$$f(X) = X^4 + 6X^2 - 3X - 6$$

The polynomial f is irreducible. Let α denote a zero and put $F = \mathbf{Q}(\alpha)$. We compute the discriminant of the ring $\mathbf{Z}[\alpha]$. We have $\mathrm{Tr}(\alpha^k) = p_k$ for $k \geq 0$. We have $p_0 = 4$. The Newton identities give that $p_1 = s_1 = 0$, $p_2 = -2s_2 + p_1 s_1 = 2 \cdot 6 + 0^2 = -12$ and $p_3 = 3s_3 + p_2 s_1 - p_1 s_2 = 3 \cdot 3 + 12 \cdot 0 - 0 = 9$. For $k \geq 4$ we have $p_k = -6p_{k-2} + 3p_{k-3} + 6p_{k-4}$. Therefore p_4, p_5 and p_6 are equal to 96, -90 and -621 respectively. This gives

$$\Delta(\mathbf{Z}[\alpha]) = \det \begin{pmatrix} 4 & 0 & -12 & 9 \\ 0 & -12 & 9 & 96 \\ -12 & 9 & 96 & -90 \\ 9 & 96 & -90 & -621 \end{pmatrix} = -402219 = -3^3 14897.$$

Since f is Eisenstein at 3 and since 14897 is prime, the ring $\mathbf{Z}[\alpha]$ is integrally closed. Therefore $O_F = \mathbf{Z}[\alpha]$.

Two of the complex zeroes of $f(X)$ are real. They are approximately equal to -0.7542 and 1.1359. Let σ and σ' denote the corresponding embeddings $F \hookrightarrow \mathbf{R}$. The non-real zeroes are $-0.1908 \pm 2.6393i$. Let $\sigma'' : F \hookrightarrow \mathbf{C}$ be the embedding that maps α to $-0.1908 + 2.6393i$. Dirchlet's Unit Theorem implies that the unit group O_F^* has rank 2. Minkowski's constant is equal to

$$\frac{4!}{4^4}\frac{4}{\pi}\sqrt{402219} = 75.7029.$$

It follows that the class group of O_F is generated by the prime ideals of norm ≤ 73.

Primes of characteristic p that have norm p^k with $k \geq 2$ necessarily satisfy $p \leq 7$. We list these prime ideals in Table I.

Table I.

p		
2	$\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_4$	$\mathfrak{p}_2 = (\alpha, 2)$, $\mathfrak{p}_2' = (\alpha+1, 2)$, $\mathfrak{p}_4 = (2, 1+\alpha+\alpha^2)$
3	$\mathfrak{p}_3^4$	$\mathfrak{p}_3 = (\alpha, 3)$
5	$\mathfrak{p}_5\mathfrak{p}_{125}$	$\mathfrak{p}_5 = (5, \alpha+2)$
7	$\mathfrak{p}_7\mathfrak{p}_{343}$	$\mathfrak{p}_7 = (7, \alpha-2)$

The remaining prime ideals of norm ≤ 73 are of degree 1. These are the primes for which the polynomial f has at least one root in $\mathbf{F}_p$. We list these primes in Table II..

Table II.

p	
13	$\mathfrak{p}_{13} = (13, \alpha - 5)$
19	$\mathfrak{p}_{19} = (19, \alpha + 9)$
23	$\mathfrak{p}_{23} = (23, \alpha + 3)$, $\mathfrak{p}'_{23} = (\alpha + 16)$
29	$\mathfrak{p}_{29} = (29, \alpha - 7)$, $\mathfrak{p}'_{29} = (\alpha - 5)$
31	$\mathfrak{p}_{31} = (31, \alpha + 9)$, $\mathfrak{p}'_{31} = (\alpha - 6)$
41	$\mathfrak{p}_{41} = (41, \alpha - 17)$
43	$\mathfrak{p}_{43} = (43, \alpha + 11)$, $\mathfrak{p}'_{43} = (\alpha + 20)$
47	$\mathfrak{p}_{47} = (47, \alpha + 24)$

We claim that the class group of O_F generated by $\mathfrak{p}_2$, rather than the prime ideals of norm ≤ 73. To see this, we factor a few principal ideals. It is convenient to consider ideals generated by elements of the form $a + b\alpha$ for $a, b \in \mathbf{Z}$, because their norms are equal to $b^4 f(-a/b)$ and are easy to compute. The factorizations listed in the following table show inductively that the class of each of the prime ideals $\mathfrak{p}$ with $2 < N(\mathfrak{p}) \leq 73$ is in fact contained in the subgroup of ideal classes generated by primes of smaller norm.

Table III.

