1. Let $F$ be a number field.
   (a) Show that for any $x \in F$ there exist an integer $0 \neq m \in \mathbf{Z}$ such that $mx \in O_F$.
   (b) Show that the field of fractions of $O_F$ is $F$.
   (c) Show that there exists an element $\alpha \in O_F$ for which $F = \mathbf{Q}(\alpha)$.

2. Let $F$ be a number field. Show that every ideal $I \neq 0$ of $O_F$ contains a non-zero integer $m \in \mathbf{Z}$.

3. Let $F$ be a number field and let $\alpha \in O_F$. Show that $N(\alpha) = \pm 1$ if and only if $\alpha$ is a unit of the ring $O_F$.

4. Let $F \subset K$ be an extension of number fields. Show that $O_K \cap F = O_F$.

5. Let $d \neq 1$ be a squarefree integer.
   (a) Compute the discriminant of $F = \mathbf{Q}(\sqrt{d})$.
   (b) For $d < 0$ determine the units of the ring of integers of $\mathbf{Q}(\sqrt{d})$.

6. Let $F$ and $K$ be two quadratic number fields. Show that if $\Delta_F = \Delta_K$, then $F \cong K$.

7. Show that $25 = 5^2$ and $25 = (1 + 2\sqrt{6})(1 - 2\sqrt{6})$ are two factorizations of 25 into products of irreducible elements of the ring $\mathbf{Z}[\sqrt{-6}]$. Write the principal ideals $(5)$, $(1 + 2\sqrt{6})$ and $(1 - 2\sqrt{6})$ as products of prime ideals of the Dedekind ring $\mathbf{Z}[\sqrt{-6}]$.

8. The ring $\mathbf{Z}[\frac{1+\sqrt{-23}}{2}]$ is the ring of integers of the number field $\mathbf{Q}(\sqrt{-23})$. Show that the ideal $(2, \frac{1+\sqrt{-23}}{2})$ and its square are not principal, but its third power is.

9. Consider the properties "Noetherian", "integrally closed" and "of Krull dimension 1" that characterize Dedekind domains. Give examples of rings that have two of these properties, but not the third.

10. Let $R$ be a Dedekind domain and let $\mathfrak{p}$ and $\mathfrak{p}'$ be two different non-zero prime ideals of $R$.
    (a) Show that $\mathfrak{p} + \mathfrak{p}' = R$.
    (b) Show that the ring $R/\mathfrak{p}\mathfrak{p}'$ is isomorphic to $R/\mathfrak{p} \times R/\mathfrak{p}'$.

11. Let $R$ be a Dedekind domain and let $\mathfrak{p}$ be a non-zero prime. Show that there exists an element $x \in \mathfrak{p}$ with $x \in \mathfrak{p}^2$.

12. Let $F$ be a number field and let $\omega_1, \ldots, \omega_n \in O_F$. Put $\Delta(\omega_1, \ldots, \omega_n) = \det(\mathrm{Tr}(\omega_i \omega_j))$. Dimostrare che $\Delta(\omega_1, \ldots, \omega_n) = m^2 \Delta_F$ for some $m \in \mathbf{Z}$.

13. Let $\alpha$ be a zero of the polynomial $X^3 - X + 1$.
    (a) Compute the trace of $\alpha^i$ for $0 \leq i \leq 4$.
    (b) Show that $\mathbf{Z}[\alpha]$ is the ring of integers of $\mathbf{Q}(\alpha)$.

14. Let $a \geq 1$ and let $\mathbf{R}^a$ be the product of $a$ copies of $\mathbf{R}$. It is a ring with componentwise addition and multplication. Show that the projection maps $\mathbf{R}^a \longrightarrow \mathbf{R}$ are ring homomorphisms. Show that they are the only ring homomorphisms.