N_V) = c_I(TM|_V) = 0$ for any multi-index I of height $n =$ dim$V$.*

1.3. **General principles for residue theorems.** Before giving the actual proof of the statements written so far, we briefly digress to heuristically describe how a residue theorem is achieved.

Roughly speaking a residue theorem is a localization of a certain characteristic class near some sets outside which a vanishing theorem holds. More precisely:

I. A characteristic class is usually the obstruction to the existence of a certain intrinsic object θ on V. Thus one has some class $\phi(\theta)$ in some cohomology group of V in such a way that $\phi(\theta)$ vanishes if θ exists. A *vanishing theorem* is thus any statement saying that *if a certain intrinsic object θ exists then some classes $\phi(\theta)$* vanish.

II. It may happen that θ exists outside some closed set $\Sigma \subset V$. Thus $\phi(\theta|_{V-\Sigma}) = 0$, and—assuming a natural functoriality—we have $H^*(V) \ni \phi(\theta) \mapsto \phi(\theta|_{V-\Sigma}) = 0 \in H^*(V - \Sigma)$. Therefore from the piece of long exact sequence

$$H^*(V, V - \Sigma) \longrightarrow H^*(V) \longrightarrow H^*(V - \Sigma),$$

it follows that there exists a naturally defined class $\eta(\theta) \in H^*(V, V - \Sigma)$ such that $\eta(\theta) \mapsto \phi(\theta)$.

If we now suppose Σ to be a compact set admitting a regular neighborhood (for instance if Σ is a subvariety of V) then the Alexander duality gives $H^*(V, V-\Sigma) \approx H_{2n-*}(\Sigma)$. Thus $H^*(V, V-\Sigma) \ni \eta(\theta) \mapsto \mathrm{Res}(\phi(\theta), \theta, \Sigma) \in H_{2n-*}(\Sigma)$ and $\mathrm{Res}(\phi(\theta), \theta, \Sigma)$ is called the *residue* of $\phi(\theta)$ at Σ with respect to θ. Now if V is compact the Poincaré duality gives an isomorphism $H^*(V) \approx H_{2n-*}(V)$ sending $\phi(\theta)$ to $P(\phi(\theta))$. Moreover, if $i : \Sigma \hookrightarrow V$ is the embedding, the following diagram is commutative:

$$\begin{array}{ccc} H^*(V, V-\Sigma) & \longrightarrow & H^*(V) \\ \| & & \| \\ H_{2n-*}(\Sigma) & \xrightarrow{i_*} & H_{2n-*}(V) \end{array}$$

If $\Sigma = \sqcup \Sigma_\lambda$, with the Σ_λ's being connected, then $H_{2n-*}(\Sigma) = \oplus_\lambda H_{2n-*}(\Sigma_\lambda)$. Therefore $P(\phi(\theta)) = \sum_\lambda i_* \mathrm{Res}(\phi(\theta), \theta, \Sigma_\lambda)$, which is the *residue theorem*.

III. Finally, one might find an easy expression for $i_*\mathrm{Res}(\phi(\theta), \theta, \Sigma_\lambda)$, which would make the residue theorem really useful. In the previous sections we saw that for the Baum-Bott-like theorems the residues are expressed in terms of Grothendieck residues.

2. Vanishing Theorems

The aim of this section is to present several vanishing theorems, some of them will be used later to prove the residue formulas stated in the previous section.

2.1. **Holomorphic actions and special connections.**

Let V be a n-dimensional complex manifold and E a holomorphic vector bundle on V. Let $v \in \Gamma(TV)$ be a nowhere zero holomorphic vector field acting on E as $\theta_v : \Gamma(E) \to \Gamma(E)$.

Definition 2.1.1. We say that a connection ∇ for E is a *special connection* with respect to θ_v if

(1) ∇ is of type $(1,0)$, *i.e.*, $\nabla_Z \sigma = 0$ for any $Z \in \Gamma(T^{0,1}V) = \Gamma(\overline{TV})$ and σ holomorphic section of E.
(2) $\nabla_v = \theta_v$.

Note that given an action θ_v on E it is always possible to define a special connection with respect to θ_v (and this is actually the point where one needs v to be non-zero). Indeed one has the natural (partial) connection $\overline{\partial}$ for E on $T^{0,1}V$, the (partial) connection θ_v for E on the subbundle $< v >$ of TV generated by v, and taking any (partial) connection ∇^0 for E on a C^∞ complement $T'V$ of $< v >$ in TV one has the special connection $\nabla := \overline{\partial} \oplus \theta_v \oplus \nabla^0$ for E on $T^{\mathbb{R}}V \otimes \mathbb{C} = \overline{TV} \oplus TV = \overline{TV} \oplus (< v > \oplus T'V)$ (see [3]).

Theorem 2.1.2. *If I is a multi-index of height n then $c_I(\nabla) = 0$ for any special connection ∇ with respect to θ_v.*

Before proving this theorem we give a general result for the Bott operator.

2.1.1. *The Bott operator in the Chern-Weyl theory.* Let E be a rank r complex vector bundle on V and let $\nabla_0, \dots, \nabla_s$ be $s+1$ connections for E. Let $\Delta_s := \{(t_0, \dots, t_s) \in \mathbb{R}^{s+1} : t_j \geq 0, \sum_{j=1}^{s} t_j = 1\}$ the standard s-simplex. Let $p_1 : V \times \Delta_s \to V$ the projection on the first factor and let $\tilde{E} := \pi_1^*(E)$ be the pull-back bundle. Note that by definition

$$\tilde{E} := \{(\sigma, (x,t)) \in E \times (V \times \Delta_s) : \sigma \in E_x\},$$

and thus one can identify $\tilde{E} = E \times \Delta_s$. Then $\tilde{E}$ is a vector bundle over $V \times \Delta_s$ whose fiber at a point (m,t) is $E_m \times \{t\} = E_m$. Thus the sections $\tilde{\sigma} \in \Gamma(\tilde{E}) = \Gamma(E \times \Delta_s)$ which are "constant along the fibers Δ_s", *i.e.*, such that $\tilde{\sigma}(x,t) = \sigma(x)$ for some section σ of E, generate $\Gamma(\tilde{E})$ as a C^∞-module:

$$\begin{array}{ccc} p_1^*(E) = E \times \Delta_s & \longrightarrow & E \\ \tilde{\pi} \downarrow \quad \uparrow \tilde{\sigma} & & \pi \downarrow \quad \uparrow \sigma \\ V \times \Delta_s & \xrightarrow{p_1} & V \end{array}$$

This means that in order to define a connection $\tilde{\nabla}$ on $\tilde{E}$ is enough to define it on sections which are constant along Δ_s. We let

$$(\tilde{\nabla}_X \tilde{\sigma})_{(m,(t_0,\dots,t_s))} = \sum_{i=0}^{s} t_i((\nabla_i)_X \sigma)_m \text{ for } X \in \Gamma(TV), \tag{2.1}$$
$$\tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\sigma} = 0.$$

We can now define the *Bott form* $c_I(\nabla_0, \dots, \nabla_s)$ integrating along the fibers Δ_s as

$$c_I(\nabla_0, \dots, \nabla_s) := (-1)^{\left[\frac{r}{2}\right]} \int_{\Delta_s} c_I(\tilde{\nabla}). \tag{2.2}$$

From the Stokes Theorem one obtains:

Theorem 2.1.3 (Bott Formula)**.**

$$dc_I(\nabla_0, \dots, \nabla_s) = \sum_{i=o}^{s} (-1)^i c_I(\nabla_0, \dots, \widehat{\nabla_i}, \dots, \nabla_s). \tag{2.3}$$

In particular we have two corollaries:

Corollary 2.1.4. *The de Rham class $[c_I(\nabla_0)] \in H^{2|I|}(V)$ is independent of the chosen connection.*

Proof. It follows from (2.3). □

and

Corollary 2.1.5. *If the connections $\nabla_0, \dots, \nabla_s$ are special connections for E with respect to θ_v then $c_I(\nabla_0, \dots, \nabla_s) = 0$ for any multi-index I of height n.*

Proof. It is easy to see that $\tilde{\nabla}$ if special as well. Then the result follows from Theorem 2.1.2 and (2.2). □

Instead of proving directly Theorem 2.1.2 we describe a slightly different situation and then retrieve the vanishing theorem (see Remark 2.2.4).

