Minimum Energy Broadcast and Disk Cover in Grid Wireless Networks ⋆

Tiziana Calamoneri [a] Andrea E. F. Clementi [b]
Miriam Di Ianni [b] Massimo Lauria [a] Angelo Monti [a]
Riccardo Silvestri [a]

[a] *Dipartimento di Informatica, Università degli Studi di Roma "La Sapienza".*

[b] *Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata".*

Abstract

The *Minimum-Energy Broadcast* problem is to assign a transmission range to every station of an ad-hoc wireless networks so that *(i)* a given source station is allowed to perform broadcast operations and *(ii)* the overall energy consumption of the range assignment is minimized.

We prove a nearly tight asymptotical bound on the optimal cost for the Minimum-Energy Broadcast problem on square grids. We also derive near-tight bounds for the *Bounded-Hop* version of this problem. Our results imply that the best-known heuristic, the MST-based one, for the Minimum-Energy Broadcast problem is far to achieve optimal solutions (even) on very regular, well-spread instances: its worst-case approximation ratio is about π and it yields $\Omega(\sqrt{n})$ hops, where n is the number of stations.

As a by product, we get nearly tight bounds for the *Minimum Disk Cover* problem and for its restriction in which the allowed disks must have *non-constant* radius.

Finally, we emphasize that our upper bounds are obtained via polynomial time constructions.

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Email addresses: `calamo@di.uniroma1.it` (Tiziana Calamoneri), `clementi@mat.uniroma2.it` (Andrea E. F. Clementi), `diianni@mat.uniroma2.it` (Miriam Di Ianni), `lauria@di.uniroma1.it` (Massimo Lauria), `monti@di.uniroma1.it` (Angelo Monti), `silver@di.uniroma1.it` (Riccardo Silvestri).

1 Introduction

An *ad-hoc* wireless network consists of a set S of radio stations connected by wireless links. We assume that stations are located on the Euclidean plane. A transmission range is assigned to every station: a *range assignment* $r : S \rightarrow R$ determines a directed *communication graph* $G(S, E)$ where edge $(i, j) \in E$ if and only if $\texttt{dist}(i, j) \leq r(i)$ where $\texttt{dist}(i, j)$ is the Euclidean distance between i and j. In other words, $(i, j) \in E$ if and only if j belongs to the *disk* of radius $r(i)$ centered at i. The transmission range of a station depends on the energy power supplied to the station. In particular, the power P_s required by a station s to transmit data to another station t must satisfy the inequality

$$\frac{P_s}{\texttt{dist}(s,t)^{\alpha}} \geq 1$$

where $\alpha \geq 1$ is the *distance-power gradient.* In this paper, we consider the case $\alpha = 2$ that holds in the empty space (see [19]).

Stations of an ad hoc network cooperate in order to provide specific network connectivity properties by adapting their transmission ranges. A *Broadcast Range Assignment* (for short *Broadcast*) is a range assignment that yields a communication graph G containing a directed spanning tree rooted at a given source station s. A fundamental problem in the design of ad hoc wireless networks is the *Minimum-Energy Broadcast* Problem (for short *Minimum Broadcast*): it consists in finding a Broadcast of minimal *overall energy power* [6,9,17]. A range assignment r can be represented by the corresponding family $\mathcal{D} = \{D_1, \ldots, D_\ell\}$ of disks, and its overall energy power (i.e. $\mathsf{cost}(\mathcal{D})$) is defined as

$$\mathsf{cost}(\mathcal{D}) = \sum_{i=1}^{\ell} r_i^2 \text{ where } r_i \text{ is the radius of } D_i \tag{1}$$

The Minimum Broadcast problem is known to be NP-hard [4] and the best-known approximation algorithm is the MST-based heuristic [1,9]. The MST-based heuristic first computes the minimum spanning tree of the complete graph induced by S where the weight of edge (i, j) is $\texttt{dist}(i, j)^2$. Then, it assigns a direction to the edges from the source s to the leaves; finally, it assigns to each node i a range equal to the length of the longest edge outgoing from i. This heuristic is efficient and easy to implement, so its worst-case approximation analysis has been the subject of several works over the last five years. In particular, the first *constant* upper bound ($\simeq 40$) on the approximation ratio was determined in [4]. A rather sophisticated analysis, recently introduced in [1], yields the *tight* upper bound 6. The tightness follows from the lower bound proved in [3,9] by considering rather *artificial* input configurations.

The worst-case analysis is often not sufficient to evaluate the practical interest of a heuristic. It might be the case that the MST-based heuristic provides *nearly* optimal solutions *for most* of natural and practically-relevant instances. Recently, experimental studies have been presented on this issue [10,5,9].

1.1 Our results

Minimum Broadcast Problem. In this paper, we address the above issue by adopting an analytical approach: we consider Minimum Broadcast and some other related problems on *square grids*. Square grids have been often considered in wireless networks since they model some *well-spread*, practically relevant ad hoc network topologies [7,18,19]. One can see that the MST-based heuristic, on a square *grid* of n points (without loss of generality, adjacent points are placed at unit distance), returns, *in the worst-case*, a solution of cost $n-1$. Furthermore, in [10] it is experimentally observed that the MST-based heuristic has bad behavior when applied on regular instances similar to square grids. This motivates a theoretical analysis of the Minimum Broadcast problem on grid networks. Our first contribution is the following result.

Theorem [Broadcast]. *If $\mathcal{B}^*$ is any optimal Broadcast for the square grid $\mathcal{G}$ of n points, then*

$$\frac{n}{\pi} - O(\sqrt{n}) \ \leq \ \mathsf{cost}(\mathcal{B}^*) \ \leq \ 1.01013\frac{n}{\pi} + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

The above upper bound implies that the MST-based heuristic yields, in the worst-case, a solution cost which is about π times larger than the optimum.

Minimum Cover Problem. Any Broadcast yields a *(disk) cover* of the grid and a communication graph that contains a spanning tree. A cover $\mathcal{C}$ of a set S of points is a set of disks $\mathcal{C} = \{D_1, \ldots, D_\ell\}$ of radius at least 1, centered at some points of S, that covers all points in S. The cost of $\mathcal{C}$ is defined as $\mathsf{cost}(\mathcal{C})$ (see Eq. 1). The *Minimum Cover* problem consists in finding a cover for S of minimum cost. Observe that this is a variant of the well-known NP-hard *Minimum Geometric Disk Cover* [8,15].

In general, a cover does not suffice to provide a feasible solution for the Minimum Broadcast problem. A natural question here is whether (or when) the minimum cover cost is asymptotically equivalent to the minimum broadcast cost. This question is formally addressed by determining the cost of a minimum cover for square grids.

Theorem [Cover]. *If $\mathcal{C}^*$ is any optimal cover of the square grid $\mathcal{G}$ of n points, then*

$$n/5 \leq \mathsf{cost}(\mathcal{C}^*) \leq n/5 + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

From the above theorems, it turns out that the cost of the cover is significantly lower than the cost of the broadcast. However, the next theorem shows that this is not the case when we require that the disks are sufficiently large.

Theorem [Large Disk Cover] *Let $f(n) = \omega(1)$. The cost of any cover of $\mathcal{G}$ with disks of radius at least $f(n)$ is at least $\frac{n}{\pi} - o(n)$. The upper bound is achieved via a polynomial time construction.*

We emphasize that there are important network scenarios in which the *installing* cost (i.e. the cost of installing an omni-directional transmitter at a given location) is rather high and it must be "amortized" by a *relevant* use of the antenna. In such cases, it is convenient to assign positive range to a station only if such a range (so, disk) is large enough.

Bounded-Hop Broadcast. An important version of the Minimum Broadcast problem is the one in which feasible solutions must guarantee a *bounded number of hops*: The number of links (i.e. *hops*) in the path from the source to *any* other node must be not larger than a fixed bound. This problem version is relevant since the number of hops is closely related to the delay transmission time. The hop restriction finds another application in the context of *reliability*: Assume that, in a communication network, link faults happen with probability p and that all faults occur independently. Then, the probability that a multi-hop transmission fails exponentially increases with the number of hops. For further motivations in studying bounded hops communication see [2,11,13,21].

