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Exercise 1

$$\begin{array}{ll}(a): & y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + O(h^4) \\ (b): & y(x+2h) = y(x) + 2hy'(x) + \frac{4h^2}{2}y''(x) + \frac{8h^3}{6}y'''(x) + O(h^4)\end{array}$$

$$4(a)-(b): \quad 4y(x+h) - y(x+2h) = 3y(x) + 2hy'(x) - \frac{4}{6}h^3y'''(x) + O(h^4)$$

$$y'(x) = \frac{-\frac{3}{2}y(x) + 2y(x+h) - \frac{1}{2}y(x+2h)}{h} + O(h^2)$$

Exercise 2

$$Z = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{n} \end{bmatrix}, \ Z^{-1} = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix};$$

$$W = \begin{bmatrix} 1 & n+1 & \cdots & n+1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \ W^{-1} = \begin{bmatrix} 1 & -(n+1) & \cdots & -(n+1) \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix};$$

$$Y = \begin{bmatrix} \frac{1}{n} & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}, \ Y^{-1} = \begin{bmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix};$$

$$X = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \ X^{-1} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

$$A = XYWZ, \ A^{-1} = Z^{-1}W^{-1}Y^{-1}X^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix} \begin{bmatrix} 1 & -(n+1) & \cdots & -(n+1) \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -(n+1) & \cdots & -(n+1) \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix} \begin{bmatrix} n & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & & & & n+1 \\ 2 & -2 & & & \\ & 3 & -3 & & \\ & & & & \\ & & & n & -n \end{bmatrix}.$$

Show that there exists $\mathbf{p}$ positive, $\|\mathbf{p}\|_1 = 1$, such that $A^{-1}\mathbf{p} = \mathbf{p}$.

The vector equation $A^{-1}\mathbf{p} = \mathbf{p}$ is equivalent to the n scalar equations:

$$\begin{cases} -p_1 + (n+1)p_n = p_1 \\ 2(p_1 - p_2) = p_2 \\ 3(p_2 - p_3) = p_3 \\ \cdots \\ (n-1)(p_{n-2} - p_{n-1}) = p_{n-1} \\ n(p_{n-1} - p_n) = p_n \end{cases}, \quad \begin{cases} (n+1)p_n = 2p_1 \\ 2p_1 = 3p_2 \\ 3p_2 = 4p_3 \\ \cdots \\ (n-1)p_{n-2} = np_{n-1} \\ np_{n-1} = (n+1)p_n \end{cases},$$

$$p_2 = \frac{2}{3}p_1,\ p_3 = \frac{3}{4}p_2 = \frac{2}{4}p_1,\ p_4 = \frac{4}{5}p_3 = \frac{2}{5}p_1, \dots, p_n = \frac{n}{n+1}p_{n-1} = \frac{2}{n+1}p_1,\ p_j = \frac{2}{j+1}p_1,\ j = 1, 2, \dots, n.$$

$$1 = \sum_j p_j \ \Rightarrow\ p_1 = \frac{1}{2(\sum_j \frac{1}{j+1})}.$$

Thus 1 is eigenvalue of A^{-1}. Let us show that the remaining eigenvalues of A^{-1} have absolute value greater than 1. Let 1, λ_j, $j = 2, \dots, n$, be the eigenvalues of A. We now prove that $|\lambda_j|$ is smaller than 1, $j = 2, \dots, n$, and the thesis will follow.

Taking into account the suggestion, compute A:

$$A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{n} & & & \\ & -1 & & \\ & & \cdots & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & n+1 & \cdots & n+1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{n} & \frac{n+1}{2n} & \frac{n+1}{3n} & \cdots & \frac{n+1}{n^2} \\ & -\frac{1}{2} & & & \\ & & -\frac{1}{3} & & \\ & & & \ddots & \\ & & & & -\frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n} & \frac{n+1}{2n} & \frac{n+1}{3n} & \cdots & \frac{n+1}{n^2} \\ \frac{1}{n} & \frac{1}{2n} & \frac{n+1}{3n} & \cdots & \frac{n+1}{n^2} \\ \frac{1}{n} & \frac{1}{2n} & \frac{1}{3n} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{n+1}{n^2} \\ \frac{1}{n} & \frac{1}{2n} & \frac{1}{3n} & \cdots & \frac{1}{n^2} \end{bmatrix}.$$

Note that A is stochastic by columns and positive (thus, non negative and irreducible). Then, by the Perron-Frobenius theory, since A is non negative and irreducible, $1 = \rho(A)$ is a simple eigenvalue of A with corresponding uniquely defined eigenvector $\mathbf{p}$ positive, $\|\mathbf{p}\|_1 = 1$ (note that we already know $\mathbf{p}$), and, since A is positive, the remaining eigenvalues of A have absolute value smaller than 1.

Exercise 3. (i)

$\eta^{EE}(h) = \eta^{EE}(0) + hf(0, \eta^{EE}(0)) = 0 + h\sqrt{1 - 0^2} = h;$
$\eta^{EI}(h) = \eta^{EI}(0) + hf(h, \eta^{EI}(h)) = 0 + h\sqrt{1 - \eta^{EI}(h)^2},$
$\eta^{EI}(h)^2 = h^2(1 - \eta^{EI}(h)^2),\ \eta^{EI}(h) = \frac{h}{\sqrt{1+h^2}};$
$\eta^C(h) = \eta^C(0) + h(a_1K_1 + a_2K_2) = 0 + hK_2,$
$K_1 = f(0, \eta^C(0)) = f(0, 0) = \sqrt{1 - 0^2} = 1$
$K_2 = f(0 + p_1h, \eta^C(0) + p_2hf(0, \eta(0))) = f(p_1h, p_2h) = \sqrt{1 - p_2^2h^2} = \sqrt{1 - \frac{1}{4}h^2},$
$\eta^C(h) = h\sqrt{1 - \frac{1}{4}h^2};$
$\eta^C(\frac{1}{2}) = \frac{\sqrt{15}}{8}$