2.2. **Complex quasi free actions.** Let V be a complex n-dimensional manifold and E a holomorphic vector bundle on V. Let G be a complex Lie group with $\dim_{\mathbb{C}} G = r$. Let E be a holomorphic vector bundle over V and assume G acts holomorphically on E through bundle morphisms, *i.e.*, preserving the vector bundle structure $E \to V$. Therefore G naturally acts on V as well. Let $\mathfrak{g}$ be the Lie algebra of G. We recall that to any $x \in \mathfrak{g}$ is associated a vector field $\bar{x}$ over V. The vector field $\bar{x}$ is defined as the infinitesimal generator of the flow over V defined for small $t \in \mathbb{C}$ by $\phi_t : V \to V$, $\phi_t(p) = (\exp tx)p$ (where $\exp$ is the exponential map of G). Vector fields like $\bar{x}$ are called *fundamentals* and form a subalgebra $\bar{\mathfrak{g}}$ of the Lie algebra $\mathfrak{X}(V)$ of holomorphic vector fields on V. Let's now consider the diagram

$$\begin{array}{ccccccc} & & & & \bar{\mathfrak{g}} \subset \mathfrak{X}(V) & & \\ & & & \nearrow & & \overset{e_m}{\searrow} & \\ 0 & \longrightarrow & \mathfrak{h}_m & \overset{i}{\hookrightarrow} & \mathfrak{g} & \xrightarrow{\overline{e_m}} & T_m V \end{array}$$

where e_m is the map "evaluation at m", $\overline{e_m}$ is the composition with the map sending $x \in \mathfrak{g}$ to $\bar{x} \in \bar{\mathfrak{g}}$, and $\mathfrak{h}_m$ is its kernel. It is not hard to show that $\mathfrak{h}_m$ is the Lie algebra of the isotropy group H_m of G at m.

Definition 2.2.1. The action of G is called *quasi-free* if and only if $\mathfrak{h}_m = 0$ for any m or, equivalently, if H_m is discrete for each m.

Any $x \in \bar{\mathfrak{g}}$ defines an action in the sense of definition 1.2.1. Indeed, fix $x \in \mathfrak{g}$; the vector field $\bar{x} \in \bar{\mathfrak{g}} \subset \mathfrak{X}(V)$ defines a flow $(\phi_t)_{t \in \mathbb{C}}$ over V. Similarly x defines a holomorphic vector field and a flow $(\Phi_t)_{t \in \mathbb{C}}$ over E. Therefore any $x \in \mathfrak{g}$ gives rise to the following commutative diagram of holomorphic maps:

$$\begin{array}{ccc} E & \xrightarrow{\Phi_t} & E \\ \downarrow & & \downarrow \\ V & \xrightarrow{\phi_t} & V \end{array}$$

We define $\Theta_x : \Gamma(E) \longrightarrow \Gamma(E)$ by

$$\Theta_x(\sigma)(m) = \frac{d}{dt}|_{t=0} \left[\Phi_{-t} \left(\sigma \left[\phi_t(m) \right] \right) \right] = \lim_{t \to 0} \frac{\Phi_{-t} \left(\sigma \left[\phi_t(m) \right] - \Phi_t \left[\sigma(m) \right] \right)}{t} \in E_m,$$

for $\sigma \in \Gamma(E)$ and $m \in V$. One can check that Θ_x satisfies the axioms of action as in definition 1.2.1.

Moreover $\bar{x} \in \bar{\mathfrak{g}} \subset \mathfrak{X}(V)$ is a holomorphic vector field and therefore its Lie derivative $\mathcal{L}_{\bar{x}} : \Gamma(TV) \to \Gamma(TV)$ defines an action of $\bar{x}$ over TV.

From the relation

$$[\bar{x}, \bar{y}] = \overline{[x, y]} \in \mathfrak{X}(V) \text{ for any } x, y \in \mathfrak{g},$$

one gets

$$[\Theta_x, \Theta_y] = \Theta_{[x,y]} \text{ for any } x, y \in \mathfrak{g}. \tag{2.4}$$

Theorem 2.2.2. *Suppose G acts on E and the action is quasi free. Then for any multi-index I of height $|I| > n - r$, $c_I(E) = 0$.*

Remark 2.2.3. The estimate does not involve the rank of E. Moreover it is relative to Chern classes with coefficients in $\mathbb{C}$, since we use the Chern-Weyl theory in the proof.

Proof of Theorem 2.2.2. Recall the decomposition

$$T^{\mathbb{R}}V \otimes \mathbb{C} = TV \oplus \overline{(TV)} = T^{1,0}V \oplus T^{0,1}V.$$

With a slight abuse of notation we denote by the same letter $\bar{\mathfrak{g}}$ the subbundle of $T^{1,0}V$ generated by the fundamental vector fields. Note that $\bar{\mathfrak{g}}$ is a (trivial) involutive subbundle of $T^{1,0}V$. By means of an hermitian metric, we can then find a C^∞ complement F decomposing the holomorphic tangent bundle as $T^{1,0}V = \bar{\mathfrak{g}} \oplus F$ and thus

$$T^{\mathbb{R}}V \otimes \mathbb{C} = \bar{\mathfrak{g}} \oplus F \oplus T^{0,1}V. \tag{2.5}$$

Let ∇ be a connection for E of type $(1,0)$. This means that $\nabla_Z\sigma$ vanishes whenever $Z \in \Gamma(T^{0,1})$ and $\sigma \in \Gamma(E)$ is holomorphic. Note that such a connection always exists for E is a holomorphic vector bundle (see, *e.g.*, [9]).

We may also assume that $\nabla_{\bar{x}} = \Theta_x$ for any $x \in \mathfrak{g}$. This can be done starting from the decomposition (2.5) similarly to what we did for the existence of special connection for an action, see the paragraph after Definition 2.1.1.

Let now k be the curvature form of ∇. Let $\bar{x}, \bar{y} \in \bar{\mathfrak{g}}$ and $z, w \in \Gamma(T^{0,1}V)$. We claim that

$$k(\bar{x}, \bar{y}) = 0, \quad k(\bar{x}, z) = 0, \quad k(z, w) = 0. \tag{2.6}$$

Recall that for $X, Y \in \Gamma(T^{\mathbb{R}}V \otimes \mathbb{C})$,

$$k(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} : \Gamma(E) \longrightarrow \Gamma(E).$$

The first identity of (2.6) is in fact just a transcription of (2.4). For the other two identities we first observe that it is enough to prove them on holomorphic sections of E, since these generate $\Gamma(E)$ as a C^∞-module, and k is a tensor. Let therefore σ be a holomorphic section of E. Then

$$k(\bar{x}, z)(\sigma) = \nabla_{\bar{x}}(\nabla_z\sigma) - \nabla_z(\nabla_{\bar{x}}\sigma) - \nabla_{[\bar{x},z]}\sigma = 0 + 0 + 0,$$

for ∇ is of type $(1,0)$, $\nabla_{\bar{x}}\sigma = \Theta_x(\sigma)$ is holomorphic and $[\bar{x}, z] = 0$. For the same reason the third identity in (2.6) holds, just observing that $[z, w] \in \Gamma(T^{0,1}V)$.

Let $\{\xi_1, ...\xi_r, \eta_1, ..., \eta_{n-r}, d\bar{z}_1, ...d\bar{z}_n\}$ be a local basis of the dual of $T^{\mathbb{R}}V \otimes \mathbb{C}$ which respects the following decomposition:

$$T^{\mathbb{R}}V \otimes \mathbb{C} = \bar{\mathfrak{g}} \oplus F \oplus_{\mathbb{R}} T^{0,1}V.$$

By (2.6), in this basis the curvature matrix $K = (K_\alpha^\beta)$ is made up of forms belonging to the ideal generated by $\{\eta_1, ..., \eta_{n-r}\}$. Since $c_I(E) = [c_I(K)]$, the theorem follows because any product of more that $n - r$ forms $\eta_\cdot$'s vanishes. □

Remark 2.2.4. Suppose a nowhere zero holomorphic vector field $v \in \Gamma(TV)$ together an action θ_v on E are given. Then arguing as in the proof of Theorem 2.2.2 substituting $\overline{\mathfrak{g}}$ with the one-dimensional vector bundle generated by v in $T^{1,0}V$ and Θ_v with the given action θ_v, one gets Theorem 2.1.2.

