Ramanujan in his first paper states that the Bernoulli numbers satisfy the following equations

$$\begin{bmatrix} 1 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & & \\ \frac{1}{3} & 0 & 0 & 1 & & & & & & & & \\ 0 & \frac{5}{2} & 0 & 0 & 1 & & & & & & & \\ 0 & 0 & 11 & 0 & 0 & 1 & & & & & & \\ \frac{1}{5} & 0 & 0 & \frac{143}{4} & 0 & 0 & 1 & & & & & \\ 0 & 4 & 0 & 0 & \frac{286}{3} & 0 & 0 & 1 & & & & \\ 0 & 0 & \frac{204}{5} & 0 & 0 & 221 & 0 & 0 & 1 & & & \\ \frac{1}{7} & 0 & 0 & \frac{1938}{7} & 0 & 0 & \frac{3230}{7} & 0 & 0 & 1 & & \\ 0 & \frac{11}{2} & 0 & 0 & \frac{7106}{5} & 0 & 0 & \frac{3553}{4} & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ B_8(0) \\ B_{10}(0) \\ B_{12}(0) \\ B_{14}(0) \\ B_{16}(0) \\ B_{18}(0) \\ B_{20}(0) \\ B_{22}(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/30 \\ 1/42 \\ 1/45 \\ -1/132 \\ 4/455 \\ 1/120 \\ -1/306 \\ 3/665 \\ 1/231 \\ -1/552 \\ \cdot \end{bmatrix}$$

(actually, since Ramanujan Bernoulli numbers are the moduli of ours, his equations are a bit different).

Call

$$R \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix} \qquad (RamanSyst)$$

such semi-infinite Ramanujan linear system. We observe that such system is equivalent to the following linear systems:

$$R \begin{bmatrix} \frac{2!}{x} & & \\ & \frac{4!}{x^2} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \frac{x}{2!} & & \\ & \frac{x^2}{4!} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \frac{2!}{x} & & \\ & \frac{4!}{x^2} & \\ & & \ddots \end{bmatrix} \tilde{R} \begin{bmatrix} \frac{x}{2!} & & \\ & \frac{x^2}{4!} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix}$$

$$\tilde{R} \begin{bmatrix} \frac{x}{2!} & & \\ & \frac{x^2}{4!} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{x}{2!} & & \\ & \frac{x^2}{4!} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix}, \quad \tilde{R} = \sum_{j=0}^{+\infty} \alpha_j Z^j = \sum_{s=0}^{+\infty} \frac{2\, x^{3s}}{(6s+2)!(2s+1)} Z^{3s},$$

$\alpha_0 = 1$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \frac{2}{8!3}x^3$, $\alpha_4 = \alpha_5 = 0$, $\alpha_6 = \frac{2}{14!5}x^6$, $\alpha_7 = \alpha_8 = 0$, $\alpha_9 = \frac{2}{20!7}x^9, \ldots$. The symbol Z denotes the semi-infinite lower shift matrix:

$$Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \cdot & \cdot \end{bmatrix}$$

So, the Ramanujan system is equivalent to a lower triangular Toeplitz (sparse) linear system. Can the latter system be obtained from the lower triangular Toeplitz (dense) even and odd linear systems introduced in toe_1n ? First let us prepare things.

Theorem.
Let Z_{n-1} and Z_n be the upper-left $n-1 \times n-1$ and $n \times n$ submatrices of Z.
Then

$$\sum_{j=0}^{n-1} \alpha_j Z_n^j \begin{bmatrix} B_0(0) \\ \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \Box \\ \frac{x}{2!} f_1 \\ \frac{x^2}{4!} f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} f_{n-1} \end{bmatrix} + B_0(0) \begin{bmatrix} ARB \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1} \end{bmatrix} \left(= \begin{bmatrix} w_0 \\ \frac{x}{2!} w_1 \\ \frac{x^2}{4!} w_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} w_{n-1} \end{bmatrix} \right)$$

if and only if

$$\sum_{j=0}^{n-2} \alpha_j Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!} f_1 \\ \frac{x^2}{4!} f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} f_{n-1} \end{bmatrix} \left(= \begin{bmatrix} \frac{x}{2!} w_1 \\ \frac{x^2}{4!} w_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} w_{n-1} \end{bmatrix} - B_0(0) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1} \end{bmatrix} \right)$$

The implication "$\Leftarrow$" holds if $\alpha_0 B_0(0) = \Box + B_0(0) \cdot ARB$ ($\alpha_0 B_0(0) = w_0$).

