

1. (a) Show that the ring of integers of $\mathbf{Q}(i)$ is $\mathbf{Z}[i]$.
- (b) Show that $\mathbf{Z}[i]$ is a Euclidean domain and hence a PID.
- (c) Factor all prime numbers p into products of prime ideals in $\mathbf{Z}[i]$ and show

$$\begin{aligned}(2) &= (1+i)^2, \\ (p) &= (p), \quad \text{if } p \equiv 3 \pmod{4}; \\ (p) &= (\pi)(\bar{\pi}), \quad \text{if } p \equiv 1 \pmod{4}.\end{aligned}$$

In the case $p \equiv 1 \pmod{4}$ we have $\pi = a + bi$ with $a, b \in \mathbf{Z}$ satisfying $a^2 + b^2 = p$.

2. Let F be a number field of degree n and let $\alpha \in O_F$ so that $F = \mathbf{Q}(\alpha)$.
 - (a) Show that the index $m = [O_F : \mathbf{Z}[\alpha]]$ is finite.
 - (b) Show that $\Delta(1, \alpha, \alpha^2, \dots, \alpha^{n-1}) = m^2 \Delta_F$.
 - (c) Show that $O_F = \mathbf{Z}[\alpha]$ when $\Delta(1, \alpha, \alpha^2, \dots, \alpha^{n-1})$ is a squarefree integer.
3. Let $f = X^3 - X + 1$ and let $F = \mathbf{Q}(\alpha)$ where α denotes a zero of f . Let $\alpha_1, \alpha_2, \alpha_3$ denote the complex zeroes of f .
 - (a) Show that f is an irreducible element of $\mathbf{Q}[X]$.
 - (b) Show that $\text{Tr}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\text{Tr}(\alpha^2) = \text{Tr}(\alpha)^2 - 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) = 0 - 2 \cdot (-1) = 2$.
 - (c) Show that $\text{Tr}(\alpha^{k+3}) = \text{Tr}(\alpha^{k+1}) + \text{Tr}(\alpha^k)$ for every $k \in \mathbf{Z}$.
 - (d) Compute the discriminant $\Delta(1, \alpha, \alpha^2)$.
 - (e) Show that $O_F = \mathbf{Z}[\alpha]$.
4. Let $\alpha \in \mathbf{R}$ and let X be a (large) positive real number. Prove that for every $X > 0$ there exist integers x, y with $x \neq 0$ and $|\alpha - \frac{y}{x}| < \frac{1}{xX}$. (Hint: Consider $\{(x, y) \in \mathbf{R}^2 : |x| \leq X + \frac{1}{2} \text{ and } |y - \alpha x| < \frac{1}{X}\}$.)
5. (Jensen's inequality) Let $n \geq 1$, let x_1, \dots, x_n in the interval $[a, b] \subset \mathbf{R}_{>0}$ and let $\lambda_1, \dots, \lambda_n$ be 'weights' in the interval $(0, 1)$ satisfying $\sum_{i=1}^n \lambda_i = 1$. Suppose f is a convex function on the interval $[a, b]$. A function is called convex on $[a, b]$ if for each $x, x' \in [a, b]$ we have $f(\frac{x+x'}{2}) \leq \frac{f(x)+f(x')}{2}$.
 - (a) Show that we have $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$.
 - (b) Apply (a) to the convex function " $-\log$ " and prove the arithmetic-geometric mean inequality

$$\left(\prod_{i=1}^n x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n x_i,$$

with equality if and only if all x_i are equal.

6. Let $\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}$ denote Euler's gamma function.
 - (a) Prove that the integral converges absolutely for $s \in \mathbf{C}$ with $\text{Re}(s) > 0$.
 - (b) Show that $s\Gamma(s) = \Gamma(s+1)$ for $s \in \mathbf{C}$ with $\text{Re}(s) > 0$.
 - (c) Show that $\Gamma(n+1) = n!$ for $n \in \mathbf{Z}_{\geq 0}$.
 - (d) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.