A main question here is the following: Does broadcasting with a bounded number of hops require a *significantly* larger cost than broadcasting with an unbounded number of hops? Intuitively speaking, one might figure out that the right answer is the positive one since the cost is proportional to the area of the solution disks and bounded-hop solutions require larger disks. Observe also that the use of *large* disks yields *large* disk overlapping. Rather surprisingly, this is not the case: we derive a broadcast for grids that uses only a constant (i.e. not depending on n) number of disks and thus yields a *constant* number of hops. This solution has a cost which is very close to that of the unbounded-hops version.

Theorem [Broadcast with few Hops]. *A positive constant c exists such that it is possible to construct in polynomial time a broadcast $\mathcal{B}$ for $\mathcal{G}$ with (only) c disks (of radius $\Omega(\sqrt{n})$) and such that*

$$\mathsf{cost}(\mathcal{B}) < 1.1171\frac{n}{\pi} + O(\sqrt{n}).$$

By comparing the above theorem with Theorem [Large Disks Cover], we can state that covering and broadcasting over grids have almost asymptotically-equivalent cost when the solution disks have *non-constant* radius (notice that any broadcast is also a cover). We also remark that the MST-based heuristic *always* returns a solution for the grid that has an *unbounded* (i.e. $\Omega(\sqrt{n})$) number of hops. So, our almost optimal polynomial-time construction yields bounded-hop solutions whose structure significantly departs from that of the MST-based solutions.

Square grids are thus the first family of well-spread, natural instances that perfectly capture the "hardness" of solving the Minimum Broadcast problem via the MST-based heuristic. It is our opinion that the set of results presented in this paper provides strong theoretical arguments that open new possibilities in the design of an efficient heuristic that significantly improves over the MST-based one (at least) in the case of *well-spread* and *uniform-random* instances.

1.2 Preliminaries

We consider a Cartesian coordinates system and a square grid $\mathcal{G}$ of side length $m-1$ with its bottom left vertex in the origin. $\mathcal{G}$ contains $n = m^2$ points at integer coordinates; the coordinates of point P of the grid will be denoted as x_p and y_p. A *$\mathcal{G}$-disk D* is a disk centered at any point of the grid and having at least one point of the grid on its boundary. By an abusing of notation, we also denote as D the set of points of grid $\mathcal{G}$ covered by D.

2 The Minimum Cover Problem on the Grid

In this section we study two versions of the disk cover problem of the grid $\mathcal{G}$. In the first version, we consider coverings by disks of arbitrary radius, while, in the second one, disks are required to have a minimal non constant radius. For both versions, we need to evaluate the number $N(r)$ of points of the *infinite* grid covered by a $\mathcal{G}$-disk of radius r. This problem, known as *Gauss' Circle problem*, has been extensively studied [14,16] in order to derive the best exponent $\delta < 1$ such that $N(r) \leq \pi r^2 + cr^\delta$ for some constant c. However, all these studies are not useful to provide a good bound on c: instead, we need an upper bound on $N(r)$ with a small constant c while the exponent δ can be 1.

Lemma 1 *For any radius $r \geq 1$, it holds that $N(r) < \pi r^2 + (\pi\sqrt{2} - 2)r + \frac{1}{5}\sqrt{r} + \frac{\pi}{2}$. Moreover, for $r > \sqrt{10}$, it holds that $N(r) < \pi r^2 + 2\sqrt{2}r - 5$.*

Proof. Let D be a $\mathcal{G}$-disk of radius r centered in the origin of the Cartesian coordinates system. For any point U in D, we consider the square centered at U and of side length 1. Let $\mathcal{P}$ be the polygon obtained by the union of all such interior disjoint squares. Notice that, in general, $\mathcal{P}$ is not contained in D, but $\mathcal{P}$ is always contained into D', the disk centered at the origin and of radius $r + \frac{\sqrt{2}}{2}$ (see Figure 1).

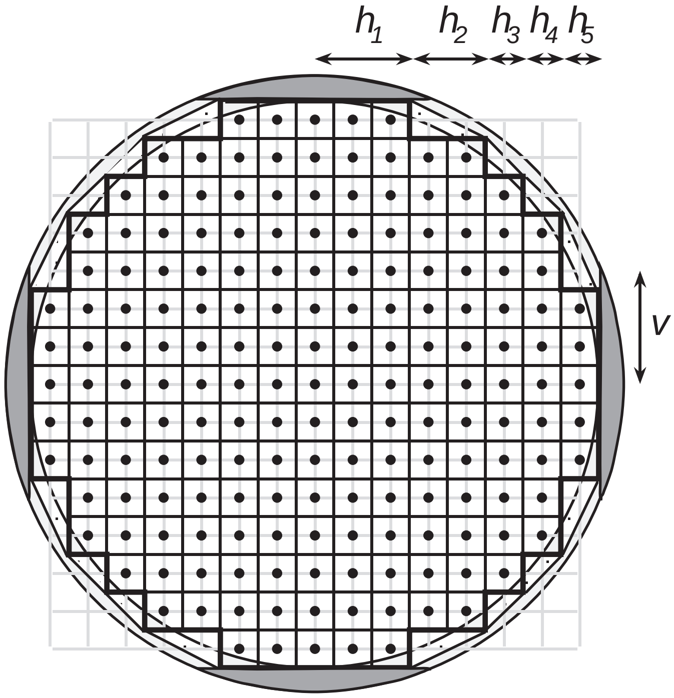

Fig. 1. Polygon $\mathcal{P}$ and D and D' when $r = 7.5$

Let R_1 be the region contained in the convex hull of $\mathcal{P}$ but not in $\mathcal{P}$ (see dotted region in Figure 1). Furthermore, let R_2 be the region in D' not contained in the square centered at the center of D and of side length $2\lfloor r \rfloor + 1$ (see the dark gray region in Figure 1). R_1 and R_2 are disjoint regions and they are both contained into D'. Hence,

$$N(r) = AREA(\mathcal{P}) < AREA(D') - (AREA(R_1) + AREA(R_2)) \qquad (2)$$

We now provide lower bounds for $AREA(R_1)$ and $AREA(R_2)$. For the sake of convenience, r will be written as $\lfloor r \rfloor + \eta$, with $\eta \in [0, 1)$. Let $h_1, h_2, \dots h_t$ and $v_1, v_2, \dots v_t$ ($t \geq 1$) be the lengths of the horizontal and vertical segments, respectively, on the boundary of $\mathcal{P}$ in Quadrant I, in clockwise order. Observe that

$$\sum_{i=2}^{t} h_i = \lfloor r \rfloor + \frac{1}{2} - h_1,\ \ h_1 = \left\lfloor \sqrt{r^2 - \lfloor r \rfloor^2} \right\rfloor + \frac{1}{2},\ \text{ and } v_i \geq 1,\ \text{ for } 1 \leq i < t$$

It follows that

$$AREA(R_1) = 4\sum_{i=1}^{t-1} \frac{v_i \cdot h_{i+1}}{2} \geq 2\lfloor r \rfloor - 2\left\lfloor \sqrt{r^2 - \lfloor r \rfloor^2} \right\rfloor \geq 2r - 2\eta - 2\sqrt{2r\eta - \eta^2} \tag{3}$$

The value $AREA(R_2)$ is computed by summing up the contributions of four identical circular caps; each of these areas is lower bounded by the area of an isosceles triangle having, respectively, bases

$$v = 2\sqrt{\left(r + \frac{\sqrt{2}}{2}\right)^2 - \left(\lfloor r \rfloor + \frac{1}{2}\right)^2} = 2\sqrt{2r\eta - \eta^2 + (\sqrt{2}-1)r + \eta + \frac{1}{4}}$$

and height

$$h = r + \frac{\sqrt{2}}{2} - \left(\lfloor r \rfloor + \frac{1}{2}\right) = \eta + \frac{\sqrt{2}-1}{2}$$