Application of the Theorem:
We have shown that the Ramanujan system is equivalent to the system

$$(*) \quad \sum_{j=0}^{n-2} \alpha_j Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!} f_1 \\ \frac{x^2}{4!} f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} f_{n-1} \end{bmatrix}, \ \alpha_j = \delta_{j=0 \bmod 3} \frac{2x^j}{(2j+2)!(\frac{2}{3}j+1)},$$

$$f_1 = \frac{1}{6},\ f_2 = -\frac{1}{30},\ f_3 = \frac{1}{42},\ f_4 = \frac{1}{45},\ f_5 = -\frac{1}{132},\ f_6 = \frac{4}{455},$$
$$f_7 = \frac{1}{120},\ f_8 =, -\frac{1}{306}\ f_9 = \frac{3}{665},\ f_{10} = \frac{1}{231},\ f_{11} = -\frac{1}{552}, \ldots$$

Then, by the Theorem,

$$\sum_{j=0}^{n-1} \alpha_j Z_n^j \begin{bmatrix} B_0(0) \\ \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \Box \\ \frac{x}{2!} f_1 \\ \frac{x^2}{4!} f_2 \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} f_{n-1} \end{bmatrix} + B_0(0) \begin{bmatrix} ARB \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_{n-1} \end{bmatrix}, \ \Box + B_0(0) \cdot ARB = \alpha_0 B_0(0),$$

$$(I + \frac{2}{8!3} x^3 Z^3 + \frac{2}{14!5} x^6 Z^6 + \frac{2}{20!7} x^9 Z^9 + \ldots) \begin{bmatrix} B_0(0) \\ \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \frac{x^3}{6!} B_6(0) \\ \frac{x^4}{8!} B_8(0) \\ \frac{x^5}{10!} B_{10}(0) \\ \frac{x^6}{12!} B_{12}(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{x}{2!} \frac{1}{6} \\ \frac{x^2}{4!} (-\frac{1}{30}) \\ \frac{x^3}{6!} \frac{1}{42} + \frac{2x^3}{8!3} \\ \frac{x^4}{8!} \frac{1}{45} \\ \frac{x^5}{10!} (-\frac{1}{132}) \\ \frac{x^6}{12!} \frac{4}{455} + \frac{2x^6}{14!5} \\ \frac{x^7}{14!} \frac{1}{120} \\ \frac{x^8}{16!} (-\frac{1}{306}) \\ \frac{x^9}{18!} \frac{3}{665} + \frac{2x^9}{20!7} \\ \frac{x^{10}}{20!} \frac{1}{231} \\ \frac{x^{11}}{22!} (-\frac{1}{552}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{0!} (\frac{1}{1 \cdot 1}) \\ \frac{x}{2!} (\frac{1}{3 \cdot 2}) \\ \frac{x^2}{4!} (\frac{1}{5 \cdot 3} - \frac{1}{5 \cdot 2}) \\ \frac{x^3}{6!} (\frac{1}{7 \cdot 4}) \\ \frac{x^4}{8!} (\frac{1}{9 \cdot 5}) \\ \frac{x^5}{10!} (\frac{1}{11 \cdot 6} - \frac{1}{11 \cdot 4}) \\ \frac{x^6}{12!} (\frac{1}{13 \cdot 7}) \\ \frac{x^7}{14!} (\frac{1}{15 \cdot 8}) \\ \frac{x^8}{16!} (\frac{1}{17 \cdot 9} - \frac{1}{17 \cdot 6}) \\ \frac{x^9}{18!} (\frac{1}{19 \cdot 10}) \\ \frac{x^{10}}{20!} (\frac{1}{21 \cdot 11}) \\ \frac{x^{11}}{22!} (\frac{1}{23 \cdot 12} - \frac{1}{23 \cdot 8}) \\ \vdots \end{bmatrix},$$

$$\sum_{j=0}^{+\infty} \alpha_j Z^j D_x \mathbf{b} = D_x \mathbf{q}^R,$$

$$\alpha_j = \delta_{j=0 \bmod 3} \frac{2x^j}{(2j+2)!(\frac{2}{3}j+1)}, \ \mathbf{q}^R = (\frac{1}{(2i+1)(i+1)}(1 - \delta_{i=2 \bmod 3} \frac{3}{2})), \ i = 0, 1, 2, 3, \ldots .$$

Note that from the explicit expression of $\mathbf{q}^R$ it follows an explicit expression for the f_i of the Ramanujan system:

$$f_i = \frac{1}{(2i+1)(i+1)}(1 - \delta_{i=2 \bmod 3}\frac{3}{2} - \delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}), \quad i = 1, 2, 3, \ldots.$$

Note also that (*) can be rewritten as

$$L(\underline{\alpha}_{n-1}) I_n^2 D_x \mathbf{b} = \operatorname{diag}(z_i,\ i = 1, 2, \ldots, n-1) I_n^2 D_x \mathbf{q}^R$$

for suitable z_i. Let us look for such z_i:

$$\begin{aligned}
&(1 - \delta_{i=2 \bmod 3}\tfrac{3}{2} - \delta_{i=0 \bmod 3}\tfrac{1}{\frac{2}{3}i+1}) = z_i(1 - \delta_{i=2 \bmod 3}\tfrac{3}{2}),\\
&z_i = 1 - \frac{\delta_{i=0 \bmod 3}\frac{1}{\frac{2}{3}i+1}}{1-\delta_{i=2 \bmod 3}\frac{3}{2}}\\
&= 1 - \delta_{i=0 \bmod 3}\tfrac{1}{\frac{2}{3}i+1}.
\end{aligned}$$