(ii)

$$\begin{aligned}
&\eta^T(h) = \eta^T(0) + \tfrac{h}{2}[f(0,\eta^T(0)) + f(h,\eta^T(h))] \\
&= 0 + \tfrac{h}{2}[\sqrt{1-\eta^T(0)^2} + \sqrt{1-\eta^T(h)^2}] \\
&= \tfrac{h}{2}[1 + \sqrt{1-\eta^T(h)^2}], \\
&\xi_0 = \eta^C(h) = h\sqrt{1-\tfrac{1}{4}h^2},\ \xi_{i+1} = \tfrac{h}{2}[1+\sqrt{1-\xi_i^2}],\ i = 0,1,2,\dots, \\
&\xi_i \ \to \ \eta^T(h),\ i \ \to \ +\infty, \\
&\eta^T(h) = \tfrac{h}{1+\frac{h^2}{4}}
\end{aligned}$$

(iii)

$$\begin{aligned}
\eta(1) &= \eta(\tfrac{1}{2}) + \tfrac{1}{2}[\tfrac{3}{2}f(\tfrac{1}{2},\eta(\tfrac{1}{2})) - \tfrac{1}{2}f(0,\eta(0))] \\
&= \eta^C(\tfrac{1}{2}) + \tfrac{1}{2}[\tfrac{3}{2}\sqrt{1-\eta^C(\tfrac{1}{2})^2} - \tfrac{1}{2}\sqrt{1-\eta^C(0)^2}] \\
&= \tfrac{\sqrt{15}}{8} + \tfrac{3}{4}\sqrt{1-\tfrac{15}{64}} - \tfrac{1}{4} \\
&= \tfrac{4\sqrt{15}+13}{32}
\end{aligned}$$

MULTISTEP

Since $y(t)$ solves the equation $y'(t) = f(t, y(t))$, $t \in a, b$, we have

$$\begin{aligned}
y(x+rh) - y(x) &= \textstyle\int_x^{x+rh} y'(t)\,dt \\
&= \textstyle\int_x^{x+rh} f(t,y(t))\,dt \\
&= h\textstyle\int_0^r f(x+\xi h, y(x+\xi h))\,d\xi \\
&= h[a_0 f(x,y(x)) + a_1 f(x+h, y(x+h)) \\
&\quad + \ldots + a_r f(x+rh, y(x+rh)) \\
&\quad + a_{-1} f(x-h, y(x-h)) + a_{-2} f(x-2h, y(x-2h)) \\
&\quad + \ldots + a_{-s} f(x-sh, y(x-sh))] + E
\end{aligned}$$

MULTISTEP method: given $y(x+jh)$, $j = -s,\ldots,-1,0,1,\ldots,r-1$, define $\eta(x+rh)$, approximation of $y(x+rh)$, by the identity

$$\begin{aligned}
\eta(x+rh) - y(x) &= h[a_0 f(x,y(x)) + \ldots + a_{r-1} f(x+(r-1)h, y(x+(r-1)h)) \\
&\quad + a_r f(x+rh, \eta(x+rh)) + a_{-1} f(x-h, y(x-h)) + a_{-2} f(x-2h, y(x-2h)) \\
&\quad + \ldots + a_{-s} f(x-sh, y(x-sh))]
\end{aligned}$$

where a_j are chosen such that $\eta(x+rh) = y(x+rh)$ when $y(t) = (t-x)^i$, $i = 0, 1, \ldots$.

Characteristic polynomial: $z^{r+s} - z^s = z^s(z^r - 1)$, thus 0-stable.

$r = 1$: Adams. Bashforth: Explicit Adams (EE). Moulton: Implicit Adams (EI, T)

$r = s = 1$:

$$\begin{aligned}
y(x+h) &= y(x) + h[a_{-1} f(x-h, y(x-h)) + a_0 f(x, y(x)) + a_1 f(x+h, y(x+h))] + E \\
&= y(x) + h[a_{-1} y'(x-h) + a_0 y'(x) + a_1 y'(x+h)] + E
\end{aligned}$$

By imposing $E = 0$ for $y(t) = (t-x)^i$, $i = 1, 2, 3$, one obtains, respectively, the following conditions:

$$a_{-1} + a_0 + a_1 = 1,\ a_1 - a_{-1} = \frac{1}{2},\ a_1 + a_{-1} = \frac{1}{3}$$

(for $y(t) = (t-x)^0$ we have $E = 0$ for any choice of the parameters).

Example: $a_1 = 0$, $a_0 = \frac{3}{2}$, $a_{-1} = -\frac{1}{2}$ (p.83,85 Lambert).

Example: $a_1 = \frac{5}{12}$, $a_0 = \frac{2}{3}$, $a_{-1} = -\frac{1}{12}$.

Preliminaries

Space of all matrices with constant column sums:

$$L = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \sum_i a_{i1} - \sum_{i\neq 2} a_{i2} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \sum_i a_{i1} - \sum_{i\neq 3} a_{i3} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \sum_i a_{i1} - \sum_{i\neq n} a_{in} \end{bmatrix}$$

It is a vector space. It is closed by multiplication and inversion. Look for a good basis ? For instance in order to compute the best least squares approximation in L of $A \in \mathbb{C}^{n\times n}$.

If 1, eigenvalue of A stochastic by columns, is simple, and

$$AX = XJ, \quad J = \begin{bmatrix} 1 & \\ & \end{bmatrix} = Jordan \ ([J]_{11} = 1, \ [J]_{1k} = [J]_{k1} = 0, \ k = 2, \dots, n),$$

then $\sum_i (X)_{ij} = 0$ for all $j \neq 1$.

