Let F be a number field of degree $n \geq 1$ and discriminant Δ_F. We write δ for the covolume $\sqrt{|\Delta_F|}$ of $O_F \subset F_{\mathbf{R}}$. We let Φ denote the set of ring homomorphisms $\phi : F \hookrightarrow \mathbf{C}$. There are $n = [F : \mathbf{Q}]$ of them. By $\overline{\phi}$ we denote the complex conjugate of $\phi \in \Phi$. The $\mathbf{C}$-algebra $F_{\mathbf{C}}$ is isomorphic to $\prod_\phi \mathbf{C}$. Its elements are vectors with coordinates $z_\phi \in \mathbf{C}$ for $\phi \in \Phi$. We denote them by (z_ϕ). The image of the ring homomorphism $F \hookrightarrow F_{\mathbf{C}}$ given by $x \mapsto (\phi(x))$ is contained in the $\mathbf{R}$-subalgebra of $F_{\mathbf{C}}$ given by

$$F_{\mathbf{R}} \;=\; \{(z_\phi) \in F_{\mathbf{C}} : z_{\overline{\phi}} = \overline{z_\phi} \text{ for all } \phi \in \Phi\}.$$

The $\mathbf{R}$-dimension of $F_{\mathbf{R}}$ is $n = r_1 + 2r_2$, where r_1 denotes the number of homomorphisms $\phi : F \longrightarrow \mathbf{C}$ with $\phi(F) \subset \mathbf{R}$ and $2r_2$ is the number of homomorphisms that do not have this property. The elements of $F_{\mathbf{R}}$ that are invariant under its canonical involution form a subalgebra. It is given by

$$F_{\mathbf{R}}^+ \;=\; \{(z_\phi) \in F_{\mathbf{R}} : z_\phi \in \mathbf{R} \text{ for all } \phi \in \Phi\}.$$

The $\mathbf{R}$-dimension of $F_{\mathbf{R}}^+$ is $r_1 + r_2$.

We let $F_{\mathbf{R}}^\perp$ denote the orthogonal complement in $F_{\mathbf{R}}^+$ of the subspace $\mathbf{R} \cdot 1$. It is an $\mathbf{R}$-subspace of dimension $r_1 + r_2 - 1$ and it is equal to the kernel of the trace map. In other words we have

$$F_{\mathbf{R}}^\perp \;=\; \{x = (x_\phi) \in F_{\mathbf{R}}^+ : Tr(x) = \sum_{\phi \in \Phi} x_\phi = 0\}.$$

For any $x \in F_{\mathbf{R}}^0$ we let $\pi(x)$ denote the orthogonal projection of x on $F_{\mathbf{R}}^+$. It is given by

$$\pi(x) \;=\; x - \frac{Tr(x)}{n} \cdot 1.$$

Let

$$\mathrm{Log} : F_{\mathbf{R}}^* \;\longrightarrow F_{\mathbf{R}}^+$$

be the homomorphism that associates to $(x_\phi) \in F_{\mathbf{R}}^*$ the element $(\log|x_\phi|) \in F_{\mathbf{R}}^+$. For any $x \in F_{\mathbf{R}}^*$, we let $\lambda(x)$ denote the orthogonal projection of $\mathrm{Log}(x) \in \prod_{\phi \in \Phi} \mathbf{R}$ on $F_{\mathbf{R}}^0$. Explicitly we have

$$\lambda(x) \;=\; \pi(\mathrm{Log}(x)) \;=\; \mathrm{Log}(x) - \frac{Tr(\mathrm{Log}\, x)}{n} \cdot 1 \;=\; \mathrm{Log}|x/N(x)^{1/n}|.$$

In this note we study the subgroup

$$L \;=\; \mathrm{Log}(O_F^*)$$

of $F_{\mathbf{R}}^+$. Any $\varepsilon \in O_F^*$ has norm 1. Therefore we have

$$\sum_{\phi \in \Phi} \log|\phi(\varepsilon)| = 0,$$

and hence we have $L \subset F^0_{\mathbf{R}}$. Since O_F is discrete in $F_{\mathbf{R}}$, so is L in $F^0_{\mathbf{R}}$. In this note we prove that L is actually a lattice in $F^0_{\mathbf{R}}$. In other words, the quotient group $F^0_{\mathbf{R}}/L$ is compact This is a form of Dirichlet's Unit Theorem.

