Examples of unitary U, $U\mathbf{e}_1 = \frac{1}{\sqrt{n}}\mathbf{e}$, and the corresponding $\mathcal{L} = \{Ud(\mathbf{z})U^H\}$

Exercise. For such $\mathcal{L}$ compute $\mathcal{L}_A$ where A is the stochastic by columns tridiagonal matrix in toe_1qr, and compare its eigenvalues (in particular, the second one $\lambda_2(\mathcal{L}_A)$) with those of A. If they do not fill our requirements, introduce $\mu \neq I$ so that $\mathcal{L}_A$, $\mathcal{L} = \{U(\mu \circ Z)U^H\}$, does what we want (f.i. $|\lambda_2(A)| \leq |\lambda_2(\mathcal{L}_A)| < 1$)).

Circulant
Set $U_{ij} = \frac{1}{\sqrt{n}}\omega_n^{(i-1)(j-1)}$, $i, j = 1, \ldots, n$, $\omega_n = e^{\mathbf{i}\frac{2\pi}{n}}$. U is unitary symmetric. Note that $U\mathbf{e}_1 = \frac{1}{\sqrt{n}}\mathbf{e}$. Set $\mathcal{C} = \{Ud(\mathbf{z})U^H\} = \{Ud(\mathbf{z})\overline{U}\}$. Then $\mathcal{C} = \{p(\Pi)\}$, where Π is the following $n \times n$ $0,1$-matrix

$$\Pi = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & 1 \\ 1 & & & & \end{bmatrix}$$

(see [ ]). The matrices A of $\mathcal{C}$ are Toeplitz and satisfy the cross-sum condition (under suitable border conditions). The matrix A of $\mathcal{C}$ whose first row is $\mathbf{z}^T$ is here denoted $\mathcal{C}(\mathbf{z})$:

$$\mathcal{C}(\mathbf{z}) = Ud(U\mathbf{z})d(U\mathbf{e}_1)^{-1}\overline{U} = \sqrt{n}Ud(U\mathbf{z})\overline{U} = \begin{bmatrix} z_1 & z_2 & & & z_n \\ z_n & z_1 & z_2 & & \\ & z_n & \cdot & \cdot & \\ & & \cdot & & z_2 \\ z_2 & & & z_n & z_1 \end{bmatrix}.$$

An orthogonal basis for $\mathcal{C}$ is $\{J_1, J_2, \ldots, J_n\}$ where

$$J_1 = I, \ J_2 = \Pi, \ J_s = \Pi^{s-1}, \ s = 3, \ldots, n.$$

Note that the J_s are $0,1$-matrices orthogonal each other with respect the Frobenius scalar product $(\cdot,\cdot)$ for any fixed n. Given $A \in \mathbb{C}^{n\times n}$ write the best approximation of A in $\mathcal{C}$:

$$\mathcal{C}_A = \frac{1}{n}\sum_{s=1}^{n}(J_s, A)J_s = U\, d\Big(\frac{1}{n}\sum_{s=1}^{n}(J_s, A)\mathbf{z}_s\Big)\,\overline{U}$$

where $\mathbf{z}_s$ is defined as the vector of the eigenvalues of J_s. Note that $\mathbf{z}_s = \sqrt{n}U\mathbf{e}_s$ since $J_s = \sqrt{n}Ud(U\mathbf{e}_s)\overline{U}$. Thus

$$\mathcal{C}_A = \sqrt{n}Ud(U\frac{1}{n}\sum_{s=1}^{n}(J_s, A)\mathbf{e}_s)\overline{U}.$$

In the following may be we will use the symbol $U_{\mathcal{C}}$ in order to denote the above unitary matrix defining the circulant algebra $\mathcal{C}$.

Haar
Set

$$Q_1 = 1,\ Q_n = \begin{bmatrix} \mathbf{e}\mathbf{e}_1^T & Q_{\frac{n}{2}} \\ Q_{\frac{n}{2}} & -\mathbf{e}\mathbf{e}_1^T \end{bmatrix},\ n = 2, 4, 8, \ldots, 2^k.$$

$$D_1 = 1,\ D_n = \begin{bmatrix} D_{\frac{n}{2}} S_{\frac{n}{2}} & 0 \\ 0 & D_{\frac{n}{2}} S_{\frac{n}{2}} \end{bmatrix},\ S_{\frac{n}{2}} = \operatorname{diag}(\frac{1}{\sqrt{2}}, 1, \ldots, 1),\ n = 2, 4, 8, \ldots, 2^k.$$

Set $U = U_n = Q_n D_n$. Examples:

$$Q_2 D_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} \end{bmatrix};$$

$$Q_4 D_4 = \left[\begin{array}{cc|cc} 1 & & 1 & 1 \\ 1 & & 1 & -1 \\ \hline 1 & 1 & -1 & \\ 1 & -1 & -1 & \end{array}\right] \left[\begin{array}{cc|cc} \frac{1}{\sqrt{4}} & & & \\ & \frac{1}{\sqrt{2}} & & \\ \hline & & \frac{1}{\sqrt{4}} & \\ & & & \frac{1}{\sqrt{2}} \end{array}\right];$$

