1. Let F be a number field and let $\alpha \in F$. Show that there exist an integer $0 \neq m \in \mathbf{Z}$ such that $m\alpha \in O_F$.
2. Show that for every number field F there exists an *integral* element $\alpha \in O_F$ such that $F = \mathbf{Q}(\alpha)$.
3. Let F be a number field. Show that the field of fractions of O_F is F.
4. Let F be a number field. Show that every ideal $I \neq 0$ of O_F contains a non-zero integer $m \in \mathbf{Z}$.
5. Let F be a number field and let $\alpha \in O_F$. Show that $\mathrm{N}(\alpha) = \pm 1$ if and only if α is a unit of the ring O_F.
6. Let $F \subset K$ be an extension of number fields. Show that $O_K \cap F = O_F$.
7. Let F be a number field. Let r be the number of distinct embeddings $F \hookrightarrow \mathbf{R}$ and let $2s$ be the number of remaining homomorphisms $F \hookrightarrow \mathbf{C}$. Show that the sign of Δ_F is $(-1)^s$.
8. Let d be a squarefree negative integer. Determine the units of the rings of integers of $\mathbf{Q}(\sqrt{d})$.
9. Let F and K be two quadratic number fields. Show that if $\Delta_F = \Delta_K$, then $F \cong K$.
10. Let $d \neq 1$ be a squarefree integer. Compute the discriminant of $F = \mathbf{Q}(\sqrt{d})$.
11. Let $n \geq 1$ be an integer and let ζ_n denote a primitive n-th root of unity. Show that $\zeta_n - 1$ is a unit if and only if n is not the power of a prime. (Hint: substitute $T = 1$ in $(T^n - 1)/(T - 1) = \prod_{d|n, d\neq 1} \Phi_d(T)$.
12. Let $a \geq 1$ and let $\mathbf{R}^a$ be a ring with componentwise addition and multplication. Show that the projection maps $\mathbf{R}^a \longrightarrow \mathbf{R}$ are ring homomorphisms. Show that they are the only ring homomorphisms.
13. Let F be a number field and let $\omega_1, \ldots, \omega_n \in O_F$. Put $\Delta(\omega_1, \ldots, \omega_n) = \det(\mathrm{Tr}(\omega_i \omega_j))$. Dimostrare che $\Delta(\omega_1, \ldots, \omega_n) = m^2 \Delta_F$ for some $m \in \mathbf{Z}$.
14. Let α be a zero of the polynomial $X^3 - X + 1$. Show that $\mathbf{Z}[\alpha]$ is the ring of integers of $\mathbf{Q}(\alpha)$.
15. Show that every unique factorization domain is integrally closed.
16. Consider the properties "Noetherian", "integrally closed" and "of Krull dimension 1" that characterize Dedekind domains. Give examples of rings that have two of these properties, but not the third.
17. Prove the Chinese Remainder Theorem: let R be a commutative ring and suppose that I and J are two ideals of R that are relatively prime i.e. $I + J = R$. Then the canonical homomorphism
$$R/IJ \longrightarrow R/I \times R/J$$
is an isomorphism.
18. Let R be a Dedekind ring and let $\mathfrak{p}$ and $\mathfrak{p}'$ be two different non-zero prime ideals of R. Then $\mathfrak{p} + \mathfrak{p}' = R$.