Algebraic Number Theory 2. Discriminants, rings of integers. Nanjing, March 20, 2026.

1. Let p be a prime. Compute the minimum polynomial of a primitive p-th root of unity ζ_p. Show that $[\mathbf{Q}(\zeta_p):\mathbf{Q}] = p-1$.

2. Let $F = \mathbf{Q}(\sqrt{2}, \sqrt{-6})$.
   (a) Compute the degree of F over $\mathbf{Q}$.
   (b) Determine a basis of F as a $\mathbf{Q}$-vector space.
   (c) Find an element $\alpha \in F$ such that $F = \mathbf{Q}(\alpha)$.
   (d) Prove that $\sqrt{-3} \in F$.
   (e) Compute the discriminant $\Delta(1, \sqrt{2}, \sqrt{-3}, \sqrt{-6})$.
   (f) Compute the discriminant $\Delta(1, \sqrt{2}, \sqrt{-3}, \sqrt{2}+\sqrt{-3})$.

3. Let F be a number field.
   (a) Prove that the fraction field of O_F is equal to F.
   (b) Sia $K \subset F$ be a subfield. Prove that $O_F \cap K = O_K$.

4. Let $F = \mathbf{Q}(\alpha)$ be a number field of degree n and let O_F be its ring of integers.
   (a) Prove that the additive group $\mathbf{Z}[\alpha]$ has finite index in O_F.
   (b) Prove that $m^2\Delta_F = \Delta(1, \alpha, \ldots, \alpha^{n-1})$ for some non-zero $m \in \mathbf{Z}$.
   (c) Prove: if $\Delta(1, \alpha, \ldots, \alpha^{n-1})$ is squarefree then $O_F = \mathbf{Z}[\alpha]$.

5. Let $f(X) = X^3 - X - 1$.
   (a) Prove that f is irreducible in $\mathbf{Q}[X]$.
   Let α denote a zero of f and let $F = \mathbf{Q}(\alpha)$.
   (b) Compute the discriminant $\Delta(1, \alpha, \alpha^2)$.
   (c) Prove that the ring of integers of F is $\mathbf{Z}[\alpha]$.

6. Prove that the ring of continuous functions $[0,1] \longrightarrow \mathbf{R}$ is not Noetherian.

7. Prove that every principal ideal domain is a Dedekind domain.

8. Let $A = \mathbf{Z}[\sqrt{-6}]$.
   (a) Show that A is the ring of integers of $F = \mathbf{Q}(\sqrt{-6})$.
   (b) Let I be the O_F-ideal $(2, \sqrt{-6})$. Show that I is not principal, but I^2 is.

9. (a) Show that $25 = 5^2$ e $25 = (1+2\sqrt{-6})(1-2\sqrt{-6})$ are factorizations of 25 into a product of irreducible elements of the ring $\mathbf{Z}[\sqrt{-6}]$.
   (b) Show that the factorizations are *distinct* in the sense that the irreducible factors are different, also up to a unit of $\mathbf{Z}[\sqrt{-6}]$.
   (c) Write the ideals (5), $(1+2\sqrt{6})$ e $(1-2\sqrt{6})$ as products of prime ideals of the Dedekind domain $\mathbf{Z}[\sqrt{-6}]$.

10. Let R be the ring $\mathbf{Z}[2i] = \mathbf{Z}[X]/(X^2+4)$. Prove that R is not integrally closed.