As a corollary we also have:

Corollary 2.2.5. *If $v \in \Gamma(TV)$ is a nowhere zero holomorphic vector field, then $c_I(V) = 0$ for $|I| = n$.*

Proof. Let $E = TV$ and θ_v be the action on TV given by the Lie derivative action as in Example 1.2.2.(1). Then the result follows from Theorem 2.1.2. □

There are two more types of vanishing theorems which we want to discuss in here. The first is a real counterpart of Theorem 2.2.2 and the second one is a vanishing theorem for the case of one-dimensional foliations.

2.3. **Real quasi free actions.** Let V be a *real* n-dimensional manifold, G a real Lie group of dimension r and $E \to V$ a complex vector bundle. As before, we suppose that G acts through bundle morphism over E and such an action is quasi free. This simply means that for any point $m \in V$, the composite map $\mathfrak{g} \longrightarrow \bar{\mathfrak{g}} \subset \mathfrak{X}(V) \xrightarrow{e_m} T_m V$ is injective. Again with a slight abuse of notation we denote by the same letter $\bar{\mathfrak{g}}$ the subbundle of the tangent bundle TV generated by the fundamental vector fields. As before one can define Θ_x for any $x \in \bar{\mathfrak{g}}$. The operator Θ_x is a derivation of $\Gamma(E)$ satisfying:

1. Θ_x is $\mathbb{C}$ -linear,
2. $\Theta_x(f\sigma) = f\Theta_x(\sigma) + v(f)\sigma$ for any $f \in C^\infty(V)$,
3. $[\Theta_x, \Theta_y] = \Theta_{[x,y]}$.

Theorem 2.3.1. *In the previous hypothesis, the following hold:*
a) $c_I(E) = 0$ for any multi-index I such that $|I| > n - r$.
b) If G is compact, then $c_I(E) = 0$ for any multi-index I such that $|I| > \left[\frac{n-r}{2}\right]$.

As before, we remark that the inequalities do not depend on the rank of E, and we consider Chern classes with real coefficients. In the noncompact case, moreover, the formula is nontrivial for $n \leq 2r - 2$, while in the compact case one has $2[(n-r)/2] \leq n - 2$ for avoiding triviality.

Proof of Theorem 2.3.1. We follow the same path of the proof in the complex case. We observe that $\bar{\mathfrak{g}}$ is a trivial integrable subbundle of TV. We consider a complement F of $\bar{\mathfrak{g}}$ given by some Riemannian metric on V and thus we can construct a connection ∇ such that

$$\nabla_{\bar{x}} = \Theta_x \text{ for any } x \in \mathfrak{g}.$$

The formula $[\Theta_x, \Theta_y] = \Theta_{[x,y]}$ assures that $k(\bar{x}, \bar{y}) = 0$ for any $x, y \in \mathfrak{g}$, where k is the curvature of ∇. If now $\{\xi_1, ...\xi_r, \eta_1, ..., \eta_{n-r}\} \in \Gamma(TV)$ is a local set of generators, with $\xi_i \in \bar{\mathfrak{g}}$, and $\eta_i \in \Gamma(F)$, and $\left\{\xi'_1, ...\xi'_r, \eta'_1, ..., \eta'_{n-r}\right\}$ is the dual basis, then the curvature form of ∇ belongs to the ideal generated by $\{\eta'_i\}$, proving the estimate.

In the case b), compactness of G allows one to pick up the Riemannian metric h defining F to be G-invariant. Also we can choose ∇ in such a way that it is a *G-invariant metric connection* for E, *i.e.* such that $\nabla h = 0$ and $g(\nabla_X \sigma) = \nabla_{gX}(g\sigma)$ for any $X \in \Gamma(TV)$, $\sigma \in \Gamma(E)$ and $g \in G$. In other words

$$Xh(Y,Z) = h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.7}$$

and

$$\Theta_x\left(\nabla_y \sigma\right)=\nabla_{[\bar{x}, y]} \sigma+\nabla_y\left(\Theta_x \sigma\right) \text{ for any } x \in \mathfrak{g}\text{ , } y \in \Gamma(TV) \text{ and } \sigma \in \Gamma\left(E\right),$$

or, which is the same,

$$\nabla_x\left(\nabla_y \sigma\right)=\nabla_{[\bar{x}, y]} \sigma+\nabla_y\left(\nabla_x \sigma\right). \tag{2.8}$$

From these one has $k(\overline{x},y)=0$ for $x\in\mathfrak{g}$ and $y\in\Gamma(F)$ as well, and the statement follows. □

2.4. **Holomorphic non-singular foliations.** Let V be a complex manifold of dimension n. Let $\mathcal{L}$ be a holomorphic line bundle over V generating a foliation $\mathcal{F}$ by means of a holomorphic vector bundle morphism $h:\mathcal{L}\to TV$.

Theorem 2.4.1. *If $\mathcal{F}$ has no singularities on V then $c_I(TM-\mathcal{L})=0$ for any multi-index I of height n.*

Note that $\mathcal{F}$ has no singularities on V if and only if h is injective, *i.e.*, $h(\mathcal{L})$ is a one dimensional subbundle of TV and the virtual bundle $TV-\mathcal{L}$ coincides with the quotient bundle $\mathcal{V}_{\mathcal{F}}:=TV/h(\mathcal{L})$, the *normal bundle* to $\mathcal{F}$.

More generally a *non-singular foliation* $\mathcal{F}$ of dimension p on V is given by a rank p holomorphic involutive vector subbundle $T\mathcal{F}\subset TV$. The bundle $T\mathcal{F}$ is called the *tangent bundle of* $\mathcal{F}$. There exists the exact sequence of (holomorphic) vector bundles:

$$0\to T\mathcal{F}\to TV\to\mathcal{V}_{\mathcal{F}}\xrightarrow{\pi}0,$$

where $\mathcal{V}_{\mathcal{F}}$ is a rank $n-p$ vector bundle on V which is called the *normal bundle* to $\mathcal{F}$.

Theorem 2.4.2 (Bott). *If $\mathcal{F}$ is a non-singular foliation of dimension p on V then $c_I(\mathcal{V}_{\mathcal{F}})=0$ for any multi-index I of height $|I|>(n-p)$.*

Proof. The technique being the same as in the previous vanishing theorems we just sketch the proof in here. First we write a C^∞ decomposition of $T^{1,0}V=T\mathcal{F}\oplus H$. Then we choose a connection ∇ on $\mathcal{V}_{\mathcal{F}}$ which is of type $(1,0)$, *i.e.*, $\nabla_X(\pi Y)=0$ for $X\in\Gamma(T^{0,1}V)$ and πY holomorphic with $Y\in\Gamma(TV)$. Moreover we require that for any $X\in\Gamma(T\mathcal{F})$ and any $Y\in\Gamma(TV)$

$$\nabla_X\left(\pi Y\right)=\pi\left(\left[X,Y\right]\right). \tag{2.9}$$

Note that if $\pi Y_1=\pi Y_2$ then $Y_1-Y_2\in\Gamma(T\mathcal{F})$ and for any $X\in\Gamma(T\mathcal{F})$ it follows $[X,Y_1-Y_2]\in\Gamma(T\mathcal{F})$, too (for $T\mathcal{F}$ is involutive). Thus $\pi([X,Y_1])=\pi([X,Y_2])$ and ∇ is well defined.

In the splitting $T^{\mathbb{R}}V\otimes\mathbb{C}=T\mathcal{F}\oplus H\oplus T^{0,1}$ the curvature k of ∇ satisfies:

$$k\left(x_1,x_2\right)=0 \text{ for any } x_1,x_2\in\Gamma\left(T\mathcal{F}\right), \tag{2.10}$$

$$k\left(x,z\right)=0 \text{ for any } x\in\Gamma\left(T\mathcal{F}\right) \text{ and } z\in\Gamma\left(T^{0,1}V\right), \tag{2.11}$$

$$k\left(z_1,z_2\right)=0 \text{ for any } z_1,z_2\in\Gamma\left(T^{0,1}V\right). \tag{2.12}$$

The first comes from (2.9) and the Jacobi identity. As for the second, one can show it holds for holomorphic sections of $\mathcal{V}_{\mathcal{F}}$ using (2.9), the type-$(1,0)$ property of ∇, and the fact that $[x,z]$

vanishes for $x \in \Gamma(T^{1,0}V)$ and $z \in \Gamma(T^{0,1}V)$. The third comes from the "type $(1,0)$"-property of ∇, using holomorphic sections of $\mathcal{V}_{\mathcal{F}}$.