$a + b\alpha$	$\lvert N(a + b\alpha)\rvert$	
$1 + 2\alpha$	47	$\mathfrak{p}_{47}$
$1 + 4\alpha$	$29 \cdot 43$	$\mathfrak{p}_{29}\mathfrak{p}_{43}$
$3 - 2\alpha$	$3 \cdot 43$	$\mathfrak{p}_3\mathfrak{p}'_{43}$
$3 - 5\alpha$	$2^2 \cdot 3 \cdot 7 \cdot 41$	$\mathfrak{p}'^2_2\mathfrak{p}_3\mathfrak{p}_7\mathfrak{p}_{41}$
$4 - 3\alpha$	$2^2 \cdot 5 \cdot 31$	$\mathfrak{p}_2^2\mathfrak{p}_5\mathfrak{p}_{31}$
$6 - \alpha$	$2^4 \cdot 3 \cdot 31$	$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}'_{31}$
$7 - \alpha$	$2^2 \cdot 23 \cdot 29$	$\mathfrak{p}'^2_2\mathfrak{p}'_{23}\mathfrak{p}_{29}$
$5 - \alpha$	$2 \cdot 13 \cdot 29$	$\mathfrak{p}'_2\mathfrak{p}_{13}\mathfrak{p}'_{29}$
$3 + \alpha$	$2 \cdot 3 \cdot 23$	$\mathfrak{p}'_2\mathfrak{p}_3\mathfrak{p}_{23}$
$2 + 3\alpha$	$2^2 \cdot 23$	$\mathfrak{p}_2^2\mathfrak{p}'_{23}$
$1 - 2\alpha$	$5 \cdot 19$	$\mathfrak{p}_5\mathfrak{p}_{19}$
$2 - 3\alpha$	$2^5 \cdot 13$	$\mathfrak{p}_2^5\mathfrak{p}_{13}$
$2 - \alpha$	$2^2 \cdot 7$	$\mathfrak{p}_2^2\mathfrak{p}_7$
$2 + \alpha$	$2^5 \cdot 5$	$\mathfrak{p}_2^5\mathfrak{p}_5$
2	2^4	$\mathfrak{p}_2\mathfrak{p}'_2\mathfrak{p}_4$
α	$2 \cdot 3$	$\mathfrak{p}_2\mathfrak{p}_3$
$1 - \alpha$	2	$\mathfrak{p}'_2$

It follows that the ideal class group of O_F is generated by the class of $\mathfrak{p}_2$. Since $(\alpha) = \mathfrak{p}_2\mathfrak{p}_3$, it is also generated by $\mathfrak{p}_3$ and since $(3) = \mathfrak{p}_3^4$ we see that the class group is cyclic of order dividing 4. To prove that the class group is cyclic of order 4, we first compute the unit group O_F^* modulo squares. To find units, we factor some more elements of the form $a + b\alpha$ with $a, b \in \mathbf{Z}$.

Table IV.

$a+b\alpha$	$N(a+b\alpha)$	
$1-\alpha$	-2	$\mathfrak{p}_2'$
$1+\alpha$	2^2	${\mathfrak{p}_2'}^2$
$1-3\alpha$	-2^9	${\mathfrak{p}_2'}^9$

Let $\varepsilon_1 = (1-\alpha)^2/(1+\alpha) = 7-6\alpha+\alpha^2-\alpha^3$. This is a unit whose vector of logarithms is given by $(2.5276, -4.7494, 1.1109)$. A second unit is given by $\varepsilon_2 = (1+\alpha)^4(1-\alpha)/(1-3\alpha) = 1-\alpha^2+\alpha^3$ whose vector of logarithms is equal to $(-6.2345, 0.1616, 3.0364)$. If the class group does not have order 4, then $\mathfrak{p}_3^2$ is principal, generated by $\beta \in O_F$ say. This means that $\beta^2 = 3\varepsilon$ for some $\varepsilon \in O_F^*$. In other words, the number 3 is equal to a unit modulo squares. We show now that the group generated by 3, -1, ε_1 and ε_2 modulo squares has $\mathbf{F}_2$-dimension 4.