Hence,

$$\begin{aligned} AREA(R_2) &> 4\frac{h \cdot v}{2} = (2\eta + \sqrt{2} - 1) \cdot 2\sqrt{2r\eta - \eta^2 + (\sqrt{2}-1)r + \eta + \frac{1}{4}} \\ &> (2\eta + \sqrt{2} - 1) \cdot 2\sqrt{2r\eta + (\sqrt{2}-1)r} \end{aligned} \tag{4}$$

By combining Inequalities (3) and (4), we get

$$\begin{aligned} &AREA(R_1) + AREA(R_2) > \\ 2r + 2\sqrt{r}&\left((2\eta + \sqrt{2} - 1)\sqrt{2\eta + \sqrt{2} - 1} - \sqrt{2\eta - \frac{\eta^2}{r}} - \frac{\eta}{\sqrt{r}}\right) \geq \\ &2r + 2\sqrt{r}\left((2\eta + \sqrt{2} - 1)^{\frac{3}{2}} - \sqrt{2\eta} - \eta\right) > \\ &2r - \frac{1}{5}\sqrt{r} \end{aligned} \tag{5}$$

where the last inequality follows since $f(\eta) = (2\eta + \sqrt{2} - 1)^{\frac{3}{2}} - \sqrt{2\eta} - \eta$ gets a minimum value in $\eta_{min} \sim 0.18$ and $f(\eta_{min}) > -1/10$. Finally, Inequalities

(2) and (5) imply that

$$N(r) < \pi\left(r + \frac{1}{\sqrt{2}}\right)^2 - 2r + \frac{1}{5}\sqrt{r} = \pi r^2 + (\pi\sqrt{2} - 2)r + \frac{1}{5}\sqrt{r} + \frac{\pi}{2} \tag{6}$$

This proves the first statement of the lemma.

Inequality (6) implies that $N(r) < \pi r^2 + 2\sqrt{2}r - 5$ when $r > 20$. The second statement of the lemma is then exhaustively verified for any $r \in (\sqrt{10}, 20]$.

□

The above lemma is now exploited to prove asymptotically tight lower and upper bounds on the minimum cost of a cover of grid $\mathcal{G}$.

Theorem 2 *If $\mathcal{C}^*$ is any minimum cover of the square grid $\mathcal{G}$ of n points, then*

$$n/5 \leq \mathsf{cost}(\mathcal{C}^*) \leq n/5 + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

Proof. We first observe that, for any $r > 0$, it holds that

$$N(r) \leq 5r^2. \tag{7}$$

Indeed, $N(1) = 5$, $N(\sqrt{2}) = 9$, and Lemma 1 implies that $N(r) \leq 5r^2$, for any $r \geq 2$. Let $D_1, D_2, \dots D_t$ be the $\mathcal{G}$-disks of an optimal cover and let $\texttt{cost}^*$ be its cost. Let r_i be the radius of D_i, $1 \leq i \leq t$. Since D_i covers $N(r_i)$ points, Inequality (7) implies that

$$n \leq \sum_{i=1}^{t} N(r_i) \leq \sum_{i=1}^{t} 5r_i^2 = 5 \cdot \texttt{cost}^*$$

and so $\texttt{cost}^* \geq \frac{n}{5}$.

A cover of $\mathcal{G}$ with cost $\frac{n}{5} + O(\sqrt{n})$ is shown in Figure 2 for $m = 11$. Observe that the number of grey $\mathcal{G}$-disks (i.e. disks not completely contained in $\mathcal{G}$) is $O(\sqrt{n})$, and the number of white $\mathcal{G}$-disks (i.e. disks completely contained in $\mathcal{G}$) is not greater than $\frac{n}{5}$. Since all $\mathcal{G}$-disks have unit radius, then the cost $\frac{n}{5} + O(\sqrt{n})$ follows. The above construction can be clearly computed in linear time in n. □

The cover resulting by the construction in Theorem 2 uses only $\mathcal{G}$-disks of unit radius. The next theorem investigates the cost of covers using only $\mathcal{G}$-disks of large, non constant radius.

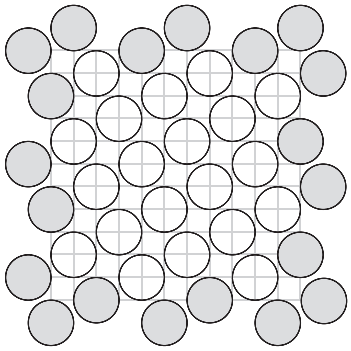

Fig. 2. An asymptotically optimum disk cover for $\mathcal{G}$ with $m = 11$.

Theorem 3 *Let $f(n) = \omega(1)$. The cost of any cover of $\mathcal{G}$ with $\mathcal{G}$-disks of radius at least $f(n)$ is at least $\frac{n}{\pi} - o(n)$.*

PROOF. Let $D_1, D_2, \ldots D_t$ be the $\mathcal{G}$-disks of a cover of $\mathcal{G}$ and let $\texttt{cost}$ be its cost. Let r_i be the radius of D_i, $1 \le i \le t$. As D_i covers $N(r_i)$ points, Lemma 1 implies that

$$n < \sum_{i=1}^{t} N(r_i) < \sum_{i=1}^{t} \left(\pi {r_i}^2 + (\pi\sqrt{2})r_i + \frac{1}{5}\sqrt{r_i} + \frac{\pi}{2}\right) < \tag{8}$$

$$< \sum_{i=1}^{t} \left(\pi {r_i}^2 + 2\pi r_i\right) = \pi\texttt{cost} + 2\pi \sum_{i=1}^{t} r_i$$

By hypothesis $r_i \ge f(n)$, hence we get

$$\texttt{cost} = \sum_{i=1}^{t} {r_i}^2 \ge f(n) \sum_{i=1}^{t} r_i$$

and thus

$$\sum_{i=1}^{t} r_i \le \frac{\texttt{cost}}{f(n)}$$

From the above inequality and from Inequality 8, we get $n < \pi\texttt{cost} + 2\pi\frac{\texttt{cost}}{f(n)}$ and, finally,

$$\texttt{cost} > n\left(\frac{f(n)}{\pi f(n) + 2\pi}\right) = \frac{n}{\pi}\left(1 - \frac{2}{f(n)+2}\right) = \frac{n}{\pi} - o(n).$$

$\square$

As we shall see in the next section, the lower bound of this theorem is almost tight.

3 The Minimum Broadcast Problem on the Grid

The aim of this section is to prove lower and upper bounds on the cost of an optimal broadcast. In particular, in order to prove the lower bound, we introduce the following definitions. A *chain* $\mathcal{H}$ is a sequence of $\mathcal{G}$-disks $D_1, D_2, \ldots, D_k$, $k \geq 1$, such that D_{i+1} is centered at some point contained in D_i for $1 \leq i < k$. We also say that a chain $\mathcal{H}$ *activates* a disk D if (i) $\mathcal{D}$ does not belong to $\mathcal{H}$, (ii) the center of D is contained in D_k, and (iii) D does not contain the center of D_1. Furthermore, we define

$$\mathcal{U}(\mathcal{H}) = \bigcup_{i=1}^{k} D_i$$

where the union refers to points of the infinite grid contained in disks D_i.

For any $r \geq 1$, consider any disk D of radius r; we define

$$M(r) = \min\{|\mathcal{U}(\mathcal{H}) \cap D| \quad : \quad \mathcal{H} \text{ is a chain that activates } D\}.$$

Notice that $M(r)$ does not depend on the choice of D and that any disk of a broadcast tree not containing the source is activated by a chain of disks belonging to the tree. The cardinality of the intersection between the disk and the chain is at least $M(r)$, where r is the radius of the disk. In order to evaluate the broadcast cost, we need a lower bound on $M(r)$.