Now let us consider the two even and odd systems introduced in toe_1n whose solution is again the vector of Bernoulli numbers:

$$\sum_{j=0}^{+\infty} \frac{x^j}{(2j+2)!} Z^j D_x \mathbf{b} = D_x \mathbf{q}^e,\ \mathbf{q}^e = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{10} \\ \cdot \end{bmatrix},\ \sum_{j=0}^{+\infty} \frac{x^j}{(2j+1)!} Z^j D_x \mathbf{b} = D_x \mathbf{q}^o,\ \mathbf{q}^o = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ \cdot \end{bmatrix};$$

$$\sum_{j=0}^{n-1} \frac{x^j}{(2j+2)!} Z_n^j D_x \mathbf{b} = D_x \mathbf{q}^e, \quad \sum_{j=0}^{n-1} \frac{x^j}{(2j+1)!} Z_n^j D_x \mathbf{b} = D_x \mathbf{q}^o.$$

And apply to them the Theorem:

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+2)!} Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!}\frac{1}{6} \\ \frac{x^2}{4!}\frac{1}{10} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{1}{4n-2} \end{bmatrix} - B_0(0) \begin{bmatrix} \frac{x}{4!} \\ \frac{x^2}{6!} \\ \cdot \\ \frac{x^{n-1}}{(2n)!} \end{bmatrix} = \begin{bmatrix} x\frac{1}{4!} \\ x^2\frac{2}{6!} \\ x^3\frac{3}{8!} \\ \cdot \\ x^{n-1}\frac{n-1}{(2n)!} \end{bmatrix}$$

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+1)!} Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!}\frac{1}{2} \\ \frac{x^2}{4!}\frac{1}{2} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{1}{2} \end{bmatrix} - B_0(0) \begin{bmatrix} \frac{x}{3!} \\ \frac{x^2}{5!} \\ \cdot \\ \frac{x^{n-1}}{(2n-1)!} \end{bmatrix} = \begin{bmatrix} x\frac{1}{3!2} \\ x^2\frac{3}{5!2} \\ x^3\frac{5}{7!2} \\ \cdot \\ x^{n-1}\frac{2n-3}{(2n-1)!2} \end{bmatrix}$$

from which it follows

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+2)!} Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!}\frac{1}{4\cdot 3} \\ \frac{x^2}{4!}\frac{2}{6\cdot 5} \\ \frac{x^3}{6!}\frac{3}{8\cdot 7} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{n-1}{2n(2n-1)} \end{bmatrix} = \operatorname{diag}(\frac{i}{i+1}, i = 1 \ldots n-1) I_{n-1}^1 (Z_n^T D_x \mathbf{q}^e)$$

$$\sum_{j=0}^{n-2} \frac{x^j}{(2j+1)!} Z_{n-1}^j \begin{bmatrix} \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!} B_{2(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \frac{x}{2!}\frac{1}{2\cdot 3} \\ \frac{x^2}{4!}\frac{3}{2\cdot 5} \\ \frac{x^3}{6!}\frac{5}{2\cdot 7} \\ \cdot \\ \frac{x^{n-1}}{(2(n-1))!}\frac{2n-3}{2(2n-1)} \end{bmatrix} = \operatorname{diag}(\frac{2i-1}{2i+1}, i = 1 \ldots n-1) I_{n-1}^1 (Z_n^T D_x \mathbf{q}^o)$$

Set

$$\mathbf{b} = \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ \vdots \end{bmatrix}, \quad D_x = \operatorname{diag}\left(\frac{x^i}{(2i)!}, \; i = 0, 1, 2, \ldots\right).$$

Then

$$L(\alpha) D_x \mathbf{b} = D_x \mathbf{q}, \quad L(\alpha) Z^T D_x \mathbf{b} = d(\mathbf{z}) Z^T D_x \mathbf{q},$$

where the vectors $\alpha = (\alpha_i)_{i=0}^{+\infty}$, $\mathbf{q} = (q_i)_{i=0}^{+\infty}$, and $\mathbf{z} = (z_i)_{i=1}^{+\infty}$, can assume respectively the values:

$$\alpha_i^R = \delta_{i=0 \bmod 3} \frac{2x^i}{(2i+2)!(\frac{2}{3}i+1)}, \quad q_i^R = \frac{1}{(2i+1)(i+1)}(1 - \delta_{i=2 \bmod 3} \tfrac{3}{2}), \; i = 0, 1, 2, 3, \ldots$$
$$z_i^R = 1 - \delta_{i=0 \bmod 3} \frac{1}{\frac{2}{3}i+1}, \; i = 1, 2, 3, \ldots,$$

$$\alpha_i^e = \frac{x^i}{(2i+2)!}, \quad q_i^e = \frac{1}{2(2i+1)}, \; i = 0, 1, 2, 3, \ldots$$
$$z_i^e = \frac{i}{i+1}, \; i = 1, 2, 3, \ldots,$$

$$\alpha_i^o = \frac{x^i}{(2i+1)!}, \; i = 0, 1, 2, 3, \ldots, \; q_0^o = 1, \; q_i^o = \frac{1}{2}, \; i = 1, 2, 3, \ldots$$
$$z_i^o = \frac{2i-1}{2i+1}, \; i = 1, 2, 3, \ldots.$$