If 1, eigenvalue of A stochastic by columns, is not simple, and

$$AX = XJ, \quad J = \begin{bmatrix} 1 & & & & & & & \\ & 1 & 1 & & & & & \\ & & 1 & & & & & \\ & & & 1 & 1 & & & \\ & & & & 1 & 1 & & \\ & & & & & 1 & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = Jordan$$

then $\sum_i (X)_{ij} = 0$ for all $j \neq 2, 4, 5$.

The product of two stochastic by columns matrices is stochastic by columns ($M^T \mathbf{e} = \mathbf{e}$, $N^T \mathbf{e} = \mathbf{e} \Rightarrow (MN)^T \mathbf{e} = \mathbf{e}$). The inverse of a non singular stochastic by columns matrix is stochastic by columns ($M^T \mathbf{e} = \mathbf{e} \Rightarrow (M^{-1})^T \mathbf{e} = (M^T)^{-1} \mathbf{e} = \mathbf{e}$).

Alternative proofs:

$$\sum_i (MN)_{ij} = \sum_i \sum_k (M)_{ik} (N)_{kj} = \sum_k (N)_{kj} \sum_i (M)_{ik} = \sum_k (N)_{kj} = 1.$$

$$\begin{aligned} \textstyle\sum_{i=1}^n (A^{-1})_{ij} &= \textstyle\sum_i (-1)^{i+j} \frac{c_{ji}(A)}{\det(A)} = \frac{1}{\det(A)} \sum_i (-1)^{j+i} c_{ji}(A) \\ &= \frac{1}{\det(A)} \det(\begin{bmatrix} \cdots & A & \cdots \\ \cdots 1 \cdots & 1 & \cdots 1 \cdots \\ \cdots & A & \cdots \end{bmatrix}) \end{aligned}$$

where the ones are in the jth row.

$$= \frac{1}{\det(A)} \det\left(\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ -1 & \cdots & -1 & 1 & -1 & \cdots & -1 \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} \cdots & A & \cdots \\ \cdots 1 \cdots & 1 & \cdots 1 \cdots \\ \cdots & A & \cdots \end{bmatrix} \right)$$

$$= \frac{1}{\det(A)} \det\left(\begin{bmatrix} \cdots & A & \cdots \\ 1-\sum_{r\neq j} a_{r1} \cdots & 1-\sum_{r\neq j} a_{ri} & \cdots 1-\sum_{r\neq j} a_{rn} \\ \cdots & A & \cdots \end{bmatrix} \right) = \frac{1}{\det(A)} \det(A) = 1$$

where the last but one equality holds by the stochastic assumption.

For any value of the parameters b_j, the following matrix is well defined:

$$A^{-1} = \begin{bmatrix} \frac{1}{n} & b_2 & b_3 & \cdots & b_n \\ \frac{1}{n} & \frac{1-b_2}{n-1} & b_3 & \cdots & b_n \\ \frac{1}{n} & \frac{1-b_2}{n-1} & \frac{1-2b_3}{n-2} & \cdots & \\ & & & & b_n \\ \frac{1}{n} & \frac{1-b_2}{n-1} & \frac{1-2b_3}{n-2} & & 1-(n-1)b_n \end{bmatrix}.$$

Example 1. $b_j = \frac{1}{n}$: $A^{-1} = \frac{1}{n}\mathbf{e}\mathbf{e}^T$, $p(\lambda) = \lambda^n - \lambda^{n-1}$.
Example 2. $b_j = \frac{1}{j-1}$
Example 3. $b_j = 0$
Example 4. $b_j = \frac{n+1}{nj}$

A^{-1} is stochastic by columns. A^{-1} is non negative iff b_j and $1-(j-1)b_j$ are non negative iff $b_j \in [0, \frac{1}{j-1}]$. A^{-1} is positive iff $b_j \in (0, \frac{1}{j-1})$. A^{-1} is irreducible iff ???

In order to compute the inverse, note that

$$A^{-1} = \begin{bmatrix} 1 & \frac{b_2(n-1)}{1-b_2} & \frac{b_3(n-2)}{1-2b_3} & \cdots & \frac{b_n}{1-(n-1)b_n} \\ 1 & 1 & \frac{b_3(n-2)}{1-2b_3} & & \\ 1 & 1 & 1 & & \\ \vdots & & & \ddots & \frac{b_n}{1-(n-1)b_n} \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{n} & & & & \\ & \frac{1-b_2}{n-1} & & & \\ & & \frac{1-2b_3}{n-2} & & \\ & & & \ddots & \\ & & & & 1-(n-1)b_n \end{bmatrix}$$

where

$$\begin{bmatrix} 1 & \frac{b_2(n-1)}{1-b_2} & \frac{b_3(n-2)}{1-2b_3} & & \frac{b_n}{1-(n-1)b_n} \\ 1 & 1 & \frac{b_3(n-2)}{1-2b_3} & & \\ 1 & 1 & 1 & & \\ & & & \ddots & \frac{b_n}{1-(n-1)b_n} \\ 1 & 1 & 1 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & & \ddots & \\ 1 & 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{b_2(n-1)}{1-b_2} & \frac{b_3(n-2)}{1-2b_3} & & \frac{b_n}{1-(n-1)b_n} \\ & \frac{1-b_2n}{1-b_2} & 0 & & 0 \\ & & \frac{1-b_3n}{1-2b_3} & & \\ & & & \ddots & 0 \\ & & & & \frac{1-b_nn}{1-(n-1)b_n} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & & \ddots & \\ 1 & 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{b_2(n-1)}{1-b_2} & \frac{b_3(n-2)}{1-2b_3} & & \frac{b_n}{1-(n-1)b_n} \\ & \frac{1-b_2n}{1-b_2} & 0 & & 0 \\ & & \frac{1-b_3n}{1-2b_3} & & \\ & & & \ddots & 0 \\ & & & & \frac{1-b_nn}{1-(n-1)b_n} \end{bmatrix} \begin{bmatrix} \frac{1}{n} & & & & \\ & \frac{1-b_2}{n-1} & & & \\ & & \frac{1-2b_3}{n-2} & & \\ & & & \ddots & \\ & & & & 1-(n-1)b_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & & \ddots & \\ 1 & 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \frac{1-b_2n}{1-b_2} & & & \\ & & \frac{1-b_3n}{1-2b_3} & & \\ & & & \ddots & \\ & & & & \frac{1-b_nn}{1-(n-1)b_n} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{b_2(n-1)}{1-b_2} & \frac{b_3(n-2)}{1-2b_3} & & \frac{b_n}{1-(n-1)b_n} \\ & 1 & 0 & & 0 \\ & & 1 & & \\ & & & \ddots & 0 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{n} & & & & \\ & \frac{1-b_2}{n-1} & & & \\ & & \frac{1-2b_3}{n-2} & & \\ & & & \ddots & \\ & & & & 1-(n-1)b_n \end{bmatrix}$$