Lemma 4 *For any $r \geq 1$, it holds that $M(r) \geq 2\sqrt{2}r - 5$.*

PROOF. Given a $\mathcal{G}$-disk $\mathcal{D}$ of radius r, we first show that for any chain $\mathcal{H}$ that activates $\mathcal{D}$, there exists a chain $\bar{\mathcal{H}}$ of $\mathcal{G}$-disks of radius 1 or $\sqrt{2}$ that activates $\mathcal{D}$, and $\mathcal{H}$ contains at least as many points as $\bar{\mathcal{H}}$. Then we prove that $\bar{\mathcal{H}}$ covers at least $2\sqrt{2}r - 5$ points of $\mathcal{D}$, and this concludes the proof.

Let P be the center of any $\mathcal{G}$-disk $\mathcal{D}_P$ in the chain $\mathcal{H}$ having radius greater than $\sqrt{2}$ and let Q be the center of the next $\mathcal{G}$-disk $\mathcal{D}_Q$ in the chain $\mathcal{H}$ (Q coincides with the center of $\mathcal{D}$ if D_P is the last $\mathcal{G}$-disk in the chain).

If $x_P = x_Q$ we replace D_P by a vertical chain constituted by unit radius $\mathcal{G}$-disks from P to Q. Such vertical chain is contained into D_P. The case $y_P = y_Q$ is similar.

Let now $x_P \neq x_Q$ and $y_P \neq y_Q$. Without loss of generality, assume $x_Q < x_P$ and $y_Q < y_P$. Consider the pair of points $U = (x_Q + 1, y_P)$ and $V = (x_Q + 1, y_Q + 1)$: replace D_P by a horizontal chain (empty if $P = U$) constituted by unit radius $\mathcal{G}$-disks from P to U, followed by a vertical chain (empty if $U = V$) constituted by unit radius $\mathcal{G}$-disks from U to V followed by a $\mathcal{G}$-disk centered at V and of radius $\sqrt{2}$. Notice that the chain replacing D_P is contained in D_P.

This replacement procedure is applied to each $\mathcal{G}$-disk of radius greater than $\sqrt{2}$. Let $\bar{\mathcal{H}}$ be the suffix of the resulting chain having all its $\mathcal{G}$-disks but the first one contained in D.

In order to prove a lower bound on the number of points covered by chain $\bar{\mathcal{H}}$ inside $\mathcal{D}$, we need to prove the following claim.

Claim *Let $S = (x_S, y_S)$ and $T = (x_T, y_T)$ be two points. Any chain from S to T of $\mathcal{G}$-disks of radius 1 or $\sqrt{2}$ covers at least $3h + v + 2$ points, where $h = |x_S - x_T|$ and $v = |y_S - y_T|$. If the first $\mathcal{G}$-disk in the chain has radius $\sqrt{2}$ then the chain covers at least $3h + v + 4$ points.*

Proof. Without loss of generality, assume $S = (x_T + h, y_T + v)$. Let us consider a chain $\bar{\mathcal{H}}$ from S to T and let (x_i, y_i) be the center of the i-th $\mathcal{G}$-disk D_i in the chain. We say that a chain is *monotone* if and only if $x_S = x_1 \geq x_2 \geq \ldots \geq x_k \geq x_T$ and $y_S = y_1 \geq y_2 \geq \ldots \geq y_k \geq y_T$. It is immediate to see that, for any $\bar{\mathcal{H}}$, there exists a monotone chain $\mathcal{H}'$ covering no points but $\bar{\mathcal{H}}$, so we can assume that $\bar{\mathcal{H}}$ is monotone.

We now prove by induction on h that the number of points of $\mathcal{G}$ covered by $\cup_{i=1}^{k} \mathcal{D}_i$ is at least $3h + v + 2$.

If $h = 0$ the assertion is trivially true since the chain contains at least $\max\{1, v\}$ $\mathcal{G}$-disks. Let us thus assume that $h > 0$. In this case, without loss of generality, we can assume that $x_T < x_k \leq x_S$: otherwise, we can change the chain as shown in Figure 3.

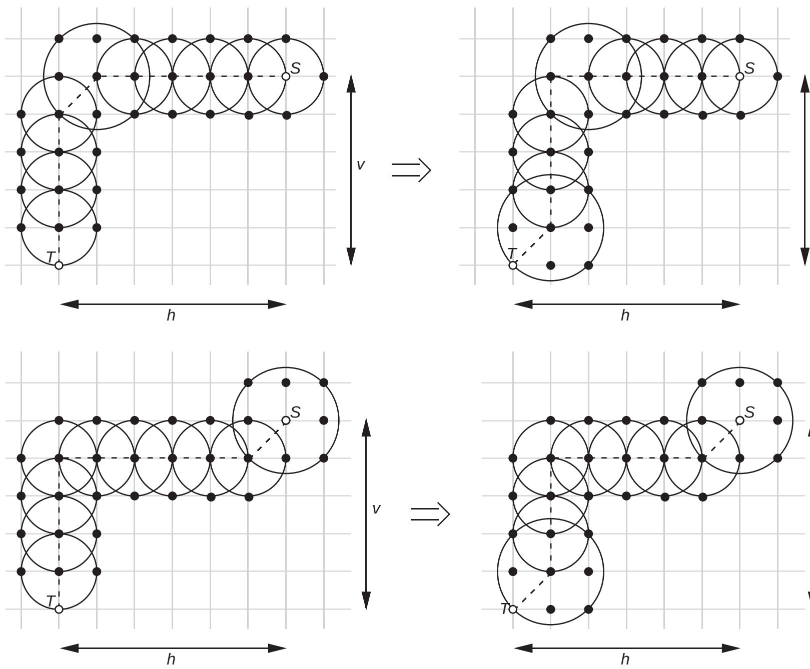

Fig. 3. Two possible transformations of chains.

Now draw an arrow from the center of $\mathcal{D}_i$ to the center of $\mathcal{D}_{i+1}$, for $i = 1, \ldots, k-1$, and draw an arrow from the center of $\mathcal{D}_k$ to T. Such arrows are of one of the following three types: $\leftarrow, \downarrow, \swarrow$. Let $n_{\downarrow}$ be the number of arrows of type $\downarrow$ and let $n_{\swarrow}$ be the number of arrows of type $\swarrow$: then, $n_{\downarrow} + n_{\swarrow} = v$.

We now focus on the sequences of consecutive arrows of type $\downarrow$. Let $j \in [x_T + 1, x_S]$ and let f_j be the number of consecutive arrows of type $\downarrow$ connecting centers of $\mathcal{G}$-disks having the x-coordinate equals to j (as the chain is monotone, for any j there exists at most one such sequence). Since the $\mathcal{G}$-disks in each sequence are consecutive, the j-th sequence covers at least $3 + f_j$ points having x-coordinate equal to j. Notice that both T and (x_S+1, y_S) are covered by the chain and that they are non not considered in any sequence. Furthermore, since the chain is monotone, if there is an arrow of type $\swarrow$ arriving at some point (j, l) then the sequence of consecutive arrows of type $\downarrow$ connecting centers of $\mathcal{G}$-disks having the x-coordinate equal to j must start at some y-coordinate smaller than l. It follows that point $(j, l+2)$ is covered by the chain but it is not considered in any sequence.