(Z is the semi-infinite lower shift matrix

$$Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \cdot & \cdot \end{bmatrix},$$

$L(\alpha)$ is the semi-infinite lower triangular Toeplitz matrix with first column α, i.e.

$$L(\alpha) = \sum_{i=0}^{+\infty} \alpha_i Z^i = \begin{bmatrix} \alpha_0 & & & & \\ \alpha_1 & \alpha_0 & & & \\ \alpha_2 & \alpha_1 & \alpha_0 & & \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.\,)$$

Exercise.

Try to prove that the system $L(\alpha^R) D_x \mathbf{b} = D_x \mathbf{q}^R$ (the lower triangular Toeplitz sparse system equivalent to the Ramanujan lower triangular sparse system) can be obtained as a consequence of the system $L(\alpha^o) D_x \mathbf{b} = D_x \mathbf{q}^o$ (or $L(\alpha^e) D_x \mathbf{b} = D_x \mathbf{q}^e$).

For example, try to find β^e, β^o such that $L(\beta^e) L(\alpha^e) = L(\alpha^R) = L(\beta^o) L(\alpha^o)$ [?? $L(\beta^e) D_x \mathbf{q}^e = D_x \mathbf{q}^R = L(\beta^o) D_x \mathbf{q}^o$??], i.e. $L(\alpha^e)\beta^e = \alpha^R = L(\alpha^o)\beta^o$.

Find γ such that $L(\gamma) L(\alpha^e) = L(\alpha)$ (or $L(\gamma) L(\alpha^o) = L(\alpha)$) with α more sparse than α^R.

Exercise.

Prove that $d(\mathbf{z}) Z^T L(\alpha)(D_x \mathbf{b}) = L(\alpha) Z^T (D_x \mathbf{b})$.

Note that

$$L(\mathbf{a}) = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ a_2 \\ -a_3 \\ a_4 \\ -a_5 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2a_2 - a_1^2 \\ 0 \\ 2a_4 - 2a_1a_3 + a_2^2 \\ 0 \\ 2a_6 - 2a_1a_5 + 2a_2a_4 - a_3^2 \\ 0 \\ 2a_8 - 2a_1a_7 + 2a_2a_6 - 2a_3a_5 + a_4^2 \\ 0 \\ \cdot \end{bmatrix},$$

$$L(\mathbf{a})\Big(\mathbf{e}_1 + \sum_{i=1}^{+\infty}(-1)^i a_i \mathbf{e}_{i+1}\Big) = \mathbf{e}_1 + \sum_{i=1}^{+\infty} \delta_{i=0 \bmod 2}\Big(2a_i + \sum_{j=1}^{i-1}(-1)^j a_j a_{i-j}\Big)\mathbf{e}_{i+1}.$$

If we introduce the following two semi-infinite matrices,

$$I_\pm = \operatorname{diag}((-1)^i,\ i = 0, 1, 2, 3, \ldots), \quad E = \begin{bmatrix} 1 & 0 & & & \\ 0 & 0 & & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

then the previous equality can be rewritten as

$$L(\mathbf{a})I_\pm \mathbf{a} = E\mathbf{a}^{(1)},\ a_0^{(1)} = 1,\ a_s^{(1)} = 2a_{2s} + \sum_{j=1}^{2s-1}(-1)^j a_j a_{2s-j},\ s = 1, 2, 3, \ldots.$$

Of course, $L(\mathbf{a})L(I_\pm\mathbf{a}) = L(I_\pm\mathbf{a})L(\mathbf{a}) = L(E\mathbf{a}^{(1)})$, so, for example, a lower triangular Toeplitz (l.t.T.) system of the type $L(\mathbf{a})\mathbf{z} = \mathbf{c}$ is equivalent to the l.t.T. system $L(E\mathbf{a}^{(1)})\mathbf{z} = L(I_\pm\mathbf{a})\mathbf{c}$ whose coefficient matrix has null diagonals alternating with non-null ones. We shall introduce, as a consequence of this result, an algorithm of complexity $O(n(\log_2 n)^2)$ for solving a lower triangular Toeplitz system (of order $n = 2^k$).

Exercise. Apply the above result to the odd (even) system (the one solved by Bernoulli numbers) in order to obtain a system where null diagonals alternate with the non-null ones. Is it possible to write an explicit formula for the entries of the non-null diagonals?

Exercise. Use twice the above result in order to show that a l.t.T. system of the type $L(\mathbf{a})\mathbf{z} = \mathbf{c}$ is equivalent to a l.t.T. system where the coefficient matrix has three-null diagonals alternating with non-null ones.

