

1. Show: If A is a Noetherian ring and $I \subset A$ is an ideal, then A/I is also Noetherian.
2. Show that every unique factorization domain is integrally closed.
3. Let R be a Dedekind domain. Let \mathfrak{p} and \mathfrak{q} be two distinct non-zero prime ideals of R .
 - (a) Show that \mathfrak{p} and \mathfrak{q} are coprime. In other words, show $\mathfrak{p} + \mathfrak{q} = R$.
 - (b) Let $m \geq 1$. Show that $\mathfrak{p}^m + \mathfrak{q}^m = R$.
- 4 (Hilbert's Basissatz) Let A be a Noetherian ring and let $I \subset A[X]$ be an ideal. For every $n \geq 0$ let $J_n \subset A$ be the set containing 0 and the leading coefficients of polynomials in I of degree at most n .
 - (a) Show that the J_n form an ascending sequence of A -ideals

$$J_1 \subset J_2 \subset J_3 \subset \dots$$

- (b) Show that there exists $N \geq 0$ be such that $J_n = J_N$ for all $n \geq N$.
 - (c) Show that for each n there exists a finite subset $S_n \subset I$ of polynomials of degree at most n , whose leading coefficients generate J_n .
 - (d) Show that the finite set $\bigcup_{n \leq N} S_n$ generates the $A[X]$ -ideal I and conclude that $A[X]$ is Noetherian.
5. Show that for every $m \geq 1$ the ring $\mathbf{Z}[X_1, X_2, \dots, X_m]$ is Noetherian. Show that for every field k and every $m \geq 1$ the ring $k[X_1, X_2, \dots, X_m]$ is Noetherian.
6. Let I and J be two fractional ideals of a Dedekind domain R .
 - (a) Show that $I \cap J$ and $I + J = \{x \in R : x = a + b \text{ for some } a \in I \text{ and } b \in J\}$ are fractional ideals.
 - (b) Show that $I^{-1} + J^{-1} = (I \cap J)^{-1}$ and that $I^{-1} \cap J^{-1} = (I + J)^{-1}$.
 - (c) Show that $I \subset J$ if and only if $J^{-1} \subset I^{-1}$.
7. Let A be a Dedekind domain and let S be a finite set of non-zero prime ideals of A . For each $\mathfrak{p} \in S$ let $m_{\mathfrak{p}} \in \mathbf{Z}$. Show that there exists an element $a \in A$ for which $v_{\mathfrak{p}}(a) = m_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$.
8. Show that every ideal of a Dedekind domain can be generated by only two elements.
9. Let $A \subset B$ be commutative rings. An element $b \in B$ is said to be integral over A if there exists a monic polynomial $f \in A[X]$ with $f(b) = 0$. Show that b is integral over A if and only if there exists a non-zero finitely generated subgroup M of the additive group of B for which $bM \subset M$.
10. Let F be a number field with $r_1 \geq 1$. In other words, F admits an embedding into \mathbf{R} . Show that the only roots of unity in F are ± 1 .