Let $\left\{x'_1, ..., x'_p, y'_1, ...y'_{n-p}, z'_1, ..., z'_n\right\}$ be a basis of $(T^{\mathbb{R}}V \otimes \mathbb{C})^*$ which respects the decomposition $T^{\mathbb{R}}V \otimes \mathbb{C} = T\mathcal{F} \oplus H \oplus T^{0,1}$. By (2.10), in such a basis the matrix of k is made up of forms which belong to the ideal generated by $\{y'_i\}$. Thus the theorem follows. $\square$

3. Existence of Residues

In this section we are going to use the previous vanishing theorems in order to localize characteristic classes.

Let V be a n-dimensional complex manifold, let E be a rank r holomorphic vector bundle on V and $v \in \Gamma(TV)$ a holomorphic vector field with isolated zeros $\Sigma = \{m_1, ..., m_\lambda, ...\}$. Suppose v acts on $E|_{V-\Sigma}$ in the sense of definition 1.2.1 as

$$\theta_v : \Gamma\left(E|_{V-\Sigma}\right) \longrightarrow \Gamma\left(E|_{V-\Sigma}\right).$$

3.1. **The Mayer-Vietoris complex.**

Let $U_0 := V - \Sigma$ and let $U_1 \subset V$ be an open neighborhood of Σ. Denote by $\mathcal{U} := \{U_0, U_1\}$. Finally let $U_{01} := U_0 \cap U_1$.

Consider the Mayer-Vietoris complex $MV^*(\mathcal{U})$ (a two-open sets Čech-de Rham complex in the terminology of [23]). Indicating by $\Omega^*_{DR}(U)$ the vector space of (complex) differential forms of degree $*$ defined on the open set $U \subseteq V$ recall that the complex $MV^*(\mathcal{U})$ is defined as

$$\begin{aligned} MV^*(\mathcal{U}) &:= \Omega^*_{DR}(U_0) \oplus \Omega^*_{DR}(U_1) \oplus \Omega^{*-1}_{DR}(U_{01}), \\ D(\alpha_0, \alpha_1, \alpha_{01}) &:= (d\alpha_0, d\alpha_1, -d\alpha_{01} + \alpha_1 - \alpha_0). \end{aligned}$$

The natural map $i : \Omega^*_{DR}(V) \to MV^*(\mathcal{U})$ given by $\alpha \mapsto (\alpha|_{U_0}, \alpha|_{U_1}, 0)$ is such that $i \circ d = D \circ i$ and induces an isomorphism in cohomology (which is also an isomorphism at the level of algebras).

We denote by $MV^*(V, V - \Sigma)$ the sub-complex of $MV^*(\mathcal{U})$ given by elements of the form $(0, \alpha_1, \alpha_{01})$, called the *relative Čech-de Rham complex*.

The advantage of using the relative Mayer-Vietoris complex is that the morphism $MV^*(\mathcal{U}) \to \Omega^*_{DR}(V - \Sigma)$ is surjective with kernel given exactly by $MV^*(V, V - \Sigma)$, whereas the map $\Omega^*_{DR}(V) \to \Omega^*_{DR}(V - \Sigma)$ given by the restriction is not surjective.

Thus one may represent $c_I(E)$ as an element in $MV^*(\mathcal{U})$. Indeed if ∇_0 is a connection for E on U_0 and ∇_1 is a connection for E on U_1 then $(c_I(\nabla_0), c_I(\nabla_1), c_I(\nabla_0, \nabla_1)) \in MV^*(\mathcal{U})$ and $D(c_I(\nabla_0), c_I(\nabla_1), c_I(\nabla_0, \nabla_1)) = 0$ by the very definition of D and the Bott operator $c_I(\nabla_0, \nabla_1)$. On the other hand one can prove that if ∇ is a connection for E then the cocycle $(c_I(\nabla|_{U_0}), c_I(\nabla|_{U_1}), 0)$ belongs to the same cohomology class of $(c_I(\nabla_0), c_I(\nabla_1), c_I(\nabla_0, \nabla_1))$ and thus this last represents $c_I(E)$ under the isomorphism i.

Now assume $\Sigma = \coprod_\lambda \Sigma_\lambda$, where Σ_λ's are the connected component of Σ. In our case Σ_λ is just a point, but the following reasoning holds for more general sets. For any λ let $U_\lambda \subset V$ be an open set such that $U_\lambda \cap \Sigma = \Sigma_\lambda$. Thus the relative Mayer-Vietoris complex

$$MV^*(V, V - \Sigma) = \bigoplus_\lambda (\Omega^*_{DR}(U_\lambda) \oplus \Omega^{*-1}_{DR}(U_\lambda - \Sigma_\lambda)).$$

Let ∇ be a connection for $E|_{U_0}$ on U_0 which is a special connection with respect to θ_v (see Definition 2.1.1). By Theorem 2.1.2 it follows that $c_I(\nabla) = 0$ in $H^{2n}(U_0, \mathbb{C})$ for any multi-index I of height n, and more generally, by Corollary 2.1.5, $c_I(\nabla^0, ..., \nabla^s) = 0$ if all the ∇^i's are special with respect to θ_v and $|I| = n$. For any λ let ∇^λ be a connection for $E|_{U_\lambda}$.

Thus, for $|I| = n$, the cocycle $\oplus_\lambda(c_I(\nabla^\lambda), c_I(\nabla, \nabla^\lambda)) \in MV^*(V, V - \Sigma)$ and it is D-closed.

Proposition 3.1.1. *The cohomology class of* $\oplus_\lambda \left(c_I\left(\nabla^\lambda\right), c_I\left(\nabla, \nabla^\lambda\right)\right)$ *in* $H^{2n}(MV^*(V, V - \Sigma))$ *does not depend on the choice of the connections* ∇^λ *nor on the connection* ∇ *provide* ∇ *is special with respect to* $\theta_v|_{U_1}$*).*

Proof. Let ∇' be a connection for E on U_1 special with respect to θ_v and let ∇'^λ be connections for E on each U_λ. Thus

$$
\begin{aligned}
&\left(c_I\left(\nabla^\lambda\right), c_I\left(\nabla, \nabla^\lambda\right)\right) - \left(c_I\left(\nabla'^\lambda\right), c_I\left(\nabla', \nabla'^\lambda\right)\right)\\
&\qquad = \left(c_I\left(\nabla^\lambda\right) - c_I\left(\nabla'^\lambda\right), c_I\left(\nabla, \nabla^\lambda\right) - c_I\left(\nabla', \nabla'^\lambda\right)\right)\\
&\qquad \overset{\text{(Bott formula)}}{=} (dc_I(\nabla'^\lambda, \nabla^\lambda), [c_I(\nabla'^\lambda, \nabla^\lambda) + c_I\left(\nabla, \nabla'^\lambda\right) - c_I\left(\nabla, \nabla^\lambda\right)\\
&\qquad\qquad - dc_I\left(\nabla, \nabla'^\lambda, \nabla^\lambda\right)] + c_I\left(\nabla, \nabla^\lambda\right) - c_I\left(\nabla', \nabla'^\lambda\right))\\
&= (dc_I(\nabla'^\lambda, \nabla^\lambda), -dc_I\left(\nabla, \nabla'^\lambda, \nabla^\lambda\right) + c_I(\nabla'^\lambda, \nabla^\lambda) + \left[c_I\left(\nabla, \nabla'^\lambda\right) - c_I\left(\nabla', \nabla'^\lambda\right)\right])\\
&\quad = (dc_I(\nabla'^\lambda, \nabla^\lambda), -dc_I\left(\nabla, \nabla'^\lambda, \nabla^\lambda\right) + c_I(\nabla'^\lambda, \nabla^\lambda) + d\left(c_I\left(\nabla', \nabla, \nabla'^\lambda\right)\right)\\
&\qquad\qquad = D\left(c_I(\nabla'^\lambda, \nabla^\lambda), c_I\left(\nabla, \nabla'^\lambda, \nabla^\lambda\right) - \left(c_I\left(\nabla', \nabla, \nabla'^\lambda\right)\right)\right)
\end{aligned}
$$

where the we used $c_I\left(\nabla', \nabla\right) = 0$ since both ∇ and ∇' are special. $\square$

For any λ let $\mathcal{T}_\lambda \subset U_\lambda$ be a $2n$-dimensional real smooth manifold with smooth boundary $\partial\mathcal{T}_\lambda$ such that $\Sigma_\lambda \subset \mathcal{T}_\lambda$.

