1. Let $L = \{(x, y, z) \in \mathbf{Z}^3 : 2x + 3y + 4z \equiv 0 \pmod 7\}$. Show that $L \subset \mathbf{R}^3$ is a lattice. Find a $\mathbf{Z}$-basis and calculate its covolume.

2. Let $L \subset \mathbf{R}^n$ be a lattice. Let A be an invertible $n \times n$-matrix. Show that $A(L)$ is a lattice. Show that $\mathrm{covol}(A(L)) = |\det(A)|\mathrm{covol}(L)$. For $m \in \mathbf{R}_{>0}$ show that $\mathrm{covol}(mL) = m^n\mathrm{covol}(L)$.

3. Identify the quaternions $\mathbf{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\}$ with $\mathbf{R}^4$ via $a + bi + cj + dk \leftrightarrow (a, b, c, d)$. We equip $\mathbf{H}$ with the usual scalar product of $\mathbf{R}^4$. What is the covolume of the ring of Hurwitz integers
$$\mathbf{Z}[i, j, k, \frac{1 + i + j + k}{2}]?$$

4. Let F be a number field. Suppose $R \subset F$ is a subring with the property that its image in $F \otimes \mathbf{R}$ is a lattice. Show that $R \subset O_F$.

5. *(Euclidean imaginary quadratic rings.)* Let F be an imaginary quadratic number field. We identify O_F with its image in $F \otimes \mathbf{R} = \mathbf{C}$.
   (a) Show that O_F is Euclidean for the norm if and only if the disks with radius 1 and centers in O_F cover $\mathbf{C}$.
   (b) Show that O_F is Euclidean for the norm if and only if $\Delta_F = -3, -4, -7, -8$ or -11.

6. *Class groups.*
   (a) Show that the class group of $\mathbf{Q}(\sqrt{-86})$ is cyclic of order 10.
   (b) Compute the structure of the class groups of $\mathbf{Q}(\sqrt{-30})$ and $\mathbf{Q}(\sqrt{-114})$.
   (c) Compute the class group of $F = \mathbf{Q}(\sqrt{229})$.

7. Let α be a zero of the irreducible polynomial $T^3 + T - 1 \in \mathbf{Q}[T]$ and let $F = \mathbf{Q}(\alpha)$.
   (a) Show that the ring of integers of F is $\mathbf{Z}[\alpha]$.
   (b) Find the factorizations in $\mathbf{Z}[\alpha]$ of the primes less than 10.
   (c) Show that the class group of $\mathbf{Q}(\alpha)$ is trivial.

8. Prove the arithmetic-geometric-mean inequality: let $a_1, \ldots, a_n \in \mathbf{R}_{\geq 0}$ then
$$(a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}.$$

9. Let d be a squarefree integer and let $F = \mathbf{Q}(\sqrt{d})$ be a quadratic field. Show that for *odd* primes p one has that p splits (is inert, ramifies) in F over $\mathbf{Q}$ if and only if d is a non-zero square (a non-square, zero) modulo p.

10. Let ζ_5 denote a primitive 5th root of unity. Determine the decomposition into prime factors in $\mathbf{Q}(\zeta_5)$ of the primes less than 14.