Hence, the number of points covered by the chain is at least

$$\sum_{j=x_T+1}^{x_S} (3 + f_j) + 2 + n_{\swarrow} = 3h + n_{\downarrow} + n_{\swarrow} + 2 = 3h + v + 2.$$

If the first $\mathcal{G}$-disk in the chain has radius $\sqrt{2}$ we have to count also $(x_S + 1, y_S + 1)$ and $(x_S + 1, y_S - 1)$. $\square$

Let W be the center of the first disk D_W in $\bar{\mathcal{H}}$. Let θ be the angle between the x-axis and the line connecting W to the center of D. Without loss of generality, we assume $\theta \leq \frac{\pi}{4}$, otherwise symmetrical reasonings hold. We consider two cases:

$W \in D$. If the radius of D_W is 1 then at most three points in D_W are outside D; if the radius of D_W is $\sqrt{2}$ then at most five points in D_W are outside D. From this observation and by the Claim, it follows that $\bar{\mathcal{H}}$ covers at least $3h + v - 1$ points in D. Hence,

$$M(r) \geq 3h+v-1 = 3\lfloor r\cos\theta\rfloor + \lfloor r\sin\theta\rfloor - 1 \geq 3r\cos\theta + r\sin\theta - 4 - 1 \geq 2\sqrt{2}r - 5$$

where the last inequality follows since the minimum of $3\cos\theta + \sin\theta$ in the interval $[0, \frac{\pi}{4}]$ is $2\sqrt{2}$.

$W \notin D$. In this case, the radius of D_W is necessarily $\sqrt{2}$, thus at most six points in D_W are outside D. From this observation and by the Claim, it follows that $\bar{\mathcal{H}}$ covers at least $3h + v - 2$ points in D. Since W is not contained in D, the distance between W and the center of D is greater than r. Hence,

$$M(r) \geq 3h + v - 2 \geq 3r\cos\theta + r\sin\theta - 2 \geq 2\sqrt{2}r - 2 > 2\sqrt{2}r - 5.$$

$\square$

Theorem 5 *The cost of any broadcast of $\mathcal{G}$ is at least $\frac{n}{\pi} - O(\sqrt{n})$.*

PROOF. Let $\mathcal{D}_1, \mathcal{D}_2, \dots \mathcal{D}_t$ be the $\mathcal{G}$-disks of an optimal broadcast of $\mathcal{G}$ and let cost^* be its cost. Let r_i be the radius of $\mathcal{D}_i$, $1 \le i \le t$. If there exists a disk $\mathcal{D}_i$ with radius $r_i \ge \sqrt{\frac{n}{\pi}}$, the thesis holds. Hence, we assume that $r_i < \sqrt{\frac{n}{\pi}}$, $1 \le i \le t$. In order to exploit Lemma 1, we partition the set $\{\mathcal{D}_1, \mathcal{D}_2, \dots \mathcal{D}_t\}$ into two sets: X and its complement $\overline{X}$, where

$$X = \{\mathcal{D}_i \mid r_i > \sqrt{10}\}$$

From Lemma 1, it follows that

$$\begin{aligned}\sum_{i=1}^{t} N(r_i) &= \sum_{\mathcal{D}_i \in X} N(r_i) + \sum_{\mathcal{D}_i \in \overline{X}} N(r_i) \le \sum_{\mathcal{D}_i \in X} (\pi {r_i}^2 + 2\sqrt{2} r_i - 5) + \sum_{\mathcal{D}_i \in \overline{X}} N(r_i) \\ &= \pi \cdot \mathsf{cost}^* + 2\sqrt{2} \sum_{\mathcal{D}_i \in X} r_i - 5|X| + \sum_{\mathcal{D}_i \in \overline{X}} \left(N(r_i) - \pi {r_i}^2\right) \qquad (9)\end{aligned}$$

As a consequence, we have that

$$\pi \cdot \mathsf{cost}^* \ge \sum_{i=1}^{t} N(r_i) - 2\sqrt{2} \sum_{\mathcal{D}_i \in X} r_i + 5|X| - \sum_{\mathcal{D}_i \in \overline{X}} \left(N(r_i) - \pi {r_i}^2\right) \qquad (10)$$

Now, we derive a lower bound on $\sum_{i=1}^{t} N(r_i)$. Observe that the communication graph yielded by the optimal broadcast contains a directed spanning tree T rooted at the source node. We partition $\{\mathcal{D}_1, \mathcal{D}_2, \dots \mathcal{D}_t\}$ into two sets Y and $\overline{Y}$, where Y is the set of $\mathcal{G}$-disks that cover the source point. We observe that every $\mathcal{G}$-disk $\mathcal{D}_i \in \overline{Y}$ is activated by a chain of $\mathcal{G}$-disks whose centers induce a directed path in T. This implies that the number of intersection points between the activating chain and $\mathcal{D}_i$ is at least $M(r_i)$. Now we prove the following inequality:

$$\sum_{i=1}^{t} N(r_i) \ge n + \sum_{\mathcal{D}_i \in \overline{Y}} M(r_i)$$

We consider a numbering of the T disks such that the disks on a root$\rightarrow$leaf path have strictly increasing numbers. Let

$$E = \{(p, i) \mid \exists i : 1 \le i \le t \wedge p \in \mathcal{D}_i\} \text{ and}$$

$$F = \{(p, j) \mid (p, j) \in E \wedge j = \min\{k \mid (p, k) \in E\}\}$$

In other words, $(p, j) \in F$ if and only if $\mathcal{D}_j$ is the "first" disk that covers p. Clearly, it holds that $|E| = \sum_i N(r_i)$, $F \subseteq E$, and $|F| \geq n$. Now, for every $i \in \overline{Y}$, let $\mathcal{H}_i$ be the chain that activates $\mathcal{D}_i$. Define $E_i = \{(p, i) \mid p \in \mathcal{U}(\mathcal{H}_i) \cap \mathcal{D}_i\}$. The following properties hold: (a) $E_i \subseteq E - F$; (b) if $i \neq j$ then $E_i \cap E_j = \emptyset$; (c) $|E_i| \geq M(r_i)$. As for (a), clearly $E_i \subseteq E$. Furthermore, if $(p, i) \in E_i$ then $p \in \mathcal{U}(\mathcal{H}_i) \cap \mathcal{D}_i$; thus, there exists a disk $\mathcal{D}_j \in \mathcal{H}_i$ such that $p \in \mathcal{D}_j$ and $j < i$. This implies that $\min\{k \mid (p, k) \in E\} \leq j < i$ and so $(p, i) \notin F$. The proofs of (b) and (c) are immediate from the definitions of E_i and $M(\cdot)$. Finally, it holds that

$$\sum_{i=1}^{t} N(r_i) = |E| = |F| + (|E| - |F|) \geq n + \sum_{i \in \overline{Y}} |E_i| \geq n + \sum_{i \in \overline{Y}} M(r_i).$$

Lemma 4 implies that

$$\sum_{\mathcal{D}_i \in \overline{Y}} M(r_i) = \sum_{\mathcal{D}_i \in \overline{Y} \cap X} M(r_i) + \sum_{\mathcal{D}_i \in \overline{Y} \cap \overline{X}} M(r_i) \geq$$

$$\geq 2\sqrt{2} \sum_{\mathcal{D}_i \in \overline{Y} \cap X} r_i - 5|\overline{Y} \cap X| + \sum_{\mathcal{D}_i \in \overline{Y} \cap \overline{X}} M(r_i)$$

From the above inequality, Inequality (10), and simple calculations, we get:

$$\pi \cdot \mathsf{cost}^* \geq n - 2\sqrt{2} \sum_{\mathcal{D}_i \in Y \cap X} r_i + 5|X| - 5|\overline{Y} \cap X| + \sum_{\mathcal{D}_i \in \overline{Y} \cap \overline{X}} M(r_i) - \sum_{\mathcal{D}_i \in \overline{X}} \left(N(r_i) - \pi {r_i}^2\right)$$

and

$$\mathsf{cost}^* > \frac{n}{\pi} - \frac{2\sqrt{2}}{\pi} \sum_{\mathcal{D}_i \in Y \cap X} r_i + \tag{11}$$

$$+ \frac{1}{\pi} \sum_{\mathcal{D}_i \in \overline{Y} \cap \overline{X}} \left(M(r_i) - N(r_i) + \pi {r_i}^2\right) - \frac{1}{\pi} \sum_{\mathcal{D}_i \in Y \cap \overline{X}} (N(r_i) - \pi {r_i}^2)$$