$$Q_8 D_8 = \left[\begin{array}{cccc|cccc} 1 & & & & 1 & & 1 & 1 \\ 1 & & & & 1 & & 1 & -1 \\ 1 & & & & 1 & 1 & -1 & \\ 1 & & & & 1 & -1 & -1 & \\ \hline 1 & & 1 & 1 & -1 & & & \\ 1 & & 1 & -1 & -1 & & & \\ 1 & 1 & -1 & & -1 & & & \\ 1 & -1 & -1 & & -1 & & & \end{array}\right] \left[\begin{array}{cccc|cccc} \frac{1}{\sqrt{8}} & & & & & & & \\ & \frac{1}{\sqrt{2}} & & & & & & \\ & & \frac{1}{\sqrt{4}} & & & & & \\ & & & \frac{1}{\sqrt{2}} & & & & \\ \hline & & & & \frac{1}{\sqrt{8}} & & & \\ & & & & & \frac{1}{\sqrt{2}} & & \\ & & & & & & \frac{1}{\sqrt{4}} & \\ & & & & & & & \frac{1}{\sqrt{2}} \end{array}\right];$$

$$Q_{16} D_{16} = \left[\begin{array}{cccccccc|cccccccc} 1 & & & & & & & & 1 & & & & 1 & & 1 & 1 \\ 1 & & & & & & & & 1 & & & & 1 & & 1 & -1 \\ 1 & & & & & & & & 1 & & & & 1 & 1 & -1 & \\ 1 & & & & & & & & 1 & & & & 1 & -1 & -1 & \\ 1 & & & & & & & & 1 & & 1 & 1 & -1 & & & \\ 1 & & & & & & & & 1 & & 1 & -1 & -1 & & & \\ 1 & & & & & & & & 1 & 1 & -1 & & -1 & & & \\ 1 & & & & & & & & 1 & -1 & -1 & & -1 & & & \\ \hline 1 & & & & 1 & & 1 & 1 & -1 & & & & & & & \\ 1 & & & & 1 & & 1 & -1 & -1 & & & & & & & \\ 1 & & & & 1 & 1 & -1 & & -1 & & & & & & & \\ 1 & & & & 1 & -1 & -1 & & -1 & & & & & & & \\ 1 & & 1 & 1 & -1 & & & & -1 & & & & & & & \\ 1 & & 1 & -1 & -1 & & & & -1 & & & & & & & \\ 1 & 1 & -1 & & -1 & & & & -1 & & & & & & & \\ 1 & -1 & -1 & & -1 & & & & -1 & & & & & & & \end{array}\right] \cdot D_{16},$$

$$D_{16} = \begin{bmatrix} D_8 S_8 & \\ & D_8 S_8 \end{bmatrix},\quad S_8 = \operatorname{diag}(\frac{1}{\sqrt{2}}, 1, 1, 1, 1, 1, 1, 1).$$

Note that

$$U_n = Q_n D_n = \begin{bmatrix} \mathbf{e}\mathbf{e}_1^T & Q_{\frac{n}{2}} \\ Q_{\frac{n}{2}} & -\mathbf{e}\mathbf{e}_1^T \end{bmatrix} \begin{bmatrix} D_{\frac{n}{2}} S_{\frac{n}{2}} & \\ & D_{\frac{n}{2}} S_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{n}} \mathbf{e}\mathbf{e}_1^T & U_{\frac{n}{2}} S_{\frac{n}{2}} \\ U_{\frac{n}{2}} S_{\frac{n}{2}} & -\frac{1}{\sqrt{n}} \mathbf{e}\mathbf{e}_1^T \end{bmatrix}.$$

(See [ ]; for a different order of the columns in Q see Appendix 2). By construction, the matrices U and U^T are unitary real matrices defining fast discrete transforms. Note that $U\mathbf{e}_1 = \frac{1}{\sqrt{n}}\mathbf{e}$.

Consider the matrix algebra $\mathcal{L}$ associated to the transform U: $\mathcal{L} = \{Ud(\mathbf{z})U^H\} = \{QDd(\mathbf{z})DQ^T\}$.

Exercise: Find an orthogonal basis $\{J_1, J_2, \dots, J_n\}$, $n = 2^k$, for $\mathcal{L}$ made up with matrices whose entries are 0, 1 or -1. Solution: the following rank-one $0, 1, -1$-matrices form a basis for $\mathcal{L}$ ($n = 4$, $n = 8$, $n = 2^k$):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = Q\mathbf{e}_1\mathbf{e}_1^T Q^T, \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = Q\mathbf{e}_3\mathbf{e}_3^T Q^T,$$

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} = Q\mathbf{e}_4\mathbf{e}_4^T Q^T, \quad \begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix} = Q\mathbf{e}_2\mathbf{e}_2^T Q^T;$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = Q\mathbf{e}_1\mathbf{e}_1^T Q^T, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} = Q\mathbf{e}_5\mathbf{e}_5^T Q^T,$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & & & & \\ 1 & 1 & -1 & -1 & & & & \\ -1 & -1 & 1 & 1 & & & & \\ -1 & -1 & 1 & 1 & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = Q\mathbf{e}_7\mathbf{e}_7^T Q^T, \quad \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & 1 & 1 & -1 & -1 \\ & & & & 1 & 1 & -1 & -1 \\ & & & & -1 & -1 & 1 & 1 \\ & & & & -1 & -1 & 1 & 1 \end{bmatrix} = Q\mathbf{e}_3\mathbf{e}_3^T Q^T,$$

$$\begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 1 & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = Q\mathbf{e}_8\mathbf{e}_8^T Q^T, \quad \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & 1 & -1 & & & & \\ & & -1 & 1 & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = Q\mathbf{e}_6\mathbf{e}_6^T Q^T,$$