It so happens that the signs of the images of these elements under σ are the same as the ones that one gets by applying σ'. Therefore we ignore σ' and consider the images of ε_1, ε_2, -1 and 3 in the group $\mathbf{R}^*/{\mathbf{R}^*}^2 \times \mathbf{F}_5^*/{\mathbf{F}_5^*}^2 \times \mathbf{F}_7^*/{\mathbf{F}_7^*}^2 \times \mathbf{F}_{13}^*/{\mathbf{F}_{13}^*}^2 \cong (\mathbf{Z}/2\mathbf{Z})^4$. Here we map the units to the first coordinate using the embedding σ. The other maps are given by reduction modulo the primes $\mathfrak{p}_5$, $\mathfrak{p}_7$ and $\mathfrak{p}_{13}$ respectively. The images of ε_1, ε_2, -1 and 3 are the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since the matrix is invertible, the group generated by -1, ε_1, ε_2 and 3 modulo ${O_F^*}^2$ has $\mathbf{F}_2$-dimension equal to 4. This implies that -1, ε_1, ε_2 generate O_F^* modulo squares and that 3 is not equal to a square times an element in O_F^*. It follows that $Cl(O_F)$ has order precisely 4.

From the relations given in Table III one finds that the ideals of norm ≤ 73 are distributed over the four ideal classes in the following way.

Table V.

class	
1	$\mathfrak{p}_2'$, $\mathfrak{p}_{47}$
c	$\mathfrak{p}_2$, $\mathfrak{p}_{19}$, $\mathfrak{p}_{23}$, $\mathfrak{p}_{29}'$, $\mathfrak{p}_{43}'$
c^2	$\mathfrak{p}_7$, $\mathfrak{p}_{23}'$, $\mathfrak{p}_{29}$, $\mathfrak{p}_{43}$
c^3	$\mathfrak{p}_3$, $\mathfrak{p}_4$, $\mathfrak{p}_5$, $\mathfrak{p}_{13}$, $\mathfrak{p}_{31}'$, $\mathfrak{p}_{31}$, $\mathfrak{p}_{41}$

Let U be the multiplicative group generated by the units -1, ε_1 and ε_2. Next we prove that O_F^* is equal to U.

Claim. If $O_F^* \neq U$, then $[O_F^* : U] \geq 17$.

Proof. We show that $[O_F^* : U]$ is not divisible by any prime number < 17. We already know that U generates O_F^* modulo squares, so that $[O_F^* : U]$ must be odd. Since ε_1 and

ε_2 are both congruent to -2 respectively modulo $\mathfrak{p}_7$, they are both equal to a non-trivial element in $\mathbf{F}_7^*$ modulo cubes. On the other hand, they are congruent to 7 and -3 modulo $\mathfrak{p}_{13}$ which are distinct in $\mathbf{F}_{13}^*/{\mathbf{F}_{13}^*}^3$. This shows that U generates O_F^* modulo cubes.

The 6-th powers of ε_1 and ε_2 are both congruent to 16 modulo $\mathfrak{p}'_{31}$, but they are distinct modulo $\mathfrak{p}_{31}$. This shows that $[O_F^* : U]$ is not divisible by 5. The 6-th powers of ε_1 and ε_2 are both congruent to 4 modulo $\mathfrak{p}_{43}$. Since they are congruent to 16 and 21 respectively modulo $\mathfrak{p}'_{43}$, the index $[O_F^* : U]$ is not divisible by 7. The units ε_1^2 and ε_2^2 are congruent to $18 = 3^9$ and $6 = 3^8$ respectively modulo $\mathfrak{p}_{23}$ and they are congruent to 3 and $16 = 3^6$ respectively modulo $\mathfrak{p}'_{23}$. Since the $\mathbf{F}_{11}$-matrix

$$\begin{pmatrix} 9 & 8 \\ 1 & 6 \end{pmatrix}$$

is invertible, the index $[O_F^* : U]$ is not divisible by 11.

Finally, we show that $[O_F^* : U]$ is not divible by 13. We have $\alpha \equiv 5 \pmod{\mathfrak{p}_{13}}$. By means of Hensel we find we have $\alpha \equiv 5 - f(5)/f'(5) \equiv 44 \pmod{\mathfrak{p}_{13}^2}$. The 12-th powers of ε_1 and ε_2 are congruent to 1 and 27 respectively in $O_F/\mathfrak{p}_{13}^2 \cong \mathbf{Z}/13^2\mathbf{Z}$. This means that with respect to the generator $1+13$ of the 1-dimensional $\mathbf{F}_{13}$-vector space $(1+13\mathbf{Z})/(1+13^2\mathbf{Z})$ the images are 0 and 2 respectively. To get a second coordinate we exploit the prime $\mathfrak{p}_{79}$. This is the unique prime of norm 79. Since $N(5+4\alpha) = 31 \cdot 79$, it appears in the factorization of $5+4\alpha$. One checks that the 6-th powers of ε_1 and ε_2 are congruent to 21 and $65 \equiv 21^9 \pmod{79}$ respectively in the residue field of $\mathfrak{p}_{79}$. This means that the images of ε_1 and ε_2 in the 2-dimensional $\mathbf{F}_{13}$-vector space $\mu_{13}(O_F/\mathfrak{p}_{13}^2) \times \mu_{13}(O_F/\mathfrak{p}_{79})$ are given by the columns of the $\mathbf{F}_{13}$-matrix

$$\begin{pmatrix} 0 & 1 \\ 2 & 9 \end{pmatrix}.$$

Since the matrix is invertible, the units ε_1 and ε_2 are independent modulo 13-th powers.