Now we bound $\sum_{\mathcal{D}_i \in Y \cap X} r_i$. Consider the sets

$$B_k = \{\mathcal{D}_j \in Y \mid 2^{k-1} \leq r_j < 2^k\},\ 1 \leq k \leq l$$

where $l = \lceil \log r_{max} \rceil + 1$ and $r_{max} = \max\{r_j \mid \mathcal{D}_j \in Y\}$. It holds that

$$\sum_{\mathcal{D}_i \in Y \cap X} r_i \leq \sum_{\mathcal{D}_i \in Y} r_i = \sum_{k=1}^{l} \sum_{\mathcal{D}_i \in B_k} r_i \leq \sum_{k=1}^{l} \frac{1}{2^{k-1}} \sum_{\mathcal{D}_i \in B_k} {r_i}^2 \tag{12}$$

Replace the $\mathcal{G}$-disks in $B_1 \cup B_2 \cup \ldots B_k$ by a $\mathcal{G}$-disk with radius (2^{k+1}) and centered in the source point. This operation produces a new broadcast with cost

$$\mathsf{cost}^* - \sum_{\mathcal{D}_i \in B_1 \cup B_2 \cup \ldots B_k} {r_i}^2 + (2 \cdot 2^k)^2$$

Hence, from the optimality of the previous broadcast it must be

$$\sum_{\mathcal{D}_i \in B_1 \cup B_2 \cup \ldots B_k} {r_i}^2 \leq (2 \cdot 2^k)^2$$

From the above inequality and from Inequality (12) we have

$$\sum_{\mathcal{D}_i \in Y \cap X} r_i \leq \sum_{k=1}^{l} \frac{2^{2k+2}}{2^{k-1}} = \sum_{k=1}^{l} 2^{k+3} < 2^{l+4} < 2^6 r_{max} = O(\sqrt{n}) \tag{13}$$

where the last step follows from the initial assumption that broadcast $\mathcal{G}$-disks have radii less than $\sqrt{\frac{n}{\pi}}$. It is possible to exhaustively prove that $M(r) - N(r) + \pi r^2 > 0$ when $r \leq \sqrt{10}$, i.e., $r \in \{1, \sqrt{2}, 2, \sqrt{5}, \sqrt{8}, 3, \sqrt{10}\}$. Hence,

$$\sum_{\mathcal{D}_i \in \overline{Y} \cap \overline{X}} \left(M(r_i) - N(r_i) + \pi {r_i}^2 \right) > 0 \tag{14}$$

Moreover, the number of $\mathcal{G}$-disks in $Y \cap \overline{X}$ is bounded by constant $N(\sqrt{10})$. Thus,

$$\sum_{\mathcal{D}_i \in Y \cap \overline{X}} \left(N(r_i) - \pi {r_i}^2 \right) = O(1) \tag{15}$$

Finally, by combining Inequality (11) with bounds (13), (14) and (15) we get the thesis. □

We now present an efficient construction of broadcasts whose cost is almost optimal.

Theorem 6 *Given any source $s \in \mathcal{G}$, it is possible to construct, in polynomial time, a Broadcast for $\mathcal{G}$ of cost $1.01013\frac{n}{\pi} + O(\sqrt{n})$.*

Proof. In order to provide a Broadcast of cost $1.01013\frac{n}{\pi} + O(\sqrt{n})$, we assume that $m - 1$ is a multiple of 6. If this is not the case, we can add $O(m)$ new unit radius $\mathcal{G}$-disks to our construction in order to broadcast to the remaining points.

Consider the Broadcast shown in figure 4. Its cost can be computed by summing up the following three contributions.

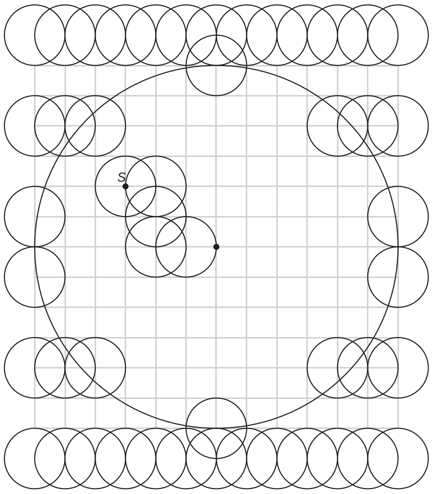

Fig. 4. An almost optimal Broadcast for the grid where $m = 19$.

- A chain of $\mathcal{G}$-disks of radius 1 from the source point to the middle point of $\mathcal{G}$. The cost of this chain is $O(m)$.
- A big $\mathcal{G}$-disk of radius $r = \frac{m-1}{2}$ centered in the middle point of $\mathcal{G}$. This disk has cost $r^2 = \frac{n}{4} - \Theta(m)$.
- A set of $\mathcal{G}$-disks of radius 1 that broadcast to all nodes of $\mathcal{G}$ out of the big $\mathcal{G}$-disk. In order to compute the cost of this set, assume that the origin of the Cartesian plane lies in the middle point of $\mathcal{G}$ and compute only the cost of the $\mathcal{G}$-disks in Quadrant I, multiplied by 4. Furthermore, observe that the contribution of Quadrant I consists of $\frac{m-1}{6}$ horizontal chains of unit-radius $\mathcal{G}$-disks whose length depends on their y-coordinates. So the cost of this contribution is:

$$C = 4\sum_{i=0}^{\frac{r}{3}} \left(r - \left\lfloor\sqrt{r^2-(3i)^2}\right\rfloor\right) < \frac{4}{3}r^2 - 4\sum_{i=0}^{\frac{r}{3}-1}\left(\sqrt{r^2-(3i)^2}-1\right) <$$

$$< \frac{4}{3}r^2 + \frac{4}{3}r - 4 - 4\int_0^{\frac{r}{3}} \sqrt{r^2-(3x)^2}dx < \frac{4}{3}r^2 + \frac{4}{3}r - 4 - \frac{4}{3}\int_0^r \sqrt{r^2-x^2}dx <$$

$$< \frac{4}{3}r^2 + \frac{4}{3}r - 4 - \frac{4}{3}\left[\frac{r^2}{2}\arcsin\frac{x}{r} + \frac{x}{2}\sqrt{r^2-x^2}\right]_0^r =$$

$$= \left(\frac{4-\pi}{3}\right)\frac{n}{4} + O(m)$$

Finally, the cost of this Broadcast is $\frac{n}{4} + \left(\frac{4-\pi}{3}\right)\frac{n}{4} + O(m) = 1.01013\frac{n}{\pi} + O(\sqrt{n})$. The construction of this solution can be clearly performed in time polynomial in n.

$\square$

We believe that the construction of optimal broadcasts for the grid is somewhat connected to the "square" version of the famous problem known as *Apollonian Circle Packing* [12,20]. The latter consists in the covering of the square by an infinite set of disks, where recursively new disks are inscribed in the enclosed space between triples of already defined mutually tangent disks and/or the sides of the square. The first disk is the one inscribed in the square (see Figure 5).

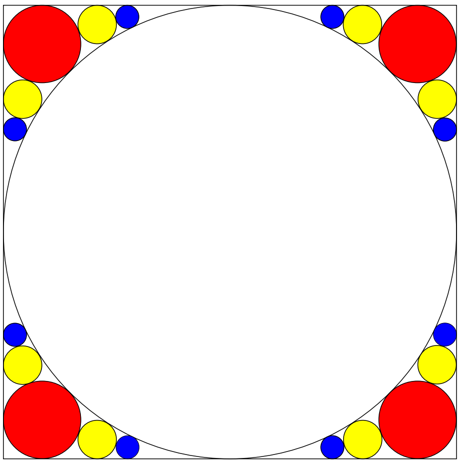

Fig. 5. The first few steps of the Apollonian Circle Packing of the square.