There are only finitely many non-zero principal ideals of O_F with norm at most δ. Let $\alpha_1, \ldots, \alpha_t \in O_F$ be generators of these ideals. For each $i = 1, \ldots, t$ let B_i denote the closed ball with center $\lambda(\alpha_i)$ and radius $\log\delta$:

$$B_i = \{z \in F^0_{\mathbf{R}} : \|z - \lambda(\alpha_i)\| \leq \log\delta\}.$$

Theorem. *We have*

$$F^0_{\mathbf{R}} = \{y + z : y \in L \text{ and } z \in B_i \text{ for some } i = 1, \ldots, t\}.$$

Since each ball B_i is compact and since the composite map

$$\bigcup_{i=1}^{t} B_i \hookrightarrow F^0_{\mathbf{R}} \longrightarrow F^0_{\mathbf{R}}/L$$

is continuous and surjective, the theorem implies that $F^0_{\mathbf{R}}/L$ is compact and we are done.

Proof of the Theorem. Let $x = (x_\phi)$ be an arbitrary element of $F^0_{\mathbf{R}}$. By e^x we denote the element (e^{x_ϕ}) of $F^*_{\mathbf{R}}$. The box

$$B = \{(z_\phi) \in F_{\mathbf{R}} : |z_\phi| \leq \delta^{1/n} \text{ for all } \phi \in \Phi\}.$$

is convex and symmetric and has has volume $\geq 2^n\delta$. Since the trace of x is zero, the norm of e^x is 1. It follows that the volume of the box $e^x B \subset F_{\mathbf{R}}$ is equal to the volume of B and is hence at least $2^n\delta$.

By Minkowski's theorem there exists therefore a non-zero vector in $O_F \cap e^x B$. In other words, there is a non-zero $a \in O_F$ for which

$$|\phi(a)| \leq e^{x_\phi}\delta^{1/n}, \qquad \text{for all } \phi \in \Phi.$$

This implies that $|N(a)| \leq \delta$. Therefore the principal ideal generated by a is one of the principal ideals (α_i) with $1 \leq i \leq t$ that were introduced above. This means that $a = \varepsilon\alpha_i$ for some unit $\varepsilon \in O^*_F$. Now we have

$$x = y + z,$$

with $y = \mathrm{Log}(\varepsilon) = \lambda(\varepsilon)$. To show that $z = x - \mathrm{Log}(\varepsilon)$ is in B, we check that $\|z - \lambda(a_i)\|$ is at most $\log\delta$. We have $z - \lambda(\alpha_i) = x - \lambda(a)$ and the inequalities

$$\lambda(a)_\phi = \log\left|\frac{\phi(a)}{N(a)^{1/n}}\right| \leq \log|\phi(a)| \leq x_\phi + \frac{1}{n}\log\delta, \qquad \text{for all } \phi \in \Phi.$$

Here we used the fact that $|N(a)| \geq 1$. Since the sum over all ϕ of $\lambda(a)_\phi$ and of x_ϕ is zero, the lemma below applies to the real numbers $\lambda(a)_\phi - x_\phi$. We get

$$\|z - \lambda(a_i)\|^2 = \|\lambda(a) - x\|^2 = \sum_{\phi\in\Phi}(\lambda(a)_\phi - x_\phi)^2 \leq \frac{n(n-1)}{n^2}\log^2\delta < \log^2\delta,$$

as required.

Lemma. *Let $n > 1$ and $M \in \mathbf{R}$. Suppose that $x_1, \ldots, x_n \in \mathbf{R}$ satisfy*

$$x_i \leq M \text{ for } i = 1, \ldots, n \qquad \text{and} \qquad \textstyle\sum_{i=1}^{n} x_i \geq 0.$$

Then we have $\sum_{i=1}^{n} x_i^2 \leq n(n-1)M^2$.

Proof. We may assume $M > 0$. The set S of $(x_1, \ldots, x_n) \in \mathbf{R}^n$ that satisfy both conditions is bounded. Let $(x_1, \ldots, x_n) \in S$ be an element for which $s = \sum_{i=1}^{n} x_i^2$ is maximal. Then at most one of the elements x_i is negative, because if $x_i < 0$ and $x_j < 0$, we replace x_i by 0 and x_j by $x_i + x_j$. Then the conditions are still satisfied, but since $(x_i + x_j)^2 > x_i^2 + x_j^2$, the sum s would be strictly larger. In addition, every $x_i \geq 0$ is equal to M. Indeed, if we replace a non-negative $x_i < M$ by M, the conditions are satisfied, but s would get larger.

So all x_i are equal to M except possibly one, say $x_1 = tM$ for some $t \in \mathbf{R}$. The conditions say that $1 - n \leq t \leq 1$. Since $s = ((n-1) + t^2)M^2$ is maximal, we have $t = 1 - n$ and the result follows. (For $n = 2$ we could also have $t = 1$; it gives the same estimate)