This proves the claim.

The vectors of logarithms of ε_1 and ε_2 have lengths 5.6048 and 7.5720 respectively. The cosine of the angle ϕ between them is -0.2304. It follows that the covolume of the rank 2 lattice generated by the vectors of logarithms of ε_1 and ε_2 is 41.2981.

By the claim, if $-1, \varepsilon_1, \varepsilon_2$ do *not* generated O_F^*, then the covolume of the logarithmic unit lattice is at most $\frac{1}{17} \cdot 41.2981 = 2.4293$. By Minkowski, a disk centered in the origin with radius squared at least $\frac{4}{\pi} \cdot 2.4293 = 3.0930$ contains a non-zero lattice point. This means that there exists a unit $\eta \neq \pm 1$ for which

$$(\log|\sigma(\eta)|)^2 + (\log|\sigma'(\eta)|)^2 + 2(\log|\sigma''(\eta)|)^2 \;\leq\; 3.0930.$$

It follows that both $|\sigma(\eta)|$ and $|\sigma'(\eta)|$ are at most $e^{\sqrt{3.0930}} = 5.8049$, while $|\sigma''(\eta)|$ is at most $e^{\sqrt{3.0930/2}} = 3.4680$. Replacing η by its inverse if necessary, we may assume that $|\sigma(\eta)| < 1$. It follows that that

$$\|\eta\|^2 \;=\; \sigma(\eta)^2 + \sigma'(\eta)^2 + 2|\sigma''(\eta)|^2 \;\leq\; 1 + 5.8049^2 + 2 \cdot 3.4680^2 < 58.7531.$$

Since $O_F = \mathbf{Z}[\alpha]$, every $x \in O_F$ is equal to $k + l\alpha + m\alpha^2 + n\alpha^3$ for unique integers k, l, m and n. In terms of these coefficients we find that $\|x\|^2 = \sigma(x)^2 + \sigma'(x)^2 + 2|\sigma''(x)|^2$ is given by the positive quadratic form

$$\begin{aligned}\|x\|^2 \;&=\; 4k^2 - 24km + 18kn + 15.8644\, l^2 + -3.2722\, lm - 190.1235\, ln + 100.0599, m^2 \\ &\quad - 34.1395\, nm + 689.0786\, n^2 \\ &= 4(k - 3m + \tfrac{9}{4}n)^2 + 15.8644(l - 0.1031\, m - 5.9921\, n)^2 + 63.8911(m + 0.0019\, n)^2 \\ &\quad + 99.2057\, n^2\end{aligned}$$

Applying this with $\eta = x = k + l\alpha + m\alpha^2 + n\alpha^3$, we see that $n = m = 0$ and hence $\eta = k + l\alpha$ so that

$$\|\eta\|^2 \;=\; 4k^2 + 15.8644\, l^2 \;<\; 58.7531.$$

This implies that $|l| \leq 1$. Since $\eta \notin \mathbf{Z}$ and replacing η by $-\eta$ if necessary, we may assume that $l = 1$, so that $\eta = k + \alpha$. Since $\sigma(\alpha) = -0.7542$ and $|\sigma(\eta)| < 1$, we must have $k = 0$ or 1 and hence $\eta = \alpha$ or $\alpha + 1$. But this is impossible because α and $\alpha + 1$ have norms 6 and 4 respectively and hence are not units. It follows that O_F^* is generated by -1, ε_1 and ε_2.

The residue of the zeta function of F is equal to

$$\frac{2^2 \cdot 2\pi \cdot h_F R_F}{2\sqrt{402219}} \;=\; 2.3144849573731740014207 05655\ldots$$

This follows from the fact that $h_F = \#Cl(O_F) = 4$ and the regulator R_F is given by

$$R_F \;=\; |\det \begin{pmatrix} \log|\sigma(\epsilon_1)| & \log|\sigma'(\epsilon_1)| \\ \log|\sigma(\epsilon_2)| & \log|\sigma'(\epsilon_2)| \end{pmatrix}| \;=\; 29.2022152689660535956766 0481\ldots$$

On the other hand, a direct evaluation of the Euler product using the primes $p < 1000000$ gives the value 2.31427982144190371594943 3572. The relative error is about 0.0001, which is quite reasonable.