Proposition 3.1.2. *The following expression, called residue,*

$$
Res_\lambda\left(\theta_v, E, c_I\right) = \int_{\mathcal{T}_\lambda} c_I\left(\nabla^\lambda\right) - \int_{\partial\mathcal{T}_\lambda} c_I\left(\nabla, \nabla^\lambda\right)
$$

is well defined (i.e. does not depend on $\mathcal{T}_\lambda$*) for any* $|I| = n$*.*

Proof. Let $\mathcal{T}'_\lambda$ be a $2n$-dimensional real smooth manifold with smooth boundary $\partial\mathcal{T}'_\lambda$ such that $\Sigma_\lambda \subset \mathcal{T}'_\lambda \subset U_\lambda$. It is always possible to choose $\mathcal{T}''_\lambda$ with the same properties but such that it contains both $\mathcal{T}_\lambda$ and $\mathcal{T}'_\lambda$ in its interior. It is enough to show that the value for $\mathcal{T}_\lambda$ coincides with that for $\mathcal{T}''_\lambda$. Indeed:

$$\int_{\mathcal{T}''_\lambda} c_I\left(\nabla^\lambda\right) - \int_{\partial\mathcal{T}''_\lambda} c_I\left(\nabla,\nabla^\lambda\right) - \int_{\mathcal{T}_\lambda} c_I\left(\nabla^\lambda\right) - \int_{\partial\mathcal{T}_\lambda} c_I\left(\nabla,\nabla^\lambda\right)$$
$$= \int_{\mathcal{T}''_\lambda-\mathcal{T}_\lambda} c_I\left(\nabla^\lambda\right) - \int_{\partial\mathcal{T}''_\lambda-\partial\mathcal{T}_\lambda} c_I\left(\nabla,\nabla^\lambda\right)$$
$$\overset{\text{(Stokes)}}{=} \int_{\mathcal{T}''_\lambda-\mathcal{T}_\lambda} \left[c_I\left(\nabla^\lambda\right) - dc_I\left(\nabla,\nabla^\lambda\right)\right] = 0,$$

where the last equality follows from the Bott formula and the fact that $c_I(\nabla) = 0$ since ∇ is special on $\mathcal{T}_\lambda - \mathcal{T}''_\lambda$. □

Now suppose $\Sigma = \{m_0\}$. Let $U \subset V$ be an open set containing m_0 such that there exist local holomorphic coordinates $\{z_1,\ldots,z_n\}$ for $V\cap U$ and $E|_U$ is holomorphically trivial by means of r holomorphic sections $\sigma_1,\ldots,\sigma_r$. In this setting let $\theta_v\sigma_\alpha = \sum_\beta \mathcal{M}_\alpha^\beta\sigma_\beta$ for some matrix $\mathcal{M} = \mathcal{M}_\alpha^\beta$ of holomorphic functions and $v|_U = \sum_i A_i\frac{\partial}{\partial z_i}$.

Theorem 3.1.3. *For any multi-index I such that $|I| = n$,*

$$Res\left(\theta_v,E,c_I\right) = \begin{bmatrix} c_I\left(\mathcal{M}\right)dz_1\wedge\ldots\wedge dz_n \\ A_1\ A_2\ \ldots\ldots\ldots\ldots A_n\end{bmatrix}_{m_0}.$$

The previous theorem expresses the residue in term of a Grothendick residue. In particular, by choosing ∇^0 a trivial connection for $E|_U$ then $c_I\left(\nabla^0\right) = 0$ and therefore, being $c_I\left(\nabla,\nabla^0\right) = -c_I\left(\nabla^0,\nabla\right)$, for any ∇ special connection, one gets:

$$Res\left(\theta_v,E,c_I\right) = \int_{\partial\mathcal{T}} c_I\left(\nabla,\nabla^0\right) = \begin{bmatrix} c_I\left(\mathcal{M}\right)dz_1\wedge\ldots\wedge dz_n \\ A_1\ A_2\ \ldots\ldots\ldots\ldots A_n\end{bmatrix}_{m_0},$$

where $\mathcal{T}$ is a $2n$-dimensional real manifold with smooth boundary such that $m_0 \in \mathcal{T} \subset U$.

3.2. **The Proof of Theorem 3.1.3.** . Let $U_i := \{m \in U | A_i(m) \neq 0\}$ for $i = 1,\ldots,n$. Let $\mathcal{U} := \{U_1,\ldots,U_n\}$.

3.2.1. *The Čech-de Rham complex.* To the covering $\mathcal{U}$ we associate the k*-nerve*

$$N_k(\mathcal{U}) := \{(j_0,\ldots,j_k) : j_0 < j_1 < \ldots < j_k, j_i \in \{1,\ldots,n\}, \bigcap_{i=0}^{k} U_{j_i} \neq \emptyset\}.$$

Thus $J \in N_k(\mathcal{U})$ means that $J = (j_0,\ldots,j_k)$ and $U_J := \cap U_{j_i} \neq \emptyset$. In particular $N_k(\mathcal{U}) = \emptyset$ for $k > n-1$. The *Čech-de Rham complex* $CDR^*(\mathcal{U})$ is the set formed by elements $\alpha := (\alpha_J)_{J\in N_k(\mathcal{U})}$ where $k = 0,\ldots,n-1$ and $\alpha_J \in \Omega_{DR}^{*-k}(U_J)$ for $J \in N_k(\mathcal{U})$ (here we set $\alpha_J = 0$ if $* - k < 0$).

Example 3.2.1. 1. If $n = 2$ then $\mathcal{U} = \{U_1, U_2\}$, $N_0(\mathcal{U}) = \{(1), (2)\}$, $N_1(\mathcal{U}) = \{((12)\}$ and $\alpha \in CDR^*(\mathcal{U})$ implies that $\alpha = (\alpha_1, \alpha_2, \alpha_{12})$ where $\alpha_1 \in \Omega^*_{DR}(U_1)$, $\alpha_2 \in \Omega^*_{DR}(U_2)$ and $\alpha_{12} \in \Omega^{*-1}_{DR}(U_1 \cap U_2)$.
2. Let $n = 3$ and assume $\mathcal{U} = \{U_1, U_2, U_3\}$ is such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$. Then $N_0(\mathcal{U}) = \{(1), (2), (3)\}$, $N_1(\mathcal{U}) = \{(12), (13), (23)\}$ and $N_2(\mathcal{U}) = \{(123)\}$. Then $\alpha \in CDR^*(\mathcal{U})$ is given by $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{123})$, where $\alpha_i \in \Omega^*_{DR}(U_i)$ for $i = 1, 2, 3$, $\alpha_{ij} \in \Omega^{*-1}_{DR}(U_i \cap U_j)$ for $i < j = 1, 2, 3$ and $\alpha_{123} \in \Omega^{*-2}_{DR}(U_1 \cap U_2 \cap U_3)$.

Now we have to define on $CDR^*(\mathcal{U})$ a structure of cohomological complex. For k, l fixed, let us indicate by

$$C^k(\mathcal{U}, \Omega^l_{DR}) := \bigoplus_{J \in N_k(\mathcal{U})} \Omega^l_{DR}(U_J).$$

The double complex $C^*(\mathcal{U}, \Omega^*_{DR}) := \oplus_{k,l} C^k(\mathcal{U}, \Omega^l_{DR})$ is equipped with two differential operators, the de Rham differential $d : C^k(\mathcal{U}, \Omega^l_{DR}) \to C^k(\mathcal{U}, \Omega^{l+1}_{DR})$ and the usual Čech differential $\delta : C^k(\mathcal{U}, \Omega^l_{DR}) \to C^{k+1}(\mathcal{U}, \Omega^l_{DR})$. Note that $d \circ \delta = \delta \circ d$. In general, given a double complex with two commuting differential one can define a cohomological complex summing up along the anti-diagonal (see [3]). In our case $CDR^*(\mathcal{U}) = \oplus_{l+k=*} C^k(\mathcal{U}, \Omega^l_{DR})$ with differential $D = \delta + (-1)^k d$.

Example 3.2.2. 1. In the case of Example 3.2.1.1, the Čech-de Rham complex is the Mayer-Vietoris complex $MV^*(\mathcal{U})$.
2. In the case of Example 3.2.1.2, for $\alpha \in CDR^*(\mathcal{U})$ it follows that $D\alpha = (d\alpha_1, d\alpha_2, d\alpha_3, -d\alpha_{12} + \alpha_2 - \alpha_1, -d\alpha_{13} + \alpha_3 - \alpha_1, -d\alpha_{23} + \alpha_3 - \alpha_2, d\alpha_{123} + \alpha_{23} - \alpha_{13} + \alpha_{12})$.