Exercise. Answer to the quotation marks in the following equality:

$$L(\mathbf{a}) = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ ? \\ ? \\ ? \\ ? \\ ? \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \\ ? \\ \cdot \end{bmatrix}.$$

As a consequence of this result,

(i) introduce an algorithm of complexity $O(n(\log_3 n)^2)$ for solving a lower triangular Toeplitz system (for example, in case the system has order $n = 27$ such algorithm is more efficient rather than embed the system in one of order $n = 32$ and use the algorithm ok for $n = 2^k$);

(ii) obtain the Ramanujan system solved by Bernoulli numbers as a consequence of the odd (even) system.

A lower triangular Toeplitz system solver of complexity $O(n(\log_2 n)^2)$
Set

$$n = 2^k,\ \mathbf{a} = \mathbf{a}^{(0)} = \begin{bmatrix} 1 \\ a_1 \\ \cdot \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ a_1^{(0)} \\ \cdot \\ a_{n-1}^{(0)} \end{bmatrix},\ L(\mathbf{a}) = \begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \\ a_{n-1} & \cdot & a_2 & a_1 & 1 \end{bmatrix},$$

and let $I_\pm$ and E denote the $n \times n$ upper-left submatrices of the previously defined semi-infinite matrices $I_\pm$ and E. In the following we multiply $L(\mathbf{a})$ on the left by suitable l.t.T matrices so to reduce the number of its non-null diagonals; we do this in $k = \log_2 n$ steps:

step 1

$$L(\mathbf{a})I_\pm\mathbf{a} =: E\mathbf{a}^{(1)},$$
$$a_0^{(1)} = 1,\ a_s^{(1)} = 2a_{2s}^{(0)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(0)} a_{2s-j}^{(0)},\ s = 1, \dots, \tfrac{n}{2} - 1,\ a_s^{(1)} = 0,\ s = \tfrac{n}{2}, \dots, n-1,$$
$$L(\mathbf{a})L(I_\pm\mathbf{a}) = L(I_\pm\mathbf{a})L(\mathbf{a}) = L(E\mathbf{a}^{(1)})$$

Note: One here needs to compute $\mathbf{a}^{(1)}$, i.e. ($\frac{n}{2} - 1$ odd entries of) the vector $L(\mathbf{a}) \cdot I_\pm\mathbf{a}$. Denote by φ_n the cost (the number of multiplications) of such operation.

step 2

$$L(E\mathbf{a}^{(1)})EI_\pm\mathbf{a}^{(1)} =: E^2\mathbf{a}^{(2)},$$
$$a_0^{(2)} = 1,\ a_s^{(2)} = 2a_{2s}^{(1)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(1)} a_{2s-j}^{(1)},\ s = 1, \dots, \tfrac{n}{4} - 1,\ a_s^{(2)} = 0,\ s = \tfrac{n}{4}, \dots, n-1,$$
$$L(E\mathbf{a}^{(1)})L(EI_\pm\mathbf{a}^{(1)}) = L(EI_\pm\mathbf{a}^{(1)})L(E\mathbf{a}^{(1)}) = L(E^2\mathbf{a}^{(2)})$$

Note: One here needs to compute $\mathbf{a}^{(2)}$, i.e. ($\frac{n}{4} - 1$ odd entries of) the order $\frac{n}{2}$ non null first part of the vector $L(\mathbf{a}^{(1)}) \cdot I_\pm\mathbf{a}^{(1)}$. Denote by $\varphi_{\frac{n}{2}}$ the cost of such operation.

step 3

$$L(E^2\mathbf{a}^{(2)})E^2 I_\pm \mathbf{a}^{(2)} =: E^3\mathbf{a}^{(3)},$$
$$a_0^{(3)} = 1,\ a_s^{(3)} = 2a_{2s}^{(2)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(2)} a_{2s-j}^{(2)},\ s = 1,\dots,\tfrac{n}{8}-1,\ a_s^{(3)} = 0,\ s = \tfrac{n}{8},\dots,n-1,$$
$$L(E^2\mathbf{a}^{(2)})L(E^2 I_\pm \mathbf{a}^{(2)}) = L(E^2 I_\pm \mathbf{a}^{(2)})L(E^2\mathbf{a}^{(2)}) = L(E^3\mathbf{a}^{(3)})$$

Note: One here needs to compute $\mathbf{a}^{(3)}$, i.e. ($\frac{n}{8}-1$ odd entries of) the order $\frac{n}{4}$ non null first part of the vector $L(\mathbf{a}^{(2)}) \cdot I_\pm \mathbf{a}^{(2)}$. Denote by $\varphi_{\frac{n}{4}}$ the cost of such operation.