$$\begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & 1 & -1 & & \\ & & & & -1 & 1 & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = Q\mathbf{e}_4\mathbf{e}_4^T Q^T, \quad \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & 1 & -1 \\ & & & & & & -1 & 1 \end{bmatrix} = Q\mathbf{e}_2\mathbf{e}_2^T Q^T;$$

$$\mathbf{e}\mathbf{e}^T = Q\mathbf{e}_1\mathbf{e}_1^T Q^T,$$

$$J_{s,i} = \begin{bmatrix} \mathit{1}_{2^{s-1}} & -\mathit{1}_{2^{s-1}} \\ -\mathit{1}_{2^{s-1}} & \mathit{1}_{2^{s-1}} \end{bmatrix} \otimes \mu_i^{(\frac{n}{2^s})} = Q\mathbf{e}_*\mathbf{e}_*^T Q^T, \quad * = (n - 2^{s-1} + 1) - 2^s(i-1),$$

$$i = 1, \dots, \tfrac{n}{2^s}, \ s = 1, 2, \dots, \log_2 n = k$$

($\mathit{1}_r$ is the $r \times r$ matrix with ones everywhere; $\mu_i^{(r)}$ is the $r \times r$ matrix with 1 in position i, i and 0 elsewhere). Note that if X, Y are two matrices from such basis, then $(X, X) = n^2, \frac{n^2}{4}, \frac{n^2}{16}, \dots, \frac{n^2}{2^{2k-2}} = 4$, and $(X, Y) = 0$ if $X \neq Y$. This implies, in particular, that we have immediately a formula for the best approximation of $A \in \mathbb{C}^{n \times n}$ in $\mathcal{L}$:

$$\mathcal{L}_A = \frac{(\mathbf{e}\mathbf{e}^T, A)}{n^2}\mathbf{e}\mathbf{e}^T + \sum_{s=1}^{\log_2 n} \frac{1}{2^{2s}} \sum_{i=1}^{\frac{n}{2^s}} (J_{s,i}, A) J_{s,i}.$$

Observe that for $n = 4$ and $n = 8$ the generic matrix in $\mathcal{L}$ has the following structure:

$$\begin{bmatrix} a+b & a-b & c & c \\ a-b & a+b & c & c \\ c & c & a+d & a-d \\ c & c & a-d & a+d \end{bmatrix},$$

$$\begin{bmatrix} (a+d)+c & (a+d)-c & a-d & a-d & b & b & b & b \\ (a+d)-c & (a+d)+c & a-d & a-d & b & b & b & b \\ a-d & a-d & (a+d)+e & (a+d)-e & b & b & b & b \\ a-d & a-d & (a+d)-e & (a+d)+e & b & b & b & b \\ b & b & b & b & (a+f)+g & (a+f)-g & a-f & a-f \\ b & b & b & b & (a+f)-g & (a+f)+g & a-f & a-f \\ b & b & b & b & a-f & a-f & (a+f)+h & (a+f)-h \\ b & b & b & b & a-f & a-f & (a+f)-h & (a+f)+h \end{bmatrix}.$$

Note also that there exists $\mathbf{v}$ such that $\mathbf{v}^T Q = \mathbf{e}^T$. See below, first for $n = 8$, and then, for a generic $n = 2^k$:

$$\mathbf{v} = \begin{bmatrix} (x+1+\frac{1}{2})+\frac{1}{4} \\ (x+\frac{1}{2})+\frac{1}{4} \\ (x+1)+\frac{1}{4} \\ (x)+\frac{1}{4} \\ ---- \\ (x+1)+\frac{1}{2} \\ (x)+\frac{1}{2} \\ ---- \\ x+1 \\ ---- \\ x \end{bmatrix}, \ x = \frac{1}{2^2} - 1, \ Q_8 = \left[\begin{array}{cccc|cccc} 1 & & & & 1 & & 1 & 1 \\ 1 & & & & 1 & & 1 & -1 \\ 1 & & & & 1 & 1 & -1 & \\ 1 & & & & 1 & -1 & -1 & \\ \hline 1 & & 1 & 1 & -1 & & & \\ 1 & & 1 & -1 & -1 & & & \\ 1 & 1 & -1 & & -1 & & & \\ 1 & -1 & -1 & & -1 & & & \end{array}\right]$$

(for any x we have $\mathbf{v}^T(Q_8\mathbf{e}_i) = 1 \; \forall\, i \geq 2$; then x is chosen so that $\mathbf{v}^T(Q_8\mathbf{e}_1) = 1$),

$$\mathbf{v} = \begin{bmatrix} (x+\cdots)+\frac{1}{2^{k-1}} \\ (x+\cdots)+\frac{1}{2^{k-1}} \\ \cdots \\ (x+\cdots)+\frac{1}{2^{k-1}} \\ ------ \\ \cdots \\ ------ \\ (x+1+\frac{1}{2})+\frac{1}{4} \\ (x+\frac{1}{2})+\frac{1}{4} \\ (x+1)+\frac{1}{4} \\ (x)+\frac{1}{4} \\ ------ \\ (x+1)+\frac{1}{2} \\ (x)+\frac{1}{2} \\ ------ \\ x+1 \\ ------ \\ x \end{bmatrix}, \; x = \frac{1}{2^{k-1}} - 1.$$