Thus the inverse is

$$A = \begin{bmatrix} n & & & & \\ & \frac{n-1}{1-b_2} & & & \\ & & \frac{n-2}{1-2b_3} & & \\ & & & \ddots & \\ & & & & \frac{1}{1-(n-1)b_n} \end{bmatrix} \begin{bmatrix} 1 & -\frac{b_2(n-1)}{1-b_2} & -\frac{b_3(n-2)}{1-2b_3} & & -\frac{b_n}{1-(n-1)b_n} \\ & 1 & 0 & & 0 \\ & & 1 & & \\ & & & \ddots & 0 \\ & & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & \\ & \frac{1-b_2}{1-b_2n} & & & \\ & & \frac{1-2b_3}{1-b_3n} & & \\ & & & \ddots & \\ & & & & \frac{1-(n-1)b_n}{1-b_nn} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n & -\frac{nb_2(n-1)}{1-b_2} & -\frac{nb_3(n-2)}{1-2b_3} & & -\frac{nb_n}{1-(n-1)b_n} \\ & \frac{n-1}{1-b_2} & 0 & & 0 \\ & & \frac{n-2}{1-2b_3} & & \\ & & & \ddots & 0 \\ & & & & \frac{1}{1-(n-1)b_n} \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & \\ -\frac{1-b_2}{1-b_2n} & \frac{1-b_2}{1-b_2n} & & & \\ & -\frac{1-2b_3}{1-b_3n} & \frac{1-2b_3}{1-b_3n} & & \\ & & & \ddots & \\ & & & -\frac{1-(n-1)b_n}{1-b_nn} & \frac{1-(n-1)b_n}{1-b_nn} \end{bmatrix}$$

$$A = \begin{bmatrix} n(1+b_2\beta_2) & n(b_3\beta_3 - b_2\beta_2) & n(b_4\beta_4 - b_3\beta_3) & & n(b_n\beta_n - b_{n-1}\beta_{n-1}) & -nb_n\beta_n \\ -\beta_2 & \beta_2 & & & & \\ & -\beta_3 & \beta_3 & & & \\ & & -\beta_4 & \beta_4 & & \\ & & & & & \\ & & & & -\beta_n & \beta_n \end{bmatrix}, \ \beta_j = \frac{n-j+1}{1-b_jn}$$

$\mathbf{p} = A\mathbf{p}$: $p_{n-1} = b_nnp_n$, $p_{n-2} = \frac{1+b_{n-1}n}{2}p_{n-1}$, $p_{n-3} = \frac{2+b_{n-2}n}{3}p_{n-2}$, $\ldots$, $p_1 = \frac{n-2+b_2n}{n-1}p_2$

Example. $b_j = \frac{n+1}{nj}$ (thus $\beta_j = -j$, $b_j\beta_j = -\frac{n+1}{n}$)

$$A = \begin{bmatrix} -1 & & & & n+1 \\ 2 & -2 & & & \\ & 3 & -3 & & \\ & & & & \\ & & & n & -n \end{bmatrix}, \; p_A(\lambda) = \Pi_{j=1}^{n}(\lambda + j) - (n+1)!$$

(compute $\det(\lambda I - A)$ with respect to the first row). For $n = 3$: $p_A(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3) - 4! = \lambda^3 + 6\lambda^2 + 11\lambda - 18 = (\lambda - 1)(\lambda^2 + 7\lambda + 18)$.

The matrix A^{-1} is invertible if and only if $b_j \neq \frac{1}{n}$ $\forall j$ ($b_j = \frac{1}{n}$ for some j implies that the first and the jth columns are equal; if $b_j \neq \frac{1}{n}$ $\forall j$, then we can define the inverse). A^{-1} is singular iff has $n-1$ eigenvalues equal to zero (and one equal to 1).

In order to have $[A]_{1j} = 0$, $j = 2, \ldots, n-1$, the parameters b_j must satisfy the following (equivalent) conditions:

$$\begin{aligned} b_j\beta_j &= b_{j+1}\beta_{j+1}, \\ (n-j+1)b_j - nb_jb_{j+1} &= (n-j)b_{j+1}, \quad j = 2, \ldots, n-1, \\ b_{j+1} &= \frac{(n-j+1)b_j}{nb_j + n - j}. \end{aligned}$$

Examples.

b_j all equal to b: then b must be such that $b - nb^2 = 0$, and thus zero or $\frac{1}{n}$.

