1. Prove the arithmetic-geometric-mean inequality: let $a_1, \ldots, a_n \in \mathbf{R}_{\geq 0}$ then

$$(a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}.$$

The equality holds if and only if $a_1 = \ldots = a_n$.(Hint: let $A = \frac{a_1+\ldots+a_n}{n}$. Show that $e^{\frac{a_i}{A}-1} \geq \frac{a_i}{A}$ for every i, with equality if and only if $a_i = A$.)

2. Prove Stirling's Formula: for every $n \geq 1$ we have

$$n! = n^n e^{-n}\sqrt{2\pi n}\, e^{\frac{\theta}{12n}} \qquad \text{for some } \theta \text{ satisfying } 0 < \theta < 1.$$

3. (a) Find all solution $X, Y \in \mathbf{Z}$ of the equation $Y^2 = X^3 - 19$.
   (b) Show that the Diophantine equation $Y^2 = X^3 - 5$ has no solutions $X, Y \in \mathbf{Z}$. (Hint: show that the class group of $\mathbf{Z}[\sqrt{-5}]$ has order 2.)

4. Show that the following three polynomials have the same discriminant:

$$T^3 - 18T - 6,$$
$$T^3 - 36T - 78,$$
$$T^3 - 54T - 150.$$

Let α, β and γ denote zeroes of the respective polynomials. Show that the fields $\mathbf{Q}(\alpha)$, $\mathbf{Q}(\beta)$ and $\mathbf{Q}(\gamma)$ have the same discriminants, but are not isomorphic. (Hint: the splitting behavior of the primes is not the same.)

5. Show that $\mathbf{Z}[\sqrt[3]{20}, \sqrt[3]{50}]$ is the ring of integers of $F = \mathbf{Q}(\sqrt[3]{20})$. Show there is no $\alpha \in O_F$ such that $O_F = \mathbf{Z}[\alpha]$.

6.* (Samuel) Let $f(T) = T^3 + T^2 - 2T + 8 \in \mathbf{Z}[T]$.
   (a) Show that f is irreducible.
   (b) Show that $\mathrm{Disc}(f) = -4 \cdot 503$. Show that the ring of integers of $F = \mathbf{Q}(\alpha)$ admits $1, \alpha, \beta = (\alpha^2 - \alpha)/2$ as a $\mathbf{Z}$-basis.
   (c) Show that O_F has precisely three distinct ideals of index 2. Conclude that 2 splits completely in F over $\mathbf{Q}$.
   (d) Show that there is no $\alpha \in F$ such that $O_F = \mathbf{Z}[\alpha]$. Show that for every $\alpha \in O_F - \mathbf{Z}$, the prime 2 divides the index $[O_F : \mathbf{Z}[\alpha]]$.

7.* Let $\mathbf{F}_q$ be a finite field with q elements. Let $\zeta(s)$ denote the ζ-function of the ring $\mathbf{F}_q[T]$:

$$\zeta_{\mathbf{F}_q(T)}(s) = \sum_{I \neq 0} \frac{1}{N(I)^s}, \qquad (s \in \mathbf{C} \text{ with } \mathrm{Re}\, s > 1).$$

(Here the product runs over the non-zero ideals I and $N(I) = [\mathbf{F}_q[T] : I]$.) Show that

$$\zeta_{\mathbf{F}_q(T)}(s) = \frac{1}{1 - q^{1-s}}, \qquad (s \in \mathbf{C} \text{ with } \mathrm{Re}\, s > 1).$$