Lecture 1: Compact Kähler manifolds

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- Lecture 1: a panoramic view of compact Kähler manifolds.
- Lecture 2: uniform integrability properties of quasi-psh functions.

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- Such a form is associated to a Riemannian metric on TX.

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Example (Hopf surface)

The surface $X = \mathbb{C}^2/\langle z \mapsto 2z\rangle \sim S^1 \times S^3$ *does not admit any Kähler form.*

Basic constructions

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- $\Rightarrow$ any compact Riemann surface is Kähler.

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- The blow up of a Kähler manifold is a Kähler manifold.

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$\hookrightarrow$ Not usually possible globally (max principle), but...

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- $PSH(X,\omega)$ =the closure of $\mathcal{K}_\omega$ in L^1 will be the hero of Lecture 2.

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$\hookrightarrow$ Picard group. In the sequel $L^j := L \otimes \cdots \otimes L$ (j times).

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- The hyperplane bundle is very ample.

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"Proof": L^j very ample $\Rightarrow L^j = \phi_{L^j}^* \mathcal{O}(1)$ has a Fubini-Study type metric.

Curvature of holomorphic line bundles

Given $s = \{s_U\} \in H^0(X, L)$, can measure the size of $s(x) \in L_x$ by using metric $h = \{h_U\}$ with $h_U = e^{-\varphi_U}$ st $\varphi_V = \varphi_U + \log|g_{UV}|$ setting

$$|s|_h(x) := |s_U(x)|e^{-\varphi_U(x)} = |s_V(x)|e^{-\varphi_V(x)}.$$

Definition

The curvature of the metric h is $\Theta_h := i\partial\overline{\partial}\varphi_U = i\partial\overline{\partial}\varphi_V$*. A line bundle is positive if it admits a smooth metric whose curvature is a Kähler form.*

Theorem (Kodaira embeding theorem)

A cpct complex manifold X is projective iff it admits a positive line bundle. In other words: $L \to X$ is positive iff it is ample.

"Proof": L^j very ample $\Rightarrow L^j = \phi_{L^j}^*\mathcal{O}(1)$ has a Fubini-Study type metric. Converse more delicate, can be proved by Hörmander's L^2 techniques. □

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The Chern class $c_1(L)$ *is the image* $L \in H^1(X, \mathcal{O}^*) \mapsto c_1(L) \in H^2(X, \mathbb{Z})$ *under the map induced by the exact sequence* $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.

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Definition

The first Chern class of X is $c_1(X) = c_1(-K_X)$.

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Definition (-Proposition)

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$\hookrightarrow$ Seeking for canonical Kähler metrics might help (next Lectures).

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The cohomology class of $\mathrm{Ric}(\omega)$ *is* $c_1(X) = -c_1(K_X)$.

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$\hookrightarrow$ Constructing K-E metrics is a main goal of these lectures.

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$\hookrightarrow$ Many more (non explicit) examples in next Lectures.

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