$b_j = 0$; $\beta_j = n - j + 1$, $b_j\beta_j = 0$,

$$\begin{bmatrix} n & & & & \\ 1-n & n-1 & & & \\ & 2-n & n-2 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

$b_j = \frac{n+1}{nj}$; $\beta_j = -j$, $b_j\beta_j = -\frac{n+1}{n}$,

$$\begin{bmatrix} -1 & & & & n+1 \\ 2 & -2 & & & \\ & 3 & -3 & & \\ & & & \ddots & \\ & & & n & -n \end{bmatrix}$$

$b_j = \frac{1}{j-1}$; $\beta_j = 1 - j$, $b_j\beta_j = -1$,

$$A = \begin{bmatrix} & & & & n \\ 1 & -1 & & & \\ & 2 & -2 & & \\ & & & \ddots & \\ & & & n-1 & 1-n \end{bmatrix}$$

Let 1, $\lambda_2, \ldots, \lambda_n$, be the eigenvalues of

$$M = (A^{-1})^T = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 1 & 0 & & & 0 \\ \frac{1}{2} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & & \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{bmatrix}$$

($b_j = \frac{1}{j-1}$) (Riccardo Fastella: for $n = 4$ the characteristic polynomial of M is $\lambda^4 - \frac{1}{4}\lambda^3 - \frac{11}{24}\lambda^2 - \frac{6}{24}\lambda - \frac{1}{24}$). Note that 1 is simple and $|\lambda_j| \le 1$, $j = 2, \ldots, n$. By using the equality $M\mathbf{e} = \mathbf{e}$, we can introduce a matrix W whose eigenvalues are 0, $\lambda_2, \ldots, \lambda_n$:

$$W = M - \frac{1}{(\mathbf{e}_1^T M)\mathbf{e}}\mathbf{e}(\mathbf{e}_1^T M) == \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \square & -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ \square & \frac{n-2}{2n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ \square & \frac{n-3}{3n} & \frac{n-3}{3n} & -\frac{1}{n} & \ddots & -\frac{1}{n} \\ \square & \vdots & & \ddots & \ddots & \vdots \\ \square & \frac{1}{n(n-1)} & \cdots & \cdots & \frac{1}{n(n-1)} & -\frac{1}{n} \end{bmatrix}$$

and thus a $(n-1) \times (n-1)$ matrix whose eigenvalues are $\lambda_2 \ldots, \lambda_n$, the $n-1$ eigenvalues of A^{-1} different from 1:

$$\begin{bmatrix} -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & & -\frac{1}{n} \\ \frac{n-2}{2n} & -\frac{1}{n} & -\frac{1}{n} & & -\frac{1}{n} \\ \frac{n-3}{3n} & \frac{n-3}{3n} & -\frac{1}{n} & & -\frac{1}{n} \\ & & & & \\ \frac{1}{n(n-1)} & & & \frac{1}{n(n-1)} & -\frac{1}{n} \end{bmatrix}$$

Gershgorin rays: $(j-1)\frac{n-j}{jn} + \frac{n-j-1}{n}$; centers: $-\frac{1}{n}$.

Result. A sufficient condition for $|\lambda_j| < 1$, $j = 2, \ldots, n$, is that

$$\frac{(n-j)(2j-1)}{jn} < 1, \quad j = 1, \ldots, n-1. \qquad (*)$$

The above inequality is satisfied for $j = 1$ and for $j = n-1$. Since the function $\varphi(x) = (n-x)(2x-1)/(nx)$, $x \in [1, n-1]$, reaches its maximum value for $x = \sqrt{n/2}$ and $\varphi(\sqrt{n/2}) < 1$ if and only if $(n+1)\sqrt{2n} < 4n$, we can say that for $n \le 5$ the inequalities (*) are certainly true. For $n = 6$ the inequalities (*) are all verified except for $j = 2$, in which case the equality holds. Gershgorin yet applies (Third G. Theorem). For $n = 7$, the inequality $j = 2$ would yield $15 \le 14$), so Gershgorin for $n \ge 7$ does not apply.

Question. Are for all n satisfied the inequalities $|\lambda_j| < 1$, $j = 2, \ldots, n$?

COMPITO D'ESAME

Let B be any positive, stochastic by columns $n \times n$ matrix. Let

$$A^{-1} = \begin{bmatrix} \frac{1}{n} & 1 & \frac{1}{2} & \cdots & \frac{1}{n-1} \\ \frac{1}{n} & 0 & \frac{1}{2} & & \frac{1}{n-1} \\ \vdots & & & \ddots & \vdots \\ \frac{1}{n} & & & & \frac{1}{n-1} \\ \frac{1}{n} & 0 & & & 0 \end{bmatrix}, \ A = \begin{bmatrix} & & & & n \\ 1 & -1 & & & \\ & 2 & -2 & & \\ & & \ddots & \ddots & \\ & & & n-1 & 1-n \end{bmatrix}.$$

Since A is invertible, the following identities are equivalent

$$(B - A)\mathbf{p} = \mathbf{0}, \ B\mathbf{p} = A\mathbf{p}, \ A^{-1}B\mathbf{p} = \mathbf{p}.$$

Note that $A^{-1}B$ is positive and stochastic by columns (product of stochastic by columns matrices), thus, by the Perron-Frobenius theory, $1 = \rho(A^{-1}B)$ is a simple eigenvalue of $A^{-1}B$ and there is a unique positive vector $\mathbf{p}$, $\|\mathbf{p}\|_1 = 1$, such that $\mathbf{p} = A^{-1}B\mathbf{p}$. Moreover, such vector can be computed by the following iterative method

$$\mathbf{p}^{(0)} \text{ positive}, \ \sum_i p_i^{(0)} = 1, \ \ \mathbf{p}^{(k+1)} = A^{-1}B\mathbf{p}^{(k)}, \ k = 0, 1, \ldots$$

(i.e. by the power method), which in fact generates a sequence of positive vectors $\mathbf{p}^{(k)}$, $\sum_i p_i^{(k)} = 1$, convergent to $\mathbf{p}$ ($A^{-1}B$ positive and stochastic by columns imply that 1 is eigenvalue and the remaining eigenvalues are smaller than 1!).