More precisely, we observe that if it were possible to evaluate the connectivity cost (see proof of Lemma 7) of the disk covering yielded by the Apollonian Circle Packing problem of the grid then it would be possible to obtain the optimal bound on the Broadcast cost.

Even when the $\mathcal{G}$-disks must be very large, we are able to provide a Broadcast whose cost is very close to the lower bound, as shown in the following result. The next construction makes use of a geometric approximation of the Apollonian Circle Packing: we use octagons rather than disks.

Lemma 7 *Let $0 < c < 1$ be a constant. For any source $s \in \mathcal{G}$, it is possible to construct, in polynomial time, a broadcast $\mathcal{B}$ for $\mathcal{G}$ with disks of radius at least $c\sqrt{n}$ and such that*

$$\mathsf{cost}(\mathcal{B}) = f(c)\frac{n}{\pi} + O(\sqrt{n})$$

where

$$f(c) < \pi\left(0.35483 + 24.6814c^{2-\log_{1+\sqrt{2}} 3} - 0.5551c + 0.5c^2\right)$$

Proof. Without loss of generality, we assume that $n > \frac{4(\sqrt{2}+1)^4}{c^2}$. The Broadcast $\mathcal{B}$ is based on a suitable partition of the grid into triangles and octagons. The partition works as follows.

First we partition the square $m \times m$ into 4 equal isosceles right-angled triangles and an octagon as shown in Fig. 6.

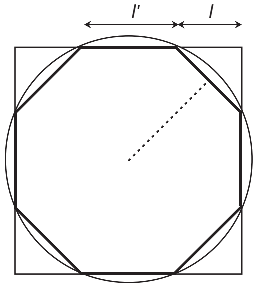

Fig. 6. The first step of the partition.

Then, while there is a triangle with a cathetus of length greater than cm, it is partitioned into 5 isosceles right-angled triangles and an octagon as shown in Figure 7. Notice that since $n > \frac{4(\sqrt{2}+1)^4}{c^2}$, every triangle and every octagon of the partition contains at least a grid point.

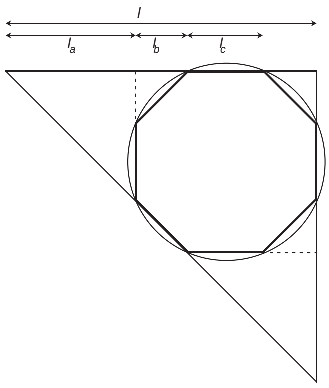

Fig. 7. The generic step of the partition.

Broadcast $\mathcal{B}$ is constructed in two phases. In the first phase a cover $\mathcal{C}$ of all grid points is obtained. In the second one, some disks are added to guarantee a Broadcast.

In order to obtain a cover, we proceed as follows.

- At the beginning, all points are uncovered.
- For each triangle, select one of its grid points and add a disk with radius $c\sqrt{2}m$ centered at the selected point. Notice that the disk covers all points contained in the triangle.
- For each octagon, let P be the grid point which is the closest to the center of the octagon and add the disk with radius $r + \frac{\sqrt{2}}{2}$ and centered in P, where r is the circumradius of the octagon. Notice that the disk covers all points contained in the octagon since P has distance at most $\frac{\sqrt{2}}{2}$ from the center of the octagon.

We first observe that the above construction can be computed in polynomial time in n. Now we evaluate the cost of $\mathcal{C}$.

Let t be the number of triangles and let OCT be the set of octagons of the partition. For each octagon $x \in OCT$, let r_x be its circumradius. It holds that

$$\mathsf{cost}(\mathcal{C}) \leq 2c^2tn + \sum_{x \in OCT} \left(r_x + \frac{\sqrt{2}}{2} \right)^2 \tag{16}$$

Let l be the length of the cathetus of the 4 initial triangles and let l' be the side length of the initial octagon. It must be that $2l + l' \leq \sqrt{n}$ and $l' = \sqrt{2}l$. So, we get

$$l \leq \tfrac{m}{2+\sqrt{2}} \qquad\qquad l' \leq \tfrac{m\sqrt{2}}{2+\sqrt{2}} \tag{17}$$

The partition of a triangle of chatetus length x (see Figure 7) yields an octagon of side length w, two (big) triangles of chatetus length y, and 3 (small) triangles with cathetus length z. Since $w = \sqrt{2}z$, $y = z + w$, and $y + 2z + w = x$, it holds that

$$y = \tfrac{x}{\sqrt{2}+1} \qquad\qquad z = \tfrac{x}{(\sqrt{2}+1)^2} \qquad\qquad w = \tfrac{\sqrt{2}x}{(\sqrt{2}+1)^2}$$

From the above equations and Inequalities (17), we can state that the cathetus lengths l_i of the triangles, yielded during the construction of the partition, satisfy

$$l_i \leq \frac{m}{(2+\sqrt{2})(\sqrt{2}+1)^i} \quad \text{with} \quad 0 \leq i \leq k+1 \tag{18}$$

where k is the minimum integer such that $l_k \leq cm$. It thus follows that

$$k \leq \left\lceil \log_{(1+\sqrt{2})} \frac{1}{(2+\sqrt{2})c} \right\rceil = O(1) \tag{19}$$

Let t_i be the number of triangles of the construction with cathetus length l_i and let r_i be the circumradius of the octagons generated by the partition of triangles with cathetus length l_i. Since all the octagons, but the initial one, are generated by the partition of triangles with cathetus length l_i, for some $0 \leq i \leq k+1$, Inequality (16) implies that

$$\mathsf{cost}(\mathcal{C}) \leq 2c^2tn + \sum_{i=0}^{k-1} t_i \left(r_i^2 + \sqrt{2}r_i + \frac{1}{2}\right) + \left(r' + \frac{\sqrt{2}}{2}\right)^2 \tag{20}$$

where r' is the circumradius of the initial octagon. Since the circumradius of an octagon of side length w is $\frac{w}{\sqrt{2-\sqrt{2}}}$, in virtue of Inequality (18), we get

$$r_i \leq \frac{\sqrt{2}l_i}{(\sqrt{2}+1)^2}\frac{1}{\sqrt{2-\sqrt{2}}} = \frac{(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{2(\sqrt{2}+1)^i}\sqrt{n} \tag{21}$$

Now, we provide a bound on t_i. Observe first that $t_0 = 4$, $t_1 = 8$, and $t_i = 2t_{i-1} + 3t_{i-2}$. Thus,

$$t_i = 3^{i+1} + (-1)^i \tag{22}$$

In virtue of the above equation and Inequality (19), it holds that

$$t = t_k + t_{k+1} = 3^{k+1} + 3^{k+2} = 12 \cdot 3^k \tag{23}$$

We are now ready to bound the terms of the right member of Inequality (20). From Inequality (23), it holds that

$$2c^2tn = 24 \cdot 3^k c^2 n < 18(2-\sqrt{2})^2((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}} 3} n \tag{24}$$

By Inequalities (21) and (22), we obtain

$$\begin{aligned}
\sum_{i=0}^{k-1} t_i r_i^2 &< \frac{(\sqrt{2}-1)^2(2-\sqrt{2})^3}{4}\sum_{i=0}^{k-1}\frac{3^{i+1}+(-1)^i}{(\sqrt{2}+1)^{2i}}n && (25)\\
&< \frac{58-41\sqrt{2}}{2}\left(3\sum_{i=0}^{k-1}\left(\frac{3}{3+2\sqrt{2}}\right)^i + 1\right)n \\
&= \frac{58-41\sqrt{2}}{2}\left(\frac{9+6\sqrt{2}}{2\sqrt{2}}\left(1-\left(\frac{3}{3+2\sqrt{2}}\right)^k\right)+1\right)n \\
&= \frac{95-67\sqrt{2}}{4}n - \frac{15\sqrt{2}-21}{4}\left(\frac{3}{3+2\sqrt{2}}\right)^k n \\
&< \frac{95-67\sqrt{2}}{4}n - \frac{261\sqrt{2}-369}{4}\left((2+\sqrt{2})c\right)^{2-\log_{1+\sqrt{2}} 3} n && (26)
\end{aligned}$$