As a consequence, we can define the matrix $\mathcal{L}_r(\mathbf{z}) = QDd(DQ^T\mathbf{z})d(DQ^T\mathbf{v})^{-1}DQ^T = QDd(DQ^T\mathbf{z})d(D\mathbf{e})^{-1}DQ^T = QD^2d(Q^T\mathbf{z})Q^T$, i.e. the matrix of $\mathcal{L}$ whose $\mathbf{v}$-row is $\mathbf{z}^T$ ($\mathbf{v}^T\mathcal{L}_r(\mathbf{z}) = \mathbf{z}^T$), and the matrix algebra $\mathcal{L}$ can be represented as $\mathcal{L} = \{\mathcal{L}_r(\mathbf{z})\}$. For example, obtain $\mathbf{z}$ such that $\mathcal{L}_r(\mathbf{z}) = QD^2d(Q^T\mathbf{z})Q^T = Q\mathbf{e}_i\mathbf{e}_i^TQ^T$. Such equality is satisfied iff $D^2Q^T\mathbf{z} = \mathbf{e}_i$ iff $((DQ^T)^{-1} = QD)$

$$\mathbf{z} = Q\mathbf{e}_i, \quad QD^2d(Q^T\mathbf{z})Q^T = Q\mathbf{e}_i\mathbf{e}_i^TQ^T.$$

In the following may be we will use the symbol $U_{\mathcal{L}}$ in order to denote the above unitary matrix defining the elle algebra $\mathcal{L}$.

Jacobi
Set $U_{ij} = \sqrt{\frac{2}{n}}\,\delta_j \cos\frac{(2i-1)(j-1)\pi}{2n}$, $i,j = 1,\ldots,n$, where $\delta_j = \frac{1}{\sqrt{2}}$ if $j = 1$, and $\delta_j = 1$ otherwise. The matrix U is unitary real. Note that $U\mathbf{e}_1 = \frac{1}{\sqrt{n}}\mathbf{e}$. Set $\Upsilon = \{Ud(\mathbf{z})U^H\} = \{Ud(\mathbf{z})U^T\}$. Then $\Upsilon = \{p(T)\}$, where T is the following tridiagonal $n \times n$ $0,1$-matrix

$$T = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \end{bmatrix}$$

(see [ ]). The matrices A of Υ are symmetric and persymmetric and satisfy the cross-sum condition (like the matrices of η, see below). The border conditions are: $a_{0,i} = a_{1,i}$, $i = 1,\ldots,n$.

Exercise. Find, at least in case $n = 2^k$, an orthogonal basis $\{J_1, J_2, \ldots, J_n\}$ for Υ made up with matrices whose entries are 0, 1 or -1. Given $A \in \mathbb{C}^{n\times n}$ write the best approximation of A in Υ:

$$\Upsilon_A = \sum_{s=1}^{n} \frac{(J_s, A)}{(J_s, J_s)} J_s = U\, d\Big(\sum_{s=1}^{n} \frac{(J_s, A)}{(J_s, J_s)} \mathbf{z}_s \Big)\, U^T$$

where $\mathbf{z}_s$ is defined as the vector of the eigenvalues of J_s. Observe that $(U^T\mathbf{e}_1)_i \neq 0\ \forall i$, thus the matrix $\Upsilon(\mathbf{z})$ in Υ with first row $\mathbf{z}^T$ is well defined, $\Upsilon(\mathbf{z}) = Ud(U^T\mathbf{z})d(U^T\mathbf{e}_1)^{-1}U^T$, and Υ can be represented as $\Upsilon = \{\Upsilon(\mathbf{z})\}$. Exercise: Find $\mathbf{z}$ such that $\Upsilon(\mathbf{z}) = J_s$ and then observe that $\mathbf{z}_s = d(U^T\mathbf{e}_1)^{-1}U^T\mathbf{z}$.

In the following may be we will use the symbol U_Υ in order to denote the above unitary matrix defining the upsilon algebra Υ.

Hartley
Set $U_{ij} = \frac{1}{\sqrt{n}}(\cos\frac{2\pi(i-1)(j-1)}{n} + \sin\frac{2\pi(i-1)(j-1)}{n})$, $i, j = 1, \ldots, n$. The matrix U is unitary real symmetric. Note that $U\mathbf{e}_1 = \frac{1}{\sqrt{n}}\mathbf{e}$. Set $\mathcal{H} = \{Ud(\mathbf{z})U^H\} = \{Ud(\mathbf{z})U\}$. Then $\mathcal{H} = C^S + J\Pi C^{SK}$, where C^S is the algebra of symmetric circulants and C^{SK} is the space of skew-symmetric circulants (see [ ]). The matrices of $\mathcal{H}$ are symmetric.

Exercise. Find, at least in case $n = 2^k$, an orthogonal basis $\{J_1, J_2, \ldots, J_n\}$ for $\mathcal{H}$ made up with matrices whose entries are 0, 1 or -1. Given $A \in \mathbb{C}^{n\times n}$ write the best approximation of A in $\mathcal{H}$:

$$\mathcal{H}_A = \sum_{s=1}^{n} \frac{(J_s, A)}{(J_s, J_s)} J_s = U\, d\Big(\sum_{s=1}^{n} \frac{(J_s, A)}{(J_s, J_s)} \mathbf{z}_s \Big)\, U$$

where $\mathbf{z}_s$ is defined as the vector of the eigenvalues of J_s. Observe that the matrix $\mathcal{H}(\mathbf{z})$ in $\mathcal{H}$ with first row $\mathbf{z}^T$ is well defined, $\mathcal{H}(\mathbf{z}) = Ud(U\mathbf{z})d(U\mathbf{e}_1)^{-1}U = \sqrt{n}Ud(U\mathbf{z})U$, and $\mathcal{H}$ can be represented as $\mathcal{H} = \{\mathcal{H}(\mathbf{z})\}$. Exercise: Find $\mathbf{z}$ such that $\mathcal{H}(\mathbf{z}) = J_s$ and then observe that $\mathbf{z}_s = \sqrt{n}U\mathbf{z}$.