The map $i : \Omega^*_{DR}(U - \{m_0\}) \to CDR^*(\mathcal{U})$ given by $\alpha \mapsto (\alpha_J)$ with $\alpha_J = \alpha|_{U_J}$ for $J \in N_0(\mathcal{U})$ and $\alpha_J = 0$ for $J \in N_k(\mathcal{U})$, $k > 0$ induces an isomorphism of algebras for some multiplicative structure which has not been specified here and in particular it induces an isomorphism in cohomology (see [3]).

On the other hand one has the natural injection $j : \Omega^*_{DR}(U_1 \cap \ldots \cap U_n) \hookrightarrow CDR^{*+n-1}(\mathcal{U})$.

3.2.2. *Outline of the proof.* Go back to the notation of Theorem 3.1.3, in particular recall that ∇ is a special connection for E on $V - \{m_0\}$ and ∇^0 is the trivial connection for E on U. Let

$$\xi = c_I\left(\nabla^0, \nabla\right) = -c_I\left(\nabla, \nabla^0\right) \in \Omega^{2n-1}_{DR}\left(U - \{m_0\}\right),$$

$$\eta = \frac{c_I\left(\mathcal{M}\right)}{A_1 \cdot A_2 \cdot \ldots \cdot A_n} dz_1 \wedge \ldots \wedge dz_n \in \Omega^n_{DR}\left(U_1 \cap \ldots \cap U_n\right).$$

The form ξ is closed, since by the Bott formula and Theorem 2.1.2

$$d\xi = c_I\left(\nabla\right) - c_I\left(\nabla^0\right) = 0 + 0.$$

The form η is closed as well, being holomorphic of top degree. For little $\epsilon > 0$ let

$$R_i = \{m \in \partial\mathcal{T} : |A_i(m)| \geq |A_j\left(m\right)| \text{ for any } j\},$$

$$R_{12\ldots n} = \{m \in \partial\mathcal{T} : |A_i\left(m\right)|^2 = \frac{\varepsilon}{n} \text{ for any } i\}.$$

We can integrate differential forms in $\Omega_{DR}^{2n-1}(U-\{m_0\})$ over $\partial\mathcal{T}$ and differential forms in $\Omega_{DR}^{n}(U_{12\ldots n})$ over $R_{12\ldots n}$.

STEP 1. There exists an operator $\widehat{\int}_{\partial\mathcal{T}} : CDR^{2n-1}(\mathcal{U}) \longrightarrow \mathbb{C}$ so that the diagram commutes:

$$\begin{array}{ccccc} \Omega_{DR}^{2n-1}(U-\{m_0\}) & \xrightarrow{i} & CDR^{2n-1}(\mathcal{U}) & \xleftarrow{j} & \Omega_{DR}^{n}(U_{12\ldots n}) \\ & \searrow \int_{\partial\mathcal{T}} & \downarrow \widehat{\int}_{\partial\mathcal{T}} & \swarrow \int_{R_{12\ldots n}} & \\ & & \mathbb{C} & & \end{array} \tag{3.1}$$

and $\widehat{\int}_{\partial\mathcal{T}}(D\beta) = 0$ for any $\beta \in CDR^{2n-2}(\mathcal{U})$.

STEP 2. There exists a $\mu \in CDR^{2n-2}(\mathcal{U})$ s.t. $D\mu = j(\eta) - i(\xi)$.

The proof of Theorem 3.1.3 follows at once from the previous steps.

3.2.3. *The proof of Step 1—Integration on honeycomb cells.* A *system of cells* adapted to $\mathcal{U}$ is given by the family $R_J := \cap_{i=0}^{k} R_{j_i}$ for $J \in N_k(\mathcal{U})$. Note that R_J is a real smooth manifold of dimension $2n-1-k$ with (oriented) boundary. The boundary $\partial R_l = \oplus_t R_{tl}$, where R_{tl} is positive oriented if $l < t$, negative oriented otherwise. Inductively one defines an orientation on each cell.

We give the proof of step 1 for $n = 3$, the higher dimensional case being essentially the same. Write $\alpha \in CDR^{2n-1}(\mathcal{U})$ as a "matrix"

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_{23} & \alpha_{13} & \alpha_{12} \\ & \alpha_{123} & \end{pmatrix}.$$

The maps i, j are given by

$$i : \Omega_{DR}^{2n-1}(U-\{m_0\}) \longrightarrow CDR^{2n-1}(\mathcal{U}) \text{ s.t. } \gamma \mapsto \begin{pmatrix} \gamma & \gamma & \gamma \\ 0 & 0 & 0 \\ & 0 & \end{pmatrix},$$

$$j : \Omega_{DR}^{n}(U_{12\ldots n}) \longrightarrow CDR^{2n-1}(\mathcal{U}) \text{ s.t. } \gamma \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & \gamma & \end{pmatrix}.$$

The cells R_{ij}, R_{123} are oriented as:

$$\begin{array}{ccc} \partial R_1 = R_{12} + R_{13}, & \partial R_2 = -R_{12} + R_{23}, & \partial R_3 = -R_{13} - R_{23}, \\ \partial R_{12} = R_{123}, & \partial R_{13} = -R_{123}, & \partial R_{23} = R_{123}. \end{array}$$

We define:

$$\widehat{\int}_{\partial\mathcal{T}} \alpha = \int_{R_1} \alpha_1 + \int_{R_2} \alpha_2 + \int_{R_3} \alpha_3 + \int_{R_{12}} \alpha_{12} + \int_{R_{13}} \alpha_{13} + \int_{R_{23}} \alpha_{23} + \int_{R_{123}} \alpha_{123}.$$

It is clear that with this definition the diagram (3.1) commutes. Moreover $\widehat{\int}_{\partial\mathcal{T}} D\alpha = 0$ from the very definition and an obvious application of the Stokes formula.

3.2.4. *The proof of Step 2.* Fix $i \in \{1, \ldots, n\}$. On U_i, since $A_i \neq 0$, the set

$$\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots \frac{\partial}{\partial z_{i-1}}, v, \frac{\partial}{\partial z_{i+1}}, \cdots \frac{\partial}{\partial z_n}\}$$

is a basis of $TV|_{U_i}$. On each U_i define a connection ∇^i of type $(1,0)$ for $E|_{U_i}$ as follows:

(1) $\nabla^i_v = \theta_v$,
(2) $\nabla^i_{\frac{\partial}{\partial z_j}} \sigma_s = 0$ for any $j \neq i$ and $s = 1, \ldots r$,

recalling that $\{\sigma_1, \ldots, \sigma_r\}$ is a fixed holomorphic trivialization of $E|_U$. Denote by $\theta_{\frac{\partial}{\partial z_j}}$ the action of $\frac{\partial}{\partial z_j}$ on $E|_U$ defined by $\theta_{\frac{\partial}{\partial z_j}} \sigma_t = 0$ for $t = 1, \ldots, r$. By the very definition it follows that ∇^i is special with respect to θ_v and with respect to $\theta_{\frac{\partial}{\partial z_j}}$ for any $j \neq i$. Obviously, even the connection ∇^0 for $E|_U$ is special with respect to any $\theta_{\frac{\partial}{\partial z_j}}$ since it is trivial in the coordinates z_j's.

We define

$$\mu = (\mu_J), \quad \mu_J = (-1)^{\left[\frac{k+1}{2}\right]} c_I\left(\nabla^0, \nabla^{j_0}, \ldots, \nabla^{j_k}, \nabla\right) \quad \text{for } J = (j_0, \ldots, j_k).$$

For example, for $n = 3$, we get

$$\mu = \left(\begin{array}{ccc} c_I\left(\nabla^0, \nabla^1, \nabla\right) & c_I\left(\nabla^0, \nabla^2, \nabla\right) & c_I\left(\nabla^0, \nabla^3, \nabla\right) \\ -c_I\left(\nabla^0, \nabla^2, \nabla^3, \nabla\right) & -c_I\left(\nabla^0, \nabla^1, \nabla^3, \nabla\right) & -c_I\left(\nabla^0, \nabla^1, \nabla^2, \nabla\right) \\ & -c_I\left(\nabla^0, \nabla^1, \nabla^2, \nabla^3, \nabla\right) & \end{array} \right).$$

Computing $D\mu$ using the Bott formula and Corollary 2.1.5, we get

$$D\mu = \left(\begin{array}{ccc} -c_I\left(\nabla^0, \nabla\right) & -c_I\left(\nabla^0, \nabla\right) & -c_I\left(\nabla^0, \nabla\right) \\ 0 & 0 & 0 \\ & -c_I\left(\nabla^0, \nabla^1, \nabla^2, \nabla^3\right) & \end{array} \right).$$

For example, the high-left term is, by the Bott formula

$$dc_I\left(\nabla^0, \nabla^1, \nabla\right) = c_I\left(\nabla^1, \nabla\right) - c_I\left(\nabla^0, \nabla\right) + c_I\left(\nabla^0, \nabla^1\right) = -c_I\left(\nabla^0, \nabla\right)$$

since the first and third addends vanish because ∇^1 and ∇ are both special for θ_v whereas ∇^0, ∇^1 are both special for $\theta_{\frac{\partial}{\partial z_2}}$.

