1. Let $F = \mathbf{Q}(\sqrt[3]{2})$ and let $x = 1 + \sqrt[3]{2}$. Compute the characteristic polynomial of x. Compute the norm and trace of x.

2. Let F be a number field, let K be any field and let $\phi : F \longrightarrow K$ be a ring homomorphism. Show that $\phi(q) = q$ for all $q \in \mathbf{Q}$.

3. Let F be a number field of degree n.
   (a) Let $q \in \mathbf{Q}$. Show that $Tr(q) = nq$ and $N(q) = q^n$. Show that for every $x \in F$ we have $Tr(qx) = qTr(x)$.
   (b) Show that the homomorphism $Tr : F \longrightarrow \mathbf{Q}$ is surjective. Show that the homomorphism $N : F^* \longrightarrow \mathbf{Q}^*$ is not surjective in general.

4. Let F be the number field $\mathbf{Q}(\sqrt[6]{5})$. How many ring homomorphisms $F \to \mathbf{C}$ are there? How many are there up to complex conjugation?

5. Let F be a number field of degree n, let $x \in F$ and let $f \in \mathbf{Q}[X]$ be the characteristic polynomial of x.
   (a) Prove that for every $q \in \mathbf{Q}$ the norm of $q - x$ is equal to $f(q)$.
   (b) Prove that for every $q, r \in \mathbf{Q}$, the norm of $q - rx$ is equal to $r^n f(\frac{q}{r})$.

6. Let $F = \mathbf{Q}(\alpha)$ be a number field and let $f \in \mathbf{Q}[X]$ be the minimum polynomial of α.
   (a) Prove that the ring homomorphism $\mathbf{Q}[X] \to F$ given by $h(X) \mapsto h(\alpha)$ induces a ring isomorphism $\mathbf{Q}[X]/(f) \cong F$.
   (b) Show that the natural homomorphisms $F \to F_{\mathbf{R}} \to F_{\mathbf{C}}$ are injective. Here we put $F_{\mathbf{R}} = \mathbf{R}[X]/(f)$ and $F_{\mathbf{C}} = \mathbf{C}[X]/(f)$. By (f) we denote the ideal generated by f.

7. (*VanderMonde*) Let R be a commutative ring with 1 and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$. Prove the equality

$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \ldots & \alpha_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i).$$

8. Let F be the number field $\mathbf{Q}(\alpha)$ and let $n = [F : \mathbf{Q}]$. Compute the discriminant $\Delta(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$ in terms of the zeroes of the minimum polynomial of α. (Hint: use Exercise 7)

9. (*The characteristic polynomial depends on the field F*) Let $F = \mathbf{Q}(\sqrt[4]{2})$ and let F' denote its subfield $\mathbf{Q}(\sqrt{2})$. Here $\sqrt{2} = \sqrt[4]{2}^2$.
   (a) Compute the characteristic polynomial of $\sqrt{2}$ viewed as an element of F.
   (b) Compute the characteristic polynomial of $\sqrt{2}$ viewed as an element of F'.
   (c) Verify that the polynomial in (a) is the square of the polynomial in (b).