1. Compute the degrees of the number fields $\mathbf{Q}(\sqrt{2}, \sqrt{-6})$ and $\mathbf{Q}(\sqrt{-2}, \sqrt{3}, \sqrt{-6})$.
2. Find an element $\alpha \in F = \mathbf{Q}(\sqrt{3}, \sqrt{-5})$ such that $F = \mathbf{Q}(\alpha)$.
3. Let p be a prime. Compute the minimum polynomial of a primitive p-th root of unity ζ_p. Show that $[\mathbf{Q}(\zeta_p) : \mathbf{Q}] = p - 1$.
4. Let $\phi : \mathbf{Q} \to \mathbf{C}$ be a ring homomorphism. Show that $\phi(q) = q$ for every $q \in \mathbf{Q}$.
5. (*VanderMonde*) Let R be a commutative ring and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$. Show that

$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \ldots & \alpha_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i).$$

6. Consider the extension $L = \mathbf{F}_p(\sqrt[p]{X}, \sqrt[p]{Y})$ of the field $K = \mathbf{F}_p(X, Y)$. Show that there does not exist any $\alpha \in L$ such that $L = K(\alpha)$. Show that there are infinitely many distinct fields F with $K \subset F \subset L$.
7. Let $F = \mathbf{Q}(\sqrt[6]{5})$. Show that $r_1 = r_2 = 2$ and describe the ring homomorphism $F \longrightarrow F_{\mathbf{R}} \cong \mathbf{R}^2 \times \mathbf{C}^2$ explicitly.
8. Find a basis (as a $\mathbf{Q}$-vector space) for $\mathbf{Q}(\sqrt{2}, \sqrt{-1})$ and $\mathbf{Q}(\sqrt[3]{2}, \zeta_3)$
9. Let F be a number field with $r_1 \ge 1$, i.e. F admits an embedding into $\mathbf{R}$. Show that the only roots of unity in F are ± 1.
10. Let F be a number field of degree n and let $x \in F$. Show that for $q \in \mathbf{Q}$ one has that

$$\begin{aligned} Tr(qx) &= qTr(x), \\ Tr(q) &= nq, \\ N(q) &= q^n. \end{aligned}$$

Show that the map $Tr : F \longrightarrow \mathbf{Q}$ is surjective. Show that the norm $N : F^* \longrightarrow \mathbf{Q}^*$ is, in general, not surjective.

11. Let F be a number field of degree n and let $\alpha \in F$. Show that for $q \in \mathbf{Q}$ one has that $N(q - \alpha) = f^{\alpha}_{\text{char}}(q)$. Show that for $q, r \in \mathbf{Q}$ one has that $N(q - r\alpha) = r^n f^{\alpha}_{\text{char}}(q/r)$. Here $f^{\alpha}_{\text{char}} \in \mathbf{Q}[X]$ denotes the characteristic polynomial of α.
12. Let $\alpha = \zeta_5 + \zeta_5^{-1} \in \mathbf{Q}(\zeta_5)$ where ζ_5 denotes a primitive 5th root of unity. Calculate the characteristic polynomial of $\alpha \in \mathbf{Q}(\zeta_5)$.
13. Consider the field $\mathbf{Q}(\sqrt{3}, \sqrt{5})$. Compute the discriminants $\Delta(1, \sqrt{3}, \sqrt{5}, \sqrt{15})$ and $\Delta(1, \sqrt{3}, \sqrt{5}, \sqrt{3} + \sqrt{5})$.