...

step $k-1 = \log_2 n - 1$

$$L(E^{k-2}\mathbf{a}^{(k-2)})E^{k-2} I_\pm \mathbf{a}^{(k-2)} =: E^{k-1}\mathbf{a}^{(k-1)},$$
$$a_0^{(k-1)} = 1,$$
$$a_s^{(k-1)} = 2a_{2s}^{(k-2)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(k-2)} a_{2s-j}^{(k-2)},\ s = 1,\dots,\tfrac{n}{2^{k-1}}-1,$$
$$a_s^{(k-1)} = 0,\ s = \tfrac{n}{2^{k-1}},\dots,n-1,$$
$$L(E^{k-2}\mathbf{a}^{(k-2)})L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) = L(E^{k-2} I_\pm \mathbf{a}^{(k-2)})L(E^{k-2}\mathbf{a}^{(k-2)}) = L(E^{k-1}\mathbf{a}^{(k-1)})$$

Note: One here needs to compute $\mathbf{a}^{(k-1)}$, i.e. ($\frac{n}{2^{k-1}}-1$ odd entries of) the order $\frac{n}{2^{k-2}}$ non null first part of the vector $L(\mathbf{a}^{(k-2)}) \cdot I_\pm \mathbf{a}^{(k-2)}$. Denote by $\varphi_{\frac{n}{2^{k-2}}}$ the cost of such operation.

step $k = \log_2 n$

$$L(E^{k-1}\mathbf{a}^{(k-1)})E^{k-1} I_\pm \mathbf{a}^{(k-1)} =: E^{k}\mathbf{a}^{(k)},$$
$$a_0^{(k)} = 1,$$
$$a_s^{(k)} = 0,\ s = \tfrac{n}{2^k},\dots,n-1,$$
$$L(E^{k-1}\mathbf{a}^{(k-1)})L(E^{k-1} I_\pm \mathbf{a}^{(k-1)}) = L(E^{k-1} I_\pm \mathbf{a}^{(k-1)})L(E^{k-1}\mathbf{a}^{(k-1)}) = L(E^{k}\mathbf{a}^{(k)})$$

($E^k = \mathbf{e}_1\mathbf{e}_1^T$, $\mathbf{a}^{(k)} = \mathbf{e}_1$, $L(E^k\mathbf{a}^{(k)}) = I$)
Note: One here needs to compute $\mathbf{a}^{(k)}$, i.e. ($\frac{n}{2^k}-1$ odd entries of) the order $\frac{n}{2^{k-1}}$ non null first part of the vector $L(\mathbf{a}^{(k-1)}) \cdot I_\pm \mathbf{a}^{(k-1)}$. Denote by $\varphi_{\frac{n}{2^{k-1}}}$ the cost of such operation ($\varphi_{\frac{n}{2^{k-1}}} = 0$).

Finally note that the following equality holds

$$L(E^{k-1} I_\pm \mathbf{a}^{(k-1)})L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) \cdots L(E^2 I_\pm \mathbf{a}^{(2)})L(E I_\pm \mathbf{a}^{(1)})L(I_\pm \mathbf{a})\Big[\,L(\mathbf{a})\,\Big] = I,$$

in other words we have transformed $L(\mathbf{a})$ into the identity matrix.

Example: $n = 8$

$$n = 2^3,\ \mathbf{a} = \mathbf{a}^{(0)} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 1 \\ a_1^{(0)} \\ a_2^{(0)} \\ a_3^{(0)} \\ a_4^{(0)} \\ a_5^{(0)} \\ a_6^{(0)} \\ a_7^{(0)} \end{bmatrix},\ L(\mathbf{a}) = \begin{bmatrix} 1 & & & & & & & \\ a_1 & 1 & & & & & & \\ a_2 & a_1 & 1 & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \end{bmatrix},$$

and let $I_\pm$ and E denote the 8×8 upper-left submatrices of the previously defined semi-infinite matrices $I_\pm$ and E.

step 1

$$L(\mathbf{a})I_\pm\mathbf{a} = \begin{bmatrix} 1 &&&&&&& \\ a_1 & 1 &&&&&& \\ a_2 & a_1 & 1 &&&&& \\ a_3 & a_2 & a_1 & 1 &&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ a_2 \\ -a_3 \\ a_4 \\ -a_5 \\ a_6 \\ -a_7 \end{bmatrix} =: E\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(1)} \\ 0 \\ a_2^{(1)} \\ 0 \\ a_3^{(1)} \\ 0 \end{bmatrix},$$

$a_0^{(1)} = 1,\ a_s^{(1)} = 2a_{2s}^{(0)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(0)} a_{2s-j}^{(0)},\ s = 1, \ldots, \frac{8}{2} - 1,\ a_s^{(1)} = 0,\ s = \frac{8}{2}, \ldots, 8-1,$
$a_1^{(1)} = 2a_2 - a_1^2,\ a_2^{(1)} = 2a_4 - 2a_1a_3 + a_2^2,\ a_3^{(1)} = 2a_6 - 2a_1a_5 + 2a_2a_4 - a_3^2,$
$L(\mathbf{a})L(I_\pm\mathbf{a}) = L(I_\pm\mathbf{a})L(\mathbf{a}) = L(E\mathbf{a}^{(1)})$

Note: One here needs to compute $\mathbf{a}^{(1)}$, i.e. ($\frac{8}{2} - 1$ odd entries of) the vector $L(\mathbf{a}) \cdot I_\pm\mathbf{a}$. Denote by φ_8 the cost of such operation ($\varphi_8 \leq 6$ if multiplications by 2 are not counted).

