Lecture 2: Quasi-plurisubharmonic functions

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Brief recap on plurisubharmonic functions

- A function $u : \Omega \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is psh if it is usc and for all z,

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  $\hookrightarrow$ psh functions are "exponentially integrable" [Skoda].

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We endow $PSH(X, \omega)$ with the L^1-topology.

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More generally if $L \to X$ is a positive holomorphic line bundle with metric $h = e^{-\phi}$ of curvature $\omega = \Theta_h = dd^c\phi > 0$ and if $s \in H^0(X,L)$, then

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$$(\omega + dd^c_z G)^n = \delta_w.$$

Compactness properties

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For all $A \geq 0$*, the sets*

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  $\hookrightarrow$ Can normalize either by $\sup_X \varphi = 0$ or by $\int_X \varphi dV = 0$.

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The proof uses Hörmander's L^2 techniques for solving the $\overline{\partial}$-equation.

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$\hookrightarrow$ These envelopes play a central role in forthcoming lectures.

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Proposition

For all $\varphi \in PSH(X, \omega)$ and $x \in X$, $\nu(\varphi, x) \leq M = M(X, \{\omega\})$.

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- Uniform control on (u, x_0) and compactness $\Rightarrow$ uniform integrability ?

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$\hookrightarrow$ Sketch of proof on the "white board".

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