The vector $\mathbf{p}^{(k+1)}$ is defined by the equation

$$A\mathbf{p}^{(k+1)} = B\mathbf{p}^{(k)}, \ k = 0, 1, \ldots$$

and thus can be computed from $\mathbf{p}^{(k)}$ by a matrix-vector multiplication involving B and by solving a linear system, $A\mathbf{z} = \mathbf{b}$, whose coefficient matrix is A. It is clear from the structure of A that such system can be solved in at most $2n$ multiplicative operations (z_n from the first equation, z_{n-1} from the last one, $\ldots$, z_1 from the second one).

Since A^{-1} is non negative and irreducible (one goes from i to j passing through 1, and thus the graph associated with A^{-1} is strongly connected), we can say that

$$\exists! \ \mathbf{p} \text{ positive}, \|\mathbf{p}\|_1 = 1, \ A^{-1}\mathbf{p} = \mathbf{p}$$

(by the Perron-Frobenius theory). For $n = 4$:

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 0 & 0 & \frac{1}{3} \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}, \ M = (A^{-1})^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

The matrix M has the same eigenvalues of A^{-1}. Moreover, $M\mathbf{e} = \mathbf{e}$. We can apply the deflation:

$$W = M - \frac{1}{\mathbf{e}_1^T M \mathbf{e}} \mathbf{e}\mathbf{e}_1^T M = M - \begin{bmatrix} \mathbf{e}_1^T M \\ \mathbf{e}_1^T M \\ \mathbf{e}_1^T M \\ \mathbf{e}_1^T M \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} \end{bmatrix}.$$

So, the remaining eigenvalues of M are the eigenvalues of the following 3×3 matrix:

$$C = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} \end{bmatrix}$$

which, by the first Gershgorin theorem, has all its eigenvalues inside the the circle with center $-\frac{1}{4}$ and ray $\frac{1}{2}$.

By the above arguments, it follows that if $1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of A^{-1}, then $|\lambda_j|$, $j = 2, 3, 4$, are smaller than 1. Thus, the procedure

$$\mathbf{p}^{(0)} \text{ positive, } \sum_i p_i^{(0)} = 1, \;\; \mathbf{p}^{(k+1)} = A^{-1}\mathbf{p}^{(k)}, \; k = 0, 1, \ldots$$

generates a sequence of positive vectors $\mathbf{p}^{(k)}$, $\sum_i p_i^{(k)} = 1$, convergent to $\mathbf{p}$ positive, $\|\mathbf{p}\|_1 = 1$, $\mathbf{p} = A^{-1}\mathbf{p}$.

In the generic case, such $\mathbf{p}$ can be computed by observing that it also solves the vector equation $\mathbf{p} = A\mathbf{p}$, or equivalently, the n equations

$$\left\{ \begin{array}{l} np_n = p_1 \\ p_1 - p_2 = p_2 \\ 2(p_2 - p_3) = p_3 \\ \cdots \\ (n-2)(p_{n-2} - p_{n-1}) = p_{n-1} \\ (n-1)(p_{n-1} - p_n) = p_n \end{array} \right. , \quad \left\{ \begin{array}{l} p_1 = np_n \\ p_2 = \frac{1}{2}\mathbf{p}_1 = \frac{n}{2}p_n \\ p_3 = \frac{2}{3}p_2 = \frac{n}{3}p_n \\ \cdots \\ p_{n-1} = \frac{n-2}{n-1}p_{n-2} = \frac{n}{n-1}p_n \end{array} \right.$$

$$p_j = \frac{n}{j}p_n, \; j = 1, \ldots, n, \; 1 = \sum_j p_j \; \Rightarrow \; p_n = \frac{1}{n(\sum_i \frac{1}{i})}.$$

The product of two stochastic by columns matrices is stochastic by columns.

$$B = \begin{bmatrix} \frac{1}{2} & b \\ \frac{1}{2} & 1-b \end{bmatrix}, \; b \in \mathbb{R}$$

$$BB^H = \begin{bmatrix} \frac{1}{4} + b^2 & \frac{1}{4} + b(1-b) \\ \frac{1}{4} + b(1-b) & \frac{1}{4} + (1-b)^2 \end{bmatrix}$$

$$B^H B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & b^2 + (1-b)^2 \end{bmatrix}$$

$$BB^H = B^H B, \;\; \frac{1}{4} + b^2 = \frac{1}{2}, \; b = \pm\frac{1}{2}, \; \frac{1}{4} + b(1-b) = \frac{1}{2}, \;\; b = \frac{1}{2}$$

A normal $\Rightarrow A = QDQ^H$, with Q unitary and D diagonal with diagonal entries equal to the eigenvalues of A. $p_A(\lambda) = \lambda^n \Rightarrow D = 0 \Rightarrow A = 0$.

The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have the same eigenvalues but are not similar (prove it!).

The inverse of a stochastic by columns matrix is stochastic by columns (see above)

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -\frac{4}{3} \\ -1 & -\frac{4}{3} & 1 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} 1 & & \\ & \alpha & \beta \\ & -\beta & \alpha \end{bmatrix}$$

$$AS_{23} = \begin{bmatrix} 3 & 2\alpha+\beta & 2\beta-\alpha \\ 2 & 3\alpha+\frac{4}{3}\beta & 3\beta-\frac{4}{3}\alpha \\ -1 & -\frac{4}{3}\alpha-\beta & -\frac{4}{3}\beta+\alpha \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{5} & 0 \\ 2 & \frac{22}{3\sqrt{5}} & \frac{1}{3\sqrt{5}} \\ -1 & -\frac{11}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} \end{bmatrix}$$

(for $\alpha = 2/\sqrt{5}$, $\beta = 1/\sqrt{5}$).