In the following may be we will use the symbol $U_{\mathcal{H}}$ in order to denote the above unitary matrix defining the Hartley algebra $\mathcal{H}$.

Eta
Set

$$U_{i,1}=\tfrac{1}{\sqrt{n}},U_{i,j}=\sqrt{\tfrac{2}{n}}\cos\tfrac{(2i-1)(j-1)\pi}{n},\ j=2,\dots,\lceil\tfrac{1}{2}n\rceil,$$
$$U_{i,\frac{1}{2}n+1}=\tfrac{(-1)^{i-1}}{\sqrt{n}}\ \text{if } n \text{ is even}, U_{ij}=\sqrt{\tfrac{2}{n}}\sin\tfrac{(2i-1)(j-1)\pi}{n},\ j=\lfloor\tfrac{1}{2}n+2\rfloor,\dots,n,$$

$i=1,\dots,n$. The matrix U is unitary real. Note that $U\mathbf{e}_1=\frac{1}{\sqrt{n}}\mathbf{e}$. Set $\eta=\{Ud(\mathbf{z})U^H\}=\{Ud(\mathbf{z})U^T\}$. Then $\eta=C^S+JC^S$, where C^S is the algebra of symmetric circulants (see [ ]). The matrices A of η are symmetric and persymmetric and satisfy the cross-sum condition. The border conditions are: $a_{0,i}=a_{1,n+1-i}$, $i=1,\dots,n$.

Exercise. Find, at least in case $n=2^k$, an orthogonal basis $\{J_1,J_2,\dots,J_n\}$ for η made up with matrices whose entries are 0, 1 or -1. A possible such basis, for $n=4,8$, is displayed here below

$$\begin{bmatrix}1&1&&\\1&1&&\\&&1&1\\&&1&1\end{bmatrix},\begin{bmatrix}&&1&1\\&&1&1\\1&1&&\\1&1&&\end{bmatrix},\begin{bmatrix}1&-1&&\\-1&1&&\\&&1&-1\\&&-1&1\end{bmatrix},\begin{bmatrix}&&1&-1\\&&-1&1\\1&-1&&\\-1&1&&\end{bmatrix};$$

$$\left[\begin{array}{cccc|cccc}1&1&&&1&1&&\\1&1&&&1&1&&\\&&1&1&&&1&1\\&&1&1&&&1&1\\\hline 1&1&&&1&1&&\\1&1&&&1&1&&\\&&1&1&&&1&1\\&&1&1&&&1&1\end{array}\right],\left[\begin{array}{cccc|cccc}&&1&1&&&1&1\\&&1&1&&&1&1\\1&1&&&1&1&&\\1&1&&&1&1&&\\\hline &&1&1&&&1&1\\&&1&1&&&1&1\\1&1&&&1&1&&\\1&1&&&1&1&&\end{array}\right],$$

$$\left[\begin{array}{cccc|cccc}1&-1&&&1&-1&&\\-1&1&&&-1&1&&\\&&1&-1&&&1&-1\\&&-1&1&&&-1&1\\\hline 1&-1&&&1&-1&&\\-1&1&&&-1&1&&\\&&1&-1&&&1&-1\\&&-1&1&&&-1&1\end{array}\right],\left[\begin{array}{cccc|cccc}&&1&-1&&&1&-1\\&&-1&1&&&-1&1\\1&-1&&&1&-1&&\\-1&1&&&-1&1&&\\\hline &&1&-1&&&1&-1\\&&-1&1&&&-1&1\\1&-1&&&1&-1&&\\-1&1&&&-1&1&&\end{array}\right],$$

$$\left[\begin{array}{cccc|cccc}1&&&1&-1&&&-1\\&1&1&&&-1&-1&\\&1&1&&&-1&-1&\\1&&&1&-1&&&-1\\\hline -1&&&-1&1&&&1\\&-1&-1&&&1&1&\\&-1&-1&&&1&1&\\-1&&&-1&1&&&1\end{array}\right],\left[\begin{array}{cccc|cccc}1&&&-1&-1&&&1\\&1&-1&&&-1&1&\\&-1&1&&&1&-1&\\-1&&&1&1&&&-1\\\hline -1&&&1&1&&&-1\\&-1&1&&&1&-1&\\&1&-1&&&-1&1&\\1&&&-1&-1&&&1\end{array}\right],$$

$$\left[\begin{array}{cccc|cccc}1&&-1&&-1&&1&\\&-1&&-1&&1&&1\\-1&&-1&&1&&1&\\&-1&&1&&1&&-1\\\hline -1&&1&&1&&-1&\\&1&&1&&-1&&-1\\1&&1&&-1&&-1&\\&1&&-1&&-1&&1\end{array}\right],\left[\begin{array}{cccc|cccc}&1&&-1&&-1&&1\\1&&1&&-1&&-1&\\&1&&1&&-1&&-1\\-1&&1&&1&&-1&\\\hline &-1&&1&&1&&-1\\-1&&-1&&1&&1&\\&-1&&-1&&1&&1\\1&&-1&&-1&&1&\end{array}\right].$$