step 2

$$L(E\mathbf{a}^{(1)})EI_\pm\mathbf{a}^{(1)} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ a_1^{(1)} && 1 &&&&& \\ & a_1^{(1)} && 1 &&&& \\ a_2^{(1)} && a_1^{(1)} && 1 &&& \\ & a_2^{(1)} && a_1^{(1)} && 1 && \\ a_3^{(1)} && a_2^{(1)} && a_1^{(1)} && 1 & \\ & a_3^{(1)} && a_2^{(1)} && a_1^{(1)} && 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -a_1^{(1)} \\ 0 \\ a_2^{(1)} \\ 0 \\ -a_3^{(1)} \\ 0 \end{bmatrix} =: E^2\mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ a_1^{(2)} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$a_0^{(2)} = 1,\ a_s^{(2)} = 2a_{2s}^{(1)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(1)} a_{2s-j}^{(1)},\ s = 1, \ldots, \frac{8}{4} - 1,\ a_s^{(2)} = 0,\ s = \frac{8}{4}, \ldots, 8-1,$
$a_1^{(2)} = 2a_2^{(1)} - (a_1^{(1)})^2,$
$L(E\mathbf{a}^{(1)})L(EI_\pm\mathbf{a}^{(1)}) = L(EI_\pm\mathbf{a}^{(1)})L(E\mathbf{a}^{(1)}) = L(E^2\mathbf{a}^{(2)})$

Note: One here needs to compute $\mathbf{a}^{(2)}$, i.e. ($\frac{8}{4} - 1$ odd entries of) the order $\frac{8}{2}$ non null first part of the vector $L(\mathbf{a}^{(1)}) \cdot I_\pm\mathbf{a}^{(1)}$. Denote by $\varphi_{\frac{8}{2}}$ the cost of such operation ($\varphi_{\frac{8}{2}} \leq 1$ if multiplications by 2 are not counted).

step $3 = \log_2 8$

$$L(E^2\mathbf{a}^{(2)})E^2I_\pm\mathbf{a}^{(2)} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ a_1^{(2)} &&&& 1 &&& \\ & a_1^{(2)} &&&& 1 && \\ && a_1^{(2)} &&&& 1 & \\ &&& a_1^{(2)} &&&& 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -a_1^{(2)} \\ 0 \\ 0 \\ 0 \end{bmatrix} =: E^3\mathbf{a}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$a_0^{(3)} = 1,\ a_s^{(3)} = 2a_{2s}^{(2)} + \sum_{j=1}^{2s-1}(-1)^j a_j^{(2)} a_{2s-j}^{(2)},\ s = 1, \ldots, \frac{8}{8} - 1,\ a_s^{(3)} = 0,\ s = \frac{8}{8}, \ldots, 8-1,$
$L(E^2\mathbf{a}^{(2)})L(E^2I_\pm\mathbf{a}^{(2)}) = L(E^2I_\pm\mathbf{a}^{(2)})L(E^2\mathbf{a}^{(2)}) = L(E^3\mathbf{a}^{(3)})$

($E^3 = \mathbf{e}_1\mathbf{e}_1^T$, $\mathbf{a}^{(3)} = \mathbf{e}_1$, $L(E^3\mathbf{a}^{(3)}) = I$) Note: One here needs to compute $\mathbf{a}^{(3)}$, i.e. ($\frac{8}{8} - 1$ odd entries of) the order $\frac{8}{4}$ non null first part of the vector $L(\mathbf{a}^{(2)}) \cdot I_\pm\mathbf{a}^{(2)}$. Denote by $\varphi_{\frac{8}{4}}$ the cost of such operation ($\varphi_{\frac{8}{4}} = 0$).

By using the obtained identities, one realizes that

$$L(E^2 I_\pm \mathbf{a}^{(2)}) L(E I_\pm \mathbf{a}^{(1)}) L(I_\pm \mathbf{a}) \big[\, L(\mathbf{a}) \, \big] = I,$$

i.e. we have performed a kind of Gaussian elimination ... .

Upper bounds for the cost $\sum_{j=k}^{1} \varphi_{2^j}$ *of the computation of* $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k-1)}, \mathbf{a}^{(k)}$

Since the cost of a matrix-vector product involving a generic vector and a $2^j \times 2^j$ lower triangular Toeplitz matrix with ones on the diagonal is obviously bounded by $1 + 2 + 3 + \ldots + (2^j - 1) = \frac{1}{2} 2^j (2^j - 1)$, we can say that

$$\sum_{j=1}^{k} \varphi_{2^j} \leq \frac{1}{2} \sum_{j=1}^{k} (2^{2j} - 2^j) = \frac{2}{3}(2^k)^2 - 2^k + \frac{1}{3}.$$