$$\begin{bmatrix} 1 & & \\ & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} AS_{23} = \begin{bmatrix} 3 & \sqrt{5} & 0 \\ \sqrt{5} & \frac{55}{15} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$f(x) = x^{-3/2},\ f'(x) = -\tfrac{3}{2}x^{-5/2},\ f''(x) = \tfrac{3}{2}\tfrac{5}{2}x^{-7/2},\ f'''(x) = -\tfrac{3}{2}\tfrac{5}{2}\tfrac{7}{2}x^{-9/2},$$

$$f^{(s)}(x) = (-1)^s \frac{3\cdot 5\cdots(2s+1)}{2^s} x^{-(s+3/2)},\ \ f^{(2j-1)}(x) = -\frac{3\cdot 5\cdot 7\cdots(4j-1)}{2^{2j-1}} x^{-(2j+\frac{1}{2})}$$

$$\int_m^n \frac{1}{x^{3/2}}\, dx = [-\frac{2}{\sqrt{x}}]_m^n = -\frac{2}{\sqrt{n}} + \frac{2}{\sqrt{m}}$$

$$\sum_{i=m}^{n} \frac{1}{i^{3/2}} = \frac{1}{2}(\frac{1}{m^{3/2}}+\frac{1}{n^{3/2}})+(-\frac{2}{\sqrt{n}}+\frac{2}{\sqrt{m}})+\sum_{j=1}^{k} \frac{B_{2j}(0)}{(2j)!}[-\frac{1}{n^{2j+\frac{1}{2}}}+\frac{1}{m^{2j+\frac{1}{2}}}]\frac{3\cdot 5\cdot 7\cdots(4j-1)}{2^{2j-1}}+u_{k+1}$$

where, since $f^{(s)}$ does not change sign in $[m,n]$, $m \geq 0$,

$$|u_{k+1}| \leq 2\frac{|B_{2k+2}(0)|}{(2k+2)!}|-\frac{1}{n^{2k+2+\frac{1}{2}}}+\frac{1}{m^{2k+2+\frac{1}{2}}}|\frac{3\cdot 5\cdot 7\cdots(4k+3)}{2^{2k+1}}$$

For $n \to +\infty$:

$$\sum_{i=m}^{+\infty} \frac{1}{i^{3/2}} = \frac{1}{2m\sqrt{m}} + \frac{2}{\sqrt{m}} + \sum_{j=1}^{k} \frac{B_{2j}(0)}{(2j)!}\frac{3\cdot 5\cdot 7\cdots(4j-1)}{m^{2j+\frac{1}{2}}2^{2j-1}} + u_{k+1}(\infty)$$

$$|u_{k+1}(\infty)| \leq 2\frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{3\cdot 5\cdot 7\cdots(4k+3)}{m^{2k+2+\frac{1}{2}}2^{2k+1}}$$

$m = 2$:

$$|u_{k+1}| \leq \frac{|B_{2k+2}(0)|3\cdot 5\cdot 7\cdots(4k+3)}{(2k+2)!2^{4k+2}\sqrt{2}}$$

$$\begin{aligned}
k=0: &\ \tfrac{1}{16\sqrt{2}} = 0.0441..\\
k=1: &\ \tfrac{7}{2^{10}3\sqrt{2}} = 0.0016..\\
k=2: &\ \tfrac{11}{2^{15}\sqrt{2}} = 2.37..\cdot 10^{-4}\\
k=3: &\ \tfrac{33\cdot 13}{2^{22}\sqrt{2}} = 7.23..\cdot 10^{-5}\\
k=4: &\ \tfrac{5\cdot 11\cdot 13\cdot 17\cdot 19}{66\cdot 2^{26}\sqrt{2}} = 3.68..\cdot 10^{-5}\\
k=5: &\ \tfrac{691\cdot 7\cdot 13\cdot 17\cdot 19\cdot 23}{2730\cdot 2^{32}\sqrt{2}} = 2.81..\cdot 10^{-5}\\
k=6: &\ \tfrac{7\cdot 17\cdot 19\cdot 23\cdot 25\cdot 27}{6\cdot 2^{37}\sqrt{2}} = 3.0..\cdot 10^{-5}
\end{aligned}$$

$$y'(t) = -200ty(t)^2, \quad y(-1) = \frac{1}{101}$$

$f(t, y(t)) = -200ty(t)^2$, $y(-1) = \frac{1}{101} = \eta^{EE}(-1) = \eta^C(-1)$.

EE:

$$\begin{aligned}
\eta^{EE}(-\tfrac{1}{2}) &= \eta^{EE}(-1) + \tfrac{1}{2}f(-1, \eta^{EE}(-1)) = \tfrac{1}{101} + \tfrac{1}{2}[-200(-1)\tfrac{1}{101^2}] \\
&= \tfrac{1}{101} + \tfrac{100}{101^2} = \tfrac{201}{101^2} = 0.0197\ldots
\end{aligned}$$

$$\begin{aligned}
\eta^{EE}(0) &= \eta^{EE}(-\tfrac{1}{2}) + \tfrac{1}{2}f(-\tfrac{1}{2}, \eta^{EE}(-\tfrac{1}{2})) = \tfrac{201}{101^2} + \tfrac{1}{2}[-200(-\tfrac{1}{2})(\tfrac{201}{101^2})^2] \\
&= \tfrac{201}{101^2} + 50(\tfrac{201}{101^2})^2 = \tfrac{201 \cdot 101^2 + 50 \cdot 201^2}{101^4} = \tfrac{4070451}{101^4} = 0.039116\ldots
\end{aligned}$$