Exercise: find such basis for generic $n = 2^k$. Note that if $n = 4 = 2^2$, $(J_s, J_s) = 8 = 2^3$; if $n = 8 = 2^3$, $(J_s, J_s) = 32 = 2^5$; and, we conjecture, if $n = 2^k$, $(J_s, J_s) = 2^{2k-1}$. Given $A \in \mathbb{C}^{n\times n}$ write the best approximation of A in η:

$$\eta_A = \frac{1}{2^{2k-1}} \sum_{s=1}^{2^k} (J_s, A) J_s = U\, d\Big(\frac{1}{2^{2k-1}} \sum_{s=1}^{2^k} (J_s, A) \mathbf{z}_s \Big)\, U^T$$

where $\mathbf{z}_s$ is defined as the vector of the eigenvalues of J_s. Observe that the matrix $\eta(\mathbf{z})$ in η with first row $\mathbf{z}^T$ is well defined, $\eta(\mathbf{z}) = Ud(U^T\mathbf{z})d(U^T\mathbf{e}_1)^{-1}U^T$, and η can be represented as $\eta = \{\eta(\mathbf{z})\}$. Exercise: Find $\mathbf{z}$ such that $\eta(\mathbf{z}) = J_s$ and then observe that $\mathbf{z}_s = d(U^T\mathbf{e}_1)^{-1}U^T\mathbf{z}$.

In the following may be we will use the symbol U_η in order to denote the above unitary matrix defining the eta algebra η.

P.S.
Another orthogonal basis of η:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \end{bmatrix}, \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \begin{bmatrix} & -1 & & 1 \\ -1 & & 1 & \\ & 1 & & -1 \\ 1 & & -1 & \end{bmatrix};$$

$$I, \begin{bmatrix} & 1 & & & & & & 1 \\ 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & 1 & & 1 & & & \\ & & & 1 & & 1 & & \\ & & & & 1 & & 1 & \\ & & & & & 1 & & 1 \\ 1 & & & & & & 1 & \end{bmatrix}, \begin{bmatrix} & & 1 & & & & 1 & \\ & & & 1 & & & & 1 \\ 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \\ & & 1 & & & & 1 & \\ & & & 1 & & & & 1 \\ 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \end{bmatrix}, \begin{bmatrix} & & & 1 & & 1 & & \\ & & & & 1 & & 1 & \\ & & & & & 1 & & 1 \\ 1 & & & & & & 1 & \\ & 1 & & & & & & 1 \\ 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & 1 & & 1 & & & \end{bmatrix},$$

$$\begin{bmatrix} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix}, \begin{bmatrix} & & & -1 & & & & 1 \\ & & -1 & & & & 1 & \\ & -1 & & & & 1 & & \\ -1 & & & & 1 & & & \\ & & & 1 & & & & -1 \\ & & 1 & & & & -1 & \\ & 1 & & & & -1 & & \\ 1 & & & & -1 & & & \end{bmatrix},$$

$$\begin{bmatrix} 1 & & -1 & & -1 & & 1 & \\ & -1 & & -1 & & 1 & & 1 \\ -1 & & -1 & & 1 & & 1 & \\ & -1 & & 1 & & 1 & & -1 \\ -1 & & 1 & & 1 & & -1 & \\ & 1 & & 1 & & -1 & & -1 \\ 1 & & 1 & & -1 & & -1 & \\ & 1 & & -1 & & -1 & & 1 \end{bmatrix}, \begin{bmatrix} & -1 & & 1 & & -1 & & 1 \\ -1 & & 1 & & -1 & & 1 & \\ & 1 & & -1 & & 1 & & -1 \\ 1 & & -1 & & 1 & & -1 & \\ & -1 & & 1 & & -1 & & 1 \\ -1 & & 1 & & -1 & & 1 & \\ & 1 & & -1 & & 1 & & -1 \\ 1 & & -1 & & 1 & & -1 & \end{bmatrix}.$$

APPENDIX (*The anti-Haar transform and the corresponding algebra*)

Set $U = DQ^T$. The matrix U is unitary real. Set $\mathcal{L} = \{Ud(\mathbf{z})U^H\} = \{Ud(\mathbf{z})U^T\}$. Since $(U^T\mathbf{e}_1)_i = (\frac{1}{\sqrt{n}}\mathbf{e})_i \neq 0 \ \forall i$, the algebra $\mathcal{L}$ can be represented as $\mathcal{L} = \{\mathcal{L}(\mathbf{z})\}$, where $\mathcal{L}(\mathbf{z})$ is the matrix of $\mathcal{L}$ with first row z^T, $\mathcal{L}(\mathbf{z}) = Ud(U^T\mathbf{z})d(U^T\mathbf{e}_1)^{-1}U^T = \sqrt{n}DQ^Td(QD\mathbf{z})QD$. Write $\mathcal{L}(\mathbf{z})$:

First problem: how to choose the order of the columns of Q ? Assume $n = 4$. First choice:

$$Q = \begin{bmatrix} 1 & 1 & 1 & \\ 1 & 1 & -1 & \\ 1 & -1 & & 1 \\ 1 & -1 & & -1 \end{bmatrix},\ D = \begin{bmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathcal{L}(z) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_1 - z_3 & z_3 & -z_4 \\ z_3 & z_3 & z_1 + z_2 & \\ z_4 & -z_4 & & z_1 - z_2 \end{bmatrix}.$$