From Inequalities (21), (22), and (19), we derive

$$\sqrt{2}\sum_{i=0}^{k-1} t_i r_i = O(m)\sum_{i=0}^{k-1}\left(\frac{3}{\sqrt{2}+1}\right)^i = O(m) \tag{27}$$

From Inequalities (22) and (19), we get

$$\frac{1}{2}\sum_{i=0}^{k-1} t_i = O(1)\sum_{i=0}^{k-1} 3^i = O(1) \tag{28}$$

Moreover, $r' = \frac{l'}{\sqrt{2-\sqrt{2}}} = \sqrt{\frac{2-\sqrt{2}}{2}}\sqrt{n}$, and so

$$\left(r' + \frac{\sqrt{2}}{2}\right)^2 = \frac{2-\sqrt{2}}{2}n + O(m) \tag{29}$$

Finally, by combining Inequality (20) with bounds (24, (23), (25), (27), (28), and (29), we obtain

$$\begin{aligned}\mathsf{cost}(\mathcal{C}) &= 18(2-\sqrt{2})^2((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}}3}n + \frac{95-67\sqrt{2}}{4}n + \\ &\quad -\frac{261\sqrt{2}-369}{4}\left((2+\sqrt{2})c\right)^{2-\log_{1+\sqrt{2}}3} n + \frac{2-\sqrt{2}}{2}n + O(m) \\ &= \left(\frac{99-69\sqrt{2}}{4} + \frac{801-549\sqrt{2}}{4}((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}}3}\right)n + \\ &\quad + O(m)\end{aligned} \tag{30}$$

Now, we describe the second phase. We add disks in order to provide a Broadcast from a source point s. We now apply the following procedure.

(1) Observe that the source s is at distance $\frac{m}{2}$ from the center P of the disk that covers the initial octagon. Hence, there exists a chain of at most $\left\lceil \frac{1}{2c} \right\rceil$ disks with radius cm connecting s to P. The centers of such disks are not necessarily grid points; however, since each center is at distance at most $\frac{\sqrt{2}}{2}$ from a grid point, then the communication from s to P is guaranteed by replacing each disk $\mathcal{D}$ with radius cm by a disk $\mathcal{D}'$ with radius $cm+\frac{\sqrt{2}}{2}$ centered at the closest grid-point to the center of $\mathcal{D}$.

(2) Notice that, by construction, each octagon O, but the initial one, is adjacent to one (and only one) octagon O' which is larger than O. Let Q be the center of O and r be its circumradius. There exists a chain of $\left\lceil \frac{r}{cm} \right\rceil$ disks with radius cm connecting Q to some point R in O'. By similar arguments to the first item, it is possible to guarantee communication from R to Q by using $\left\lceil \frac{r}{cm} \right\rceil$ disks with radius at most $cm + \frac{\sqrt{2}}{2}$.

(3) Observe that, by construction, the hypotenuse of any triangle T coincides (or is included) to the edge of one octagon O. Since the cathetus of T has size at most cm, then there exists a disk D of radius $\frac{\sqrt{2}}{2}cm + \frac{\sqrt{2}}{2}$, centered at a grid point contained in O, such that D contains the center of the disk covering T.

Again, this phase can be performed in polynomial time in n. Since the cost of each disk, added in the second phase, is $c^2n + O(m)$, we can bound the costs C_1, C_2 and C_3, due to each of the three steps of the second phase, as follows:

$$C_1 = \left\lceil \frac{1}{2c} \right\rceil \left(c^2n + O(m)\right) \leq \frac{cn}{2} + c^2n + O(m)$$

$$\begin{aligned}
C_2 &= \sum_{i=0}^{k-1} t_i \left\lceil \frac{r_i}{cm} \right\rceil (c^2n + O(m)) \\
&< \sum_{i=0}^{k-1} \left(3^{i+1} + (-1)^i\right) \left(\frac{(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{c2(\sqrt{2}+1)^i} + 1 \right) (c^2n + O(m)) \\
&< \frac{3c(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{2} \sum_{i=0}^{k-1} \left(\frac{3}{\sqrt{2}+1} \right)^i n + 3c^2 \sum_{i=0}^{k-1} 3^i n + \\
&+ \frac{(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{2} cn + c^2n + O(m) \\
&= \left(\frac{3c\sqrt{2-\sqrt{2}}}{2} \left(\left(\frac{3}{\sqrt{2}+1} \right)^k - 1 \right) + \right. \\
&+ \left. \frac{3c^2(3^k-1)}{2} + \frac{(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{2} c + c^2 \right) n + O(m) \\
&< \left(\frac{9(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{4} ((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}} 3} - \frac{3c\sqrt{2-\sqrt{2}}}{2} + \right. \\
&+ \frac{3(2-\sqrt{2})^2}{8} ((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}} 3} \\
&- \left. \frac{3c^2}{2} + \frac{(\sqrt{2}-1)(2-\sqrt{2})^{\frac{3}{2}}}{2} c + c^2 \right) n + O(m)
\end{aligned}$$

$$\begin{aligned}
C_3 = tc^2n + O(m) &= 12 \cdot 3^k c^2 n + O(m) = \\
&= 9(2-\sqrt{2})^2((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}} 3} n + O(m)
\end{aligned}$$

Finally, the cost to transform the cover into a broadcast is bounded by

$$\left(\frac{(27\sqrt{2}-36)\sqrt{2-\sqrt{2}}+225-150\sqrt{2}}{4}((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}}3}+\right.$$
$$\left.+\frac{1+(3\sqrt{2}-7)\sqrt{2-\sqrt{2}}}{2}c+\frac{c^2}{2}\right)n+O(m)$$

From this bound and Inequality (30) we get:

$$\begin{aligned}
cost(\mathcal{B}) < & \left(\frac{99-69\sqrt{2}}{4}+\right.\\
& +\frac{(27\sqrt{2}-36)\sqrt{2-\sqrt{2}}+1026-699\sqrt{2}}{4}((2+\sqrt{2})c)^{2-\log_{1+\sqrt{2}}3}+\\
& \left.+\frac{1+(3\sqrt{2}-7)\sqrt{2-\sqrt{2}}}{2}c+\frac{c^2}{2}\right)n+O(m)\\
& < \left(0.35483+24.6814c^{2-\log_{1+\sqrt{2}}3}-0.5551c+0.5c^2\right)n+O(m).
\end{aligned}$$

□

The following upper bound is an easy consequence of the previous lemma.

Theorem 8 *For any source point, there exists a (polynomial-time computable) Broadcast $\mathcal{B}$ for $\mathcal{G}$ that uses disks with radius at least $\frac{\sqrt{n}}{10^6}$ and such that*

$$\mathsf{cost}(\mathcal{B}) < 1.1171\frac{n}{\pi}+O(\sqrt{n})$$

As a consequence, $\mathcal{B}$ consists of a constant number of disks thus yielding a constant number of hops. This implies that the asymptotically optimal cost of bounded-hop solutions is very close to that of unbounded ones in grid networks. Finally, observe that Theorem 5 implies that the upper bound of Theorem 8 is almost tight.

4 Future research

Our asymptotical bounds on the Broadcast Problem on grids are not tight: achieving tight bounds here is an interesting theoretical open problem. As for determining a better upper bound, we observe that if it were possible to evaluate the cost of the Apollonian Circle Packing (see Section 3) of the grid

[12,20] then it would be possible to obtain the optimal bound on the Broadcast cost. We strongly believe that this is the *only* way to obtain such an optimal bound. The former problem is known to be a hard mathematical problem.

However, as mentioned in the Introduction, we believe that our results open new promising directions in the design of new, good heuristics for a wide and practically relevant class of input configurations: well-spread, regular instances and uniform random instances [7,18]. This is, in our opinion, the most relevant challenge in this topic.

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