1. Let $H = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} \mathbf{Z}$. So, H is the subgroup of $\mathbf{Z}^2$ generated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Prove that $\mathbf{Z}^2/H$ is a cyclic group of order 15. Exhibit a vector $\mathbf{v} \in \mathbf{Z}^2$ whose image generates $\mathbf{Z}^2/H$.

2. Let $H = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \mathbf{Z}$. Determine the structure of the group $\mathbf{Z}^2/H$. Is it cyclic?

3. Let $H = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} \mathbf{Z}$. Determine the structure of the group $\mathbf{Z}^2/H$. Is it cyclic?

4 Let $H \subset \mathbf{Z}^3$ be the subgroup generated by $(1,1,1)$, $(0,1,1)$ and $(-1,2,3)$ and let A be the matrix $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$. Show that $A(H)$ is a lattice in $\mathbf{R}^3$ and compute its covolume.

5. Let $L = \{(x,y,z) \in \mathbf{Z}^3 : 2x + 3y + 4z \equiv 0 \pmod 7\}$. Show that $L \subset \mathbf{R}^3$ is a lattice. Find a $\mathbf{Z}$-basis of L and calculate its covolume.

6. Show that $\mathbf{Q}^*_{>0}$ and the additive group of the ring $\mathbf{Z}[T]$ are isomorphic groups.

7. Let F be a number field and let I be a fractional ideal of F. Show that there is a positive integer m such that mI is an ideal of the ring of integers O_F.

8. Let $F = \mathbf{Q}(\sqrt{-10})$.
   (a) Show that O_F is equal to $\mathbf{Z}[\sqrt{-10}]$.
   (b) Show that $10 = 2 \cdot 5 = -\sqrt{-10}^2$ are two factorizations of 6 into products of irreducible elements of O_F.
   (c) Show that $\mathfrak{p} = (2, \sqrt{-10})$ is a prime ideal of norm 2. Show that $\mathfrak{p}^2 = (2)$.
   (d) Show that $\mathfrak{q} = (5, \sqrt{-10})$ is a prime ideal of norm 5. Show that $\mathfrak{q}^2 = (5)$.
   (e) Factor the ideal (10) into a product of prime ideals. Show that the ideal $(\sqrt{-10})$ is equal to $\mathfrak{p}\mathfrak{q}$.

9. Let $F = \mathbf{Q}(\sqrt{10})$.
   (a) Show that O_F is equal to $\mathbf{Z}[\sqrt{10}]$.
   (b) Show that $\mathfrak{p} = (2, \sqrt{10})$ is a prime ideal of norm 2. Show that $\mathfrak{p}^2 = (2)$.
   (c) Show that $\mathfrak{p}$ is not principal. (Hint. there are no elements in O_F with norm ± 2)

10. Let $F = \mathbf{Q}(\sqrt{7})$.
   (a) Show that O_F is equal to $\mathbf{Z}[\sqrt{7}]$.
   (b) For all prime numbers $p < 10$, factor the O_F-ideal (p) into a product of prime ideals.