Second choice:

$$Q = \begin{bmatrix} 1 & -1 & -1 & \\ 1 & 1 & -1 & \\ 1 & & 1 & -1 \\ 1 & & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} \frac{1}{2} & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathcal{L}(z) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_1 - z_3 & -z_2 & \\ z_3 & -z_2 & z_1 & z_4 \\ z_4 & & z_4 & z_1 + z_3 \end{bmatrix}.$$

Finally, the third, definitive choice:

$$Q = \begin{bmatrix} 1 & & 1 & 1 \\ 1 & & 1 & -1 \\ 1 & 1 & -1 & \\ 1 & -1 & -1 & \end{bmatrix},\ D = \begin{bmatrix} \frac{1}{2} & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathcal{L}(z) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_1 - z_3 & -z_2 & \\ z_3 & -z_2 & z_1 & z_4 \\ z_4 & & z_4 & z_1 + z_3 \end{bmatrix}.$$

$n = 8$:

$$Q = [\mathbf{e} \cdots],\ D = \begin{bmatrix} \frac{1}{2\sqrt{2}} & & & & \\ & \frac{1}{\sqrt{2}} & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{\sqrt{2}} & \\ & & & & \mathit{idem} \end{bmatrix},$$

$\mathcal{L}(z) =$

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ z_2 & z_1 - z_5 - \sqrt{2}z_3 & -\sqrt{2}z_2 & & -z_2 & & & \\ z_3 & -\sqrt{2}z_2 & z_1 - z_5 & \sqrt{2}z_4 & -z_3 & & & \\ z_4 & & \sqrt{2}z_4 & z_1 - z_5 + \sqrt{2}z_3 & -z_4 & & & \\ z_5 & -z_2 & -z_3 & -z_4 & z_1 & z_6 & z_7 & z_8 \\ z_6 & & & & z_6 & z_1 + z_5 - \sqrt{2}z_7 & -\sqrt{2}z_6 & \\ z_7 & & & & z_7 & -\sqrt{2}z_6 & z_1 + z_5 & \sqrt{2}z_8 \\ z_8 & & & & z_8 & & \sqrt{2}z_8 & z_1 + z_5 + \sqrt{2}z_7 \end{bmatrix}.$$

APPENDIX 1 (*The first investigations on the Haar matrix algebra*)

$Q=[\mathbf{e}\ \cdots]$, $D=$ diag $(\frac{1}{\sqrt{n}},\ldots)$, QD and DQ^T are real unitary. Investigate the matrix algebras $\{DQ^Td(\mathbf{z})QD : \mathbf{z}\in\mathbb{C}^n\}$ and $\{QDd(\mathbf{z})DQ^T : \mathbf{z}\in\mathbb{C}^n\}$.

$\{QDd(\mathbf{z})DQ^T : \mathbf{z}\in\mathbb{C}^n\}$ (the other one is investigated in APPENDIX)

$Md(M^T\mathbf{z})d(M^T\mathbf{v})^{-1}M^{-1}$, $\mathbf{v}^T(\cdot)=\mathbf{z}^T$, $M=QD$: $\mathbf{v}=\mathbf{e}_1$ is not ok, what $\mathbf{v}$ is ok ? ... Write $QDd(DQ^T\mathbf{z})DQ^T=QD^3d(Q^T\mathbf{z})Q^T$:

$$n=4:\ Q=\begin{bmatrix}1 & & 1 & 1\\ 1 & & 1 & -1\\ 1 & 1 & -1 & \\ 1 & -1 & -1 & \end{bmatrix},\ D=\begin{bmatrix}\frac12 & & & \\ & \frac{1}{\sqrt2} & & \\ & & \frac12 & \\ & & & \frac{1}{\sqrt2}\end{bmatrix},$$

$$QD^3d(Q^T\mathbf{z})Q^T=\begin{bmatrix}a+b & a-b & c & c\\ a-b & a+b & c & c\\ c & c & a+d & a-d\\ c & c & a-d & a+d\end{bmatrix}=\begin{bmatrix}s & t & c & c\\ t & s & c & c\\ c & c & \frac{s+t}{2}+d & \frac{s+t}{2}-d\\ c & c & \frac{s+t}{2}-d & \frac{s+t}{2}+d\end{bmatrix},$$

$$\begin{aligned}&a=\tfrac14(z_1+z_2),\ b=\tfrac{1}{2\sqrt2}(z_1-z_2),\\ &c=\tfrac14(z_3+z_4),\ d=\tfrac{1}{2\sqrt2}(z_3-z_4),\\ &a+b=s,\ a-b=t,\ a=\tfrac{s+t}{2};\end{aligned}$$

$$n=8:\ Q=[\mathbf{e}\ \cdots],\ D=\begin{bmatrix}\frac{1}{2\sqrt2} & & & & \\ & \frac{1}{\sqrt2} & & & \\ & & \frac12 & & \\ & & & \frac{1}{\sqrt2} & \\ & & & & idem\end{bmatrix},$$

$$QD^3d(Q^T\mathbf{z})Q^T$$

$$=\begin{bmatrix}(a+d)+c & (a+d)-c & a-d & a-d & b & b & b & b\\ (a+d)-c & (a+d)+c & a-d & a-d & b & b & b & b\\ a-d & a-d & (a+d)+e & (a+d)-e & b & b & b & b\\ a-d & a-d & (a+d)-e & (a+d)+e & b & b & b & b\\ b & b & b & b & (a+f)+g & (a+f)-g & a-f & a-f\\ b & b & b & b & (a+f)-g & (a+f)+g & a-f & a-f\\ b & b & b & b & a-f & a-f & (a+f)+h & (a+f)-h\\ b & b & b & b & a-f & a-f & (a+f)-h & (a+f)+h\end{bmatrix}$$