Similar calculations hold for $n > 3$. Thus to complete step 2 we need to show that

$$\eta \stackrel{def}{=} \frac{c_I(\mathcal{M})}{A_1 \cdots A_n} dz_1 \wedge \ldots \wedge dz_n = (-1)^{\left[\frac{n}{2}\right]} c_I\left(\nabla^0, \ldots, \nabla^n\right).$$

We will use the Chern-Weyl formula (2.2). Firstly we need to compute the connection forms and the curvature forms of the various connections $\nabla^0, \ldots, \nabla^n$. With respect to the local holomorphic frame $\sigma_1, \ldots, \sigma_r$ of $E|_U$, the connection one forms $\{\omega^\mu_\lambda\}_{\lambda,\mu=1,\ldots,r}$ for a connection ∇ are defined by the relations

$$\nabla_X \sigma_\lambda = \sum_\mu \omega^\mu_\lambda(X) \sigma_\mu.$$

Compactly, $\omega = (\omega^\mu_\lambda)$ is a $r \times r$ matrix of one-form which represents the connection ∇. In the local coordinates $\{z_i\}$'s, we can write $\omega = \sum_i P_i dz_i + Q_i d\overline{z}_i$, that is $\omega^\mu_\lambda = \sum_i P^\mu_{i,\lambda} dz_i + Q^\mu_{i,\lambda} d\overline{z}_i$. If the connection ∇ is of type $(1,0)$ then

$$0 = \nabla_{\frac{\partial}{\partial \overline{z}_i}} \sigma_\lambda = \sum_\mu \omega^\mu_\lambda (\frac{\partial}{\partial \overline{z}_i}) \sigma_\mu = \sum_\mu Q^\mu_{\lambda,j} \sigma_\mu,$$

and thus $Q_j = 0$ for any j. Hence $\omega = \sum P_j dz_j$.

We will denote by ω^i the matrix of connection one-forms relative to ∇^i. Now ∇^0 is flat in the coordinates z_i, i.e. we assumed $\nabla^0_{\frac{\partial}{\partial z^i}} \sigma_\lambda = 0$, therefore $\omega^0 = 0$. The connection ∇^i satisfies $\nabla_{\frac{\partial}{\partial z_j}} \sigma_\lambda = 0$ for $i \neq j$, and therefore $P_j = 0$ for $j \neq i$. Thus

$$\sum_\mu \mathcal{M}^\mu_\lambda \sigma_\mu = \nabla_v \sigma_\lambda = \sum_j \nabla_{A_j \frac{\partial}{\partial z_j}} \sigma_\lambda = \sum_j A_j \nabla_{\frac{\partial}{\partial z_j}} \sigma_\lambda = A_i \nabla_{\frac{\partial}{\partial z^i}} \sigma_\lambda = \sum_\mu A_i P^\mu_{i,\lambda} \sigma_\mu,$$

therefore $A_i P_i = \mathcal{M}$ i.e.$P_i = \frac{\mathcal{M}}{A_i}$. Hence

$$\omega^0 = 0, \ \ \omega^i = \frac{\mathcal{M}}{A_i} dz_i. \tag{3.2}$$

For sake of clearness we assume $n = 3$. We have

$$c_I \left(\nabla^0, ..., \nabla^3\right) = (-1)^{\left[\frac{3}{2}\right]} \int_{\Delta_3} c_I(\tilde{\nabla}),$$

where $\tilde{\nabla}$ is the connection for the vector bundle $p_1^*(E) = E \times \Delta_3$ on $V \times \Delta_3$ defined as in (2.1), where $p_1 : V \times \Delta_s \to V$ is the projection. The connection form of $\tilde{\nabla}$ at the point $(m,t) \in V \times \Delta_3$ is

$$\tilde{\omega} = t_0 \omega^0 + t_1 \omega^1 + t_2 \omega^2 + t_3 \omega^3 = t_1 \omega^1 + t_2 \omega^2 + t_3 \omega^3,$$

where $t_0 = 1 - (t_1 + t_2 + t_3)$. Cartan's structure equation gives the $(r \times r)$-matrix Ω of two forms representing the curvature for a connection ∇ with connection matrix ω as

$$\Omega = d\omega + \frac{1}{2} \omega \wedge \omega.$$

Therefore, if $\tilde{\Omega}$ is the matrix of the curvature of $\tilde{\nabla}$ we have

$$\begin{aligned} \tilde{\Omega} &= dt_1 \wedge \omega^1 + dt_2 \wedge \omega^2 + dt_3 \wedge \omega^3 + S = \\ &= dt_1 \wedge \frac{dz^1}{A_1} \mathcal{M} + dt_2 \wedge \frac{dz^1}{A_1} \mathcal{M} + dt_3 \wedge \frac{dz^1}{A_1} \mathcal{M} + S, \end{aligned}$$

where the forms in S do not involve differentials in the t_l variables. For $|I| = n$, we get

$$c_I(\tilde{\nabla}) \stackrel{def}{=} c_I(\tilde{\Omega}) = 3! dt_1 \wedge dt_2 \wedge dt_3 \wedge \frac{dz^1}{A_1} \wedge \frac{dz^2}{A_2} \wedge \frac{dz^3}{A_2} c_I \left(\mathcal{M}\right) + S',$$

where the forms in S' do not contain more than two differentials in the t_l variables. Thus

$$\int_{\Delta_3} c_I(\tilde{\nabla}) = \int_{(\sum_{i=0}^{3} t_i)=1} 3! dt_1 \wedge dt_2 \wedge dt_3 \wedge \frac{dz^1}{A_1} \wedge \frac{dz^2}{A_2} \wedge \frac{dz^3}{A_2} c_I\left(\mathcal{M}\right) + S' =$$

$$= \int_{(\sum_{i=0}^{3} t_i)=1} 3! dt_1 \wedge dt_2 \wedge dt_3 \left(\frac{dz^1}{A_1} \wedge \frac{dz^2}{A_2} \wedge \frac{dz^3}{A_2} c_I\left(\mathcal{M}\right) \right) =$$

$$= 3! \frac{1}{3!} \left(\frac{dz^1}{A_1} \wedge \frac{dz^2}{A_2} \wedge \frac{dz^3}{A_2} c_I\left(\mathcal{M}\right) \right),$$

which completes the proof of step 2.

Corollary 3.2.3. *If $m_0 \in V$ is an isolated, non degenerate, singular point of v then*

$$\textit{Res}(\theta_v, E, c_I)_{m_0} = \frac{c_I\left(\mathcal{M}(m_0)\right)}{\lambda_1 \cdot \ldots \cdot \lambda_n}, \tag{3.3}$$

where the λ_i's are the eigenvalues of the Jacobian matrix of v at m_0.

Proof. The eigenvalues of the Jacobian of v at m_0, $J(m_0)$ are all different from zero for $J(m_\lambda)$ is invertible by definition of non-degeneracy. Up to shrink U if necessary, we may suppose that $\det(J(m)) \neq 0$ for any $m \in U$. Now $dA_1 \wedge \ldots dA_n = \det(J) dz_1 \wedge \ldots dz_n$ and therefore

$$\frac{c_I(\mathcal{M})}{\det(J)} \left(\frac{dA_1}{A_1} \wedge \ldots \wedge \frac{dA_n}{A_n} \right) = c_I(\mathcal{M}) \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}.$$

If we set $F = \frac{c_I(\mathcal{M})}{\det(J)}$, then F is holomorphic on U and $F(m) = F(m_0) + O(m)$, where $O(m)$ is holomorphic in U and vanishing at m_0. On $R_{1\ldots n}$ we have $A_i = \epsilon e^{\sqrt{-1}\theta_i}$ for $\theta_i \in \mathbb{R}$, and therefore

$$\begin{bmatrix} c_I(\mathcal{M}) dz_1 \wedge \ldots dz_n \\ A_1, \ldots, A_n \end{bmatrix} = \left(\frac{1}{2\pi i} \right)^n F(m_0) \int_{|A_1|=\epsilon} \frac{dA_1}{A_1} \cdots \int_{|A_n|=\epsilon} \frac{dA_n}{A_n} + O(m) = F(m_0).$$

Then Theorem 3.1.3 gives the assertion. □

3.3. **Examples of Residues.** In Example 1.2.2 we saw some instances of natural actions. We are going to calculate the residues for isolated singular points of v in such cases.