C:

$$\begin{aligned}
K_1 &= f(-1, \eta^C(-1)) = -200(-1)\tfrac{1}{101^2}, \\
K_2 &= f(-1 + \tfrac{1}{2}\tfrac{1}{2}, \eta^C(-1) + \tfrac{1}{2}\tfrac{1}{2}\tfrac{200}{101^2}) = -200(-\tfrac{3}{4})(\tfrac{1}{101} + \tfrac{1}{4}\tfrac{200}{101^2})^2, \\
\eta^C(-\tfrac{1}{2}) &= \eta^C(-1) + \tfrac{1}{2}K_2 = \tfrac{1}{101} + \tfrac{300}{4}(\tfrac{1}{101} + \tfrac{1}{4}\tfrac{200}{101^2})^2 \\
&= \tfrac{1}{101} + 75\tfrac{151^2}{101^4} = \tfrac{101^3 + 75 \cdot 151^2}{101^4} = \tfrac{2740376}{101^4} = 0.026334\ldots
\end{aligned}$$

$$\begin{aligned}
K_1 &= f(-\tfrac{1}{2}, \eta^C(-\tfrac{1}{2})) = -200(-\tfrac{1}{2})\eta^C(-\tfrac{1}{2})^2, \\
K_2 &= f(-\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{2}, \eta^C(-\tfrac{1}{2}) + \tfrac{1}{2}\tfrac{1}{2}K_1) = -200(-\tfrac{1}{4})(\eta^C(-\tfrac{1}{2}) + \tfrac{1}{4}100\eta^C(-\tfrac{1}{2})^2)^2, \\
\eta^C(0) &= \eta^C(-\tfrac{1}{2}) + \tfrac{1}{2}K_2 \\
&= \eta^C(-\tfrac{1}{2}) + \tfrac{1}{2}50(\eta^C(-\tfrac{1}{2}) + 25\eta^C(-\tfrac{1}{2})^2)^2 \\
&= \eta^C(-\tfrac{1}{2}) + 25(\eta^C(-\tfrac{1}{2}) + 25\eta^C(-\tfrac{1}{2})^2)^2 = 0.074014\ldots
\end{aligned}$$

Compute an approximation of $\int_n^{n+1} f(t)\,dt$, $f(t) = \frac{1}{t}$, by using three values of f in $[n, n+1]$. (Note that the right value of the integral is $\log_e(1 + \frac{1}{n})$).

$$I_1 = 1 \cdot [\frac{1}{2}f(n) + \frac{1}{2}f(n+1)] = 1 \cdot [\frac{1}{2n} + \frac{1}{2(n+1)}] = \frac{2n+1}{2n(n+1)}$$

$$\begin{aligned}
I_{\frac{1}{2}} &= \tfrac{1}{2}[\tfrac{1}{2}f(n) + f(n + \tfrac{1}{2}) + \tfrac{1}{2}f(n+1)] = \tfrac{1}{2}[\tfrac{2n+1}{2n(n+1)} + \tfrac{2}{2n+1}] \\
&= \tfrac{1}{2}\tfrac{(2n+1)^2 + 4n(n+1)}{2n(n+1)(2n+1)} = \tfrac{8n^2 + 8n + 1}{4n(n+1)(2n+1)}
\end{aligned}$$

$$\begin{aligned}
\tilde{I} = \tfrac{2^2 I_{\frac{1}{2}} - I_1}{2^2 - 1} &= \tfrac{\frac{8n^2+8n+1}{n(n+1)(2n+1)} - \frac{2n+1}{2n(n+1)}}{3} \\
&= \tfrac{2(8n^2+8n+1) - (2n+1)^2}{6n(n+1)(2n+1)} = \tfrac{12n^2+12n+1}{6n(n+1)(2n+1)}.
\end{aligned}$$

Note that for $\tilde{I} \to +\infty$ when $n \to +\infty$.

We already know that $B_{2k+1}(0) = B_{2k+1}(\frac{1}{2}) = B_{2k+1}(1) = 0$, $k \geq 1$. Assume $B_{2k+1}(\xi) = 0$, $\xi \in (0, \frac{1}{2})$. Then $B'_{2k+1}(\mu) = B'_{2k+1}(\eta) = 0$, for some $\mu \in (0, \xi)$, $\eta \in (\xi, \frac{1}{2})$, and therefore, since $B'_{2k+1}(x) = (2k+1)B_{2k}(x)$, we have $B_{2k}(\mu) = B_{2k}(\eta) = 0$. It follows that $B'_{2k}(\rho) = 0$, for some $\rho \in (\mu, \eta) \subset (0, \frac{1}{2})$, that is, being $B'_{2k}(x) = 2kB_{2k-1}(x)$, the polynomial B_{2k-1} must be zero in $\rho \in (0, \frac{1}{2})$. Since k is generic, such arguments imply that there exists $\gamma \in (0, \frac{1}{2})$ such that B_3 is zero in γ, which is absurd, since the only zeros of B_3 in $[0, \frac{1}{2}]$ are 0 and $\frac{1}{2}$.

$$
\begin{aligned}
(a):\ y(x-h) &= y(x) - hy'(x) + \tfrac{h^2}{2}y''(x) \\
&\quad -\tfrac{h^3}{6}y'''(x) + \tfrac{h^4}{24}y''''(x) + O(h^5), \\
(b):\ y(x+h) &= y(x) + hy'(x) + \tfrac{h^2}{2}y''(x) \\
&\quad +\tfrac{h^3}{6}y'''(x) + \tfrac{h^4}{24}y''''(x) + O(h^5), \\
(c):\ y(x+2h) &= y(x) + 2hy'(x) + \tfrac{4h^2}{2}y''(x) \\
&\quad +\tfrac{8h^3}{6}y'''(x) + \tfrac{16h^4}{24}y''''(x) + O(h^5),
\end{aligned}
$$

$$
\begin{aligned}
&(d) = (b) - (a):\ y(x+h) - y(x-h) = 2hy'(x) + \tfrac{2}{6}h^3y'''(x) + O(h^5), \\
&(e) = (c) - 4(b):\ y(x+2h) - 4y(x+h) = -3y(x) - 2hy'(x) + \tfrac{4}{6}h^3y'''(x) + \tfrac{12}{24}h^4y''''(x) + O(h^5),
\end{aligned}
$$

$$
2(d) - (e):\ y'(x) = \frac{-2y(x-h) - 3y(x) + 6y(x+h) - y(x+2h)}{6h} + O(h^3)
$$