$$=\begin{bmatrix}s & t & q & q & b & b & b & b\\ t & s & q & q & b & b & b & b\\ q & q & \frac{s+t}{2}+e & \frac{s+t}{2}-e & b & b & b & b\\ q & q & \frac{s+t}{2}-e & \frac{s+t}{2}+e & b & b & b & b\\ b & b & b & b & & & & \\ b & b & b & b & & & & \\ b & b & b & b & & & & \\ b & b & b & b & & & & \end{bmatrix},$$

$$\begin{aligned}&a=\tfrac{1}{8\sqrt2}(z_1+z_2+z_3+z_4),\ b=\tfrac{1}{8\sqrt2}(z_5+z_6+z_7+z_8),\\ &c=\tfrac{1}{2\sqrt2}(z_1-z_2),\ d=\tfrac18(z_1+z_2-z_3-z_4),\ e=\tfrac{1}{2\sqrt2}(z_3-z_4),\\ &f=\tfrac18(z_5+z_6-z_7-z_8),\ g=\tfrac{1}{2\sqrt2}(z_5-z_6),\ h=\tfrac{1}{2\sqrt2}(z_7-z_8),\\ &(a+d)+c=s,\ (a+d)-c=t,\ a-d=q,\ a+d=\tfrac{s+t}{2},\\ &2a-c=q+t,\ 2a+c=q+s,\ a=\tfrac12(\tfrac{t+s}{2}+q).\end{aligned}$$

Appendix 2 (*A different position of the columns in Q*)

$$Q_2D_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$Q_4D_4 = \left[\begin{array}{cc|cc} 1 & -1 & -1 & \\ 1 & 1 & -1 & \\ \hline 1 & & 1 & -1 \\ 1 & & 1 & 1 \end{array}\right] \left[\begin{array}{cc|cc} \frac{1}{\sqrt{4}} & & & \\ & \frac{1}{\sqrt{2}} & & \\ \hline & & \frac{1}{\sqrt{4}} & \\ & & & \frac{1}{\sqrt{2}} \end{array}\right],$$

$$Q_8D_8 = \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & & -1 & & & \\ 1 & 1 & -1 & & -1 & & & \\ 1 & & 1 & -1 & -1 & & & \\ 1 & & 1 & 1 & -1 & & & \\ \hline 1 & & & & 1 & -1 & -1 & \\ 1 & & & & 1 & 1 & -1 & \\ 1 & & & & 1 & & 1 & -1 \\ 1 & & & & 1 & & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|cccc} \frac{1}{\sqrt{8}} & & & & & & & \\ & \frac{1}{\sqrt{2}} & & & & & & \\ & & \frac{1}{\sqrt{4}} & & & & & \\ & & & \frac{1}{\sqrt{2}} & & & & \\ \hline & & & & \frac{1}{\sqrt{8}} & & & \\ & & & & & \frac{1}{\sqrt{2}} & & \\ & & & & & & \frac{1}{\sqrt{4}} & \\ & & & & & & & \frac{1}{\sqrt{2}} \end{array}\right],$$

$$Q_{16}D_{16} = \left[\begin{array}{cccccccc|cccccccc} 1 & -1 & -1 & & -1 & & & & -1 & & & & & & & \\ 1 & 1 & -1 & & -1 & & & & -1 & & & & & & & \\ 1 & & 1 & -1 & -1 & & & & -1 & & & & & & & \\ 1 & & 1 & 1 & -1 & & & & -1 & & & & & & & \\ 1 & & & & 1 & -1 & -1 & & -1 & & & & & & & \\ 1 & & & & 1 & 1 & -1 & & -1 & & & & & & & \\ 1 & & & & 1 & & 1 & -1 & -1 & & & & & & & \\ 1 & & & & 1 & & 1 & 1 & -1 & & & & & & & \\ \hline 1 & & & & & & & & 1 & -1 & -1 & & -1 & & & \\ 1 & & & & & & & & 1 & 1 & -1 & & -1 & & & \\ 1 & & & & & & & & 1 & & 1 & -1 & -1 & & & \\ 1 & & & & & & & & 1 & & 1 & 1 & -1 & & & \\ 1 & & & & & & & & 1 & & & & 1 & -1 & -1 & \\ 1 & & & & & & & & 1 & & & & 1 & 1 & -1 & \\ 1 & & & & & & & & 1 & & & & 1 & & 1 & -1 \\ 1 & & & & & & & & 1 & & & & 1 & & 1 & 1 \end{array}\right] \cdot D_{16},$$

$$D_{16} = \begin{bmatrix} D_8S_8 & \\ & D_8S_8 \end{bmatrix}, \quad S_8 = \operatorname{diag}(\frac{1}{\sqrt{2}}, 1, 1, 1, 1, 1, 1, 1).$$

Note that

$$Q_nD_n = \begin{bmatrix} Q_{\frac{n}{2}} & -\mathbf{e}\mathbf{e}_1^T \\ \mathbf{e}\mathbf{e}_1^T & Q_{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} D_{\frac{n}{2}}S_{\frac{n}{2}} & \\ & D_{\frac{n}{2}}S_{\frac{n}{2}} \end{bmatrix}.$$