1. Let $E = TV$ and $\theta_v = [v, \cdot]$. Let $\sigma_i = \frac{\partial}{\partial z_i}$ for $i = 1, \ldots, n$. Then

$$\theta_v\left(\sigma_\lambda\right) = \left[A_i \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^\lambda} \right] = -\frac{\partial A_i}{\partial z^\lambda} \frac{\partial}{\partial z_i},$$

and $\mathcal{M} = -J$, where J is the Jacobian matrix of v. In particular if m_0 is a non-degenerate singularity for v then by Corollary 3.3 one has for $|I| = n$,

$$\text{Res}\left(\theta_v, TV, c_I\right) = \frac{c_I\left(\lambda_1, \ldots, \lambda_n\right)}{\lambda_1 \cdot \ldots \cdot \lambda_n},$$

where $c_I\,(\lambda_1, ..., \lambda_n)$ indicates the c_I of the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$.
2. Suppose $V \subset M$ for some complex manifold M of dimension $n+q$. Assume moreover that v is the restriction to V of a holomorphic vector field $\tilde{v}$ on M. We may assume that there exists an open set $\tilde{U} \subset M$ such that $\tilde{U} \cap V = U$ and local coordinates $\{z_1, \ldots, z_n, y_1, \ldots, y_q\}$ on $\tilde{U}$ so that $V \cap \tilde{U} = \{y_1 = \ldots = y_q = 0\}$. In such coordinates

$$\tilde{v} = \sum_{i=1}^{n} \tilde{A}_i(z,y)\frac{\partial}{\partial z_i} + \sum_{j=1}^{q} \tilde{B}_j(z,y)\frac{\partial}{\partial y_j},$$

for some holomorphic functions $\tilde{A}_i, \tilde{B}_j$ such that $\tilde{A}_i(z,0) = A_i(z)$ and $\tilde{B}_j(z,0) = 0$ (the for v is tangent to V).

Let $E = TM|_V$, and $\theta_v(Y) := [\tilde{v}, \tilde{Y}]|_V$, where $\tilde{Y}$ is a holomorphic section of TM such that $\tilde{Y}|_V = Y$. An explicit calculation shows that in such a case

$$\mathcal{M} = -\frac{D(A_1, ..., A_n, \tilde{B}_1, .., \tilde{B}_q)}{D(z_1, ..., z_n, y_1, ..., y_q)}.$$

The residue given by Theorem 3.1.3 is called the *variation index*.
3. Assume a setup as in 2. Let $E = N_V$, where the normal bundle N_V to V is given by (1.4). Let θ_v be defined as in Example 1.2.2.(3). Then $\{\pi(\frac{\partial}{\partial y_1}|_V), \ldots, \pi(\frac{\partial}{\partial y_q}|_V)\}$ is a local holomorphic frame of N_V and we have

$$\theta_v(\pi(\frac{\partial}{\partial y_\nu}|_V)) = \pi([\sum_i \tilde{A}_i \frac{\partial}{\partial z_i} + \sum_j \tilde{B}_j \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_\nu}]|_V) = -\sum_j \frac{\partial \tilde{B}_j}{\partial y_\nu}\pi(\frac{\partial}{\partial y_j}|_V).$$

If $n = q = 1$ and $\tilde{v} = A(z,y)\frac{\partial}{\partial z} + B(z,y)\frac{\partial}{\partial y}$ then Theorem 3.1.3 gives

$$\mathrm{Res}(\theta_v, N_V, c_1) = -\frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{B_y(z,0)}{A(z,0)}dz,$$

where $\gamma = \{(z,0) : |z| = \epsilon\}$. This is called the *Camacho-Sad index*.

4. Proof of the Residues Theorems and applications

We are now in the good shape to prove the residues theorems stated in the first sections. The proof is essentially stated in section 1.3.

Suppose we are in the hypothesis of Theorem 1.2.3. In particular V is compact and one can integrate the $2n$-form $c_I(E)$ over V. The singular set of v is given by $\Sigma = \{m_1, \ldots, m_h\}$. By Theorem 2.1.2 one can consider a representative $c_I(V, \Sigma)$ of $c_I(E)$ in the relative Mayer-Vietoris cohomology $H^{2n}(MV^*(V, V - \Sigma))$, which is isomorphic to the relative cohomology $H^{2n}(V, V - \Sigma, \mathbb{C})$. Integration of $c_I(E)$ on V is the same as integration of $c_I(V, \Sigma)$ on V (using honeycomb cells as in 3.2.3 or in [23]). Thus by Proposition 3.1.2

$$\int_V c_I(E) = \sum_\lambda \mathrm{Res}(\theta_v, E, c_I),$$

and Theorem 1.2.3 follows from Theorem 3.1.3.

As for Theorem 1.1.2, it follows from Theorem 1.2.3 taking θ_v to be the Lie derivative action on TV (see Example 1.2.2.(1)) and calculating the residues as in section 3.3.1.

Theorem 1.3 involves virtual bundles and the proof is slightly different from the previous ones, see, *e.g.*, [22] p.110-113. Instead of giving it here, we present an application of Theorem 2.4.1.

Proposition 4.0.1. *All one-dimensional foliations on* $\mathbb{CP}^n$ *are singular.*

Proof. Let $\mathcal{L}$ be a holomorphic line bundle over $\mathbb{CP}^n$ and $h : \mathcal{L} \to T\mathbb{CP}^n$ be an injective morphism of holomorphic vector bundle defining a non-singular one-dimensional foliations.

Now $H^2(\mathbb{CP}^n, \mathbb{C}) \simeq \mathbb{C}$ and the tautological bundle (the inverse of the hyperplane bundle) L on $\mathbb{CP}^n$ generates this cohomology. Also each line bundle on $\mathbb{CP}^n$ is determined (up to isomorphisms) by its Chern class in $H^2(\mathbb{CP}^n, \mathbb{C})$ and therefore $\mathcal{L} = L^{d-1}$ (see, *e.g.*, [8]). The number $d \in \mathbb{Z}$ is called the *degree* of the foliation. Let $\gamma = c_1(-L)$ be the generator of $H^2(\mathbb{CP}^n, \mathbb{C})$. Then, since $T(\mathbb{CP}^n) \oplus (\mathbb{CP}^n \times \mathbb{C}) = -(n+1)L$, it follows that $c(T\mathbb{CP}^n) = (1+\gamma)^{n+1}$. Moreover $c(L^{d-1}) = 1 - (d-1)\gamma$. Thus, if $\mathcal{V}_{\mathcal{F}} = T\mathbb{CP}^n / L^{d-1}$, we have

$$c(\mathcal{V}_{\mathcal{F}}) = \frac{(1+\gamma)^{n+1}}{1-(d-1)\gamma} = (1+\gamma)^{n+1}(1 + (d-1)\gamma + (d-1)^2\gamma^2 + \ldots). \tag{4.1}$$

Thus $c_1(\mathcal{V}_{\mathcal{F}}) = (d+n)\gamma$. By Theorem 2.4.1 $c_1^n(\mathcal{V}_{\mathcal{F}}) = 0$, but

$$\int_{\mathbb{CP}^n} c_1^n(\mathcal{V}_{\mathcal{F}}) = \int_{\mathbb{CP}^n} (d+n)^n \gamma^n = (d+n)^n,$$

for $\int_{\mathbb{CP}^n} \gamma^n = 1$. Hence the only possibility is $d = -n$. But $c_n(\mathcal{V}_{\mathcal{F}}) = 0$ for $\mathcal{V}_{\mathcal{F}}$ has rank $n-1$, and a straightforward calculation shows that if $d = -n$ than (4.1) gives a non-vanishing term of degree n. □

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