However, by exploiting the particularity of our matrix-vector products, we observe that $\varphi_2 = 0$, $\varphi_4 \leq 1$, $\varphi_8 \leq 3 + 1 + 2$, $\varphi_{16} \leq 7 + 1 + 2 + 3 + 4 + 5 + 6$, $\varphi_{2^j} \leq \frac{1}{2} 2^{j-1}(2^{j-1} - 1)$. So,

$$\sum_{j=1}^{k} \varphi_{2^j} \leq \frac{1}{2} \sum_{j=2}^{k} (2^{2j-2} - 2^{j-1}) = \frac{1}{6}(2^k)^2 - \frac{1}{2} 2^k + \frac{1}{3}.$$

Finally, since a matrix-vector product involving a generic vector and a $2^j \times 2^j$ lower triangular Toeplitz matrix can be computed by means of the three FFT of order 2^{j+1}, i.e. with a cost $O((j+1)2^{j+1})$, we have

$$\sum_{j=1}^{k} \varphi_{2^j} = O(k^2 2^k) = O(n (\log_2 n)^2) \quad \text{prove this!}$$

Computing the first column of $L(\mathbf{a})^{-1}$

Now observe that

$$\begin{aligned}
& L(E^{k-1} I_\pm \mathbf{a}^{(k-1)}) L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) \cdots L(E^2 I_\pm \mathbf{a}^{(2)}) L(E I_\pm \mathbf{a}^{(1)}) L(I_\pm \mathbf{a}) \Big[\, L(\mathbf{a}) \, \Big] = I, \\
& L(\mathbf{a})\mathbf{z} = \mathbf{c}, \\
& \mathbf{z} = L(E^{k-1} I_\pm \mathbf{a}^{(k-1)}) L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) \cdots L(E^2 I_\pm \mathbf{a}^{(2)}) L(E I_\pm \mathbf{a}^{(1)}) L(I_\pm \mathbf{a}) \, \mathbf{c} \\
& \quad = L(I_\pm \mathbf{a}) L(E I_\pm \mathbf{a}^{(1)}) L(E^2 I_\pm \mathbf{a}^{(2)}) \cdots L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) L(E^{k-1} I_\pm \mathbf{a}^{(k-1)}) \, \mathbf{c},
\end{aligned}$$

$$\begin{aligned}
& L(\mathbf{a})\mathbf{z} = u\mathbf{e}_1 + v\mathbf{e}_{\frac{n}{2}+1}, \\
& \mathbf{z} = L(I_\pm \mathbf{a}) L(E I_\pm \mathbf{a}^{(1)}) L(E^2 I_\pm \mathbf{a}^{(2)}) \cdots L(E^{k-2} I_\pm \mathbf{a}^{(k-2)}) L(E^{k-1} I_\pm \mathbf{a}^{(k-1)}) \, (u\mathbf{e}_1 + v\mathbf{e}_{\frac{n}{2}+1}) \\
& \quad = L(I_\pm \mathbf{a}) E L(I_\pm \mathbf{a}^{(1)}) E L(I_\pm \mathbf{a}^{(2)}) \cdots E L(I_\pm \mathbf{a}^{(k-2)}) E L(I_\pm \mathbf{a}^{(k-1)}) \, (u\mathbf{e}_1 + v\mathbf{e}_2).
\end{aligned}$$

The latter identity, whose proof is left to the reader, implies that the solution of the system $L(\mathbf{a})\mathbf{z} = u\mathbf{e}_1 + v\mathbf{e}_{\frac{n}{2}+1}$ (the first column of $L(\mathbf{a})^{-1}$, if $u = 1$, $v = 0$) can be computed by performing $k = \log_2 n$ l.t.T.matrix-vector products where the matrices are, respectively, 2×2 (no operation, if $u = 1$, $v = 0$), 4×4 (one multiplication, if $u = 1$, $v = 0$), 8×8, ..., $2^k \times 2^k$, i.e. $2^j \times 2^j$, $j = 1, 2, \ldots, k$. If we perform such products by means of FFT (as soon as j becomes so large that using FFT is cheaper than the economized direct product), then the total amount of operations is $O(n(\log_2 n)^2)$ (prove it!).

Example: $n = 8 = 2^3$, $k = 3$

$$
\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix}
:=
\begin{bmatrix}
1 &&&&&&& \\
-a_1 & 1 &&&&&& \\
a_2 & -a_1 & 1 &&&&& \\
-a_3 & a_2 & -a_1 & 1 &&&& \\
a_4 & -a_3 & a_2 & -a_1 & 1 &&& \\
-a_5 & a_4 & -a_3 & a_2 & -a_1 & 1 && \\
a_6 & -a_5 & a_4 & -a_3 & a_2 & -a_1 & 1 & \\
-a_7 & a_6 & -a_5 & a_4 & -a_3 & a_2 & -a_1 & 1
\end{bmatrix}
\begin{bmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
-a_1^{(1)} && 1 &&&&& \\
& -a_1^{(1)} && 1 &&&& \\
a_2^{(1)} && -a_1^{(1)} && 1 &&& \\
& a_2^{(1)} && -a_1^{(1)} && 1 && \\
-a_3^{(1)} && a_2^{(1)} && -a_1^{(1)} && 1 & \\
& -a_3^{(1)} && a_2^{(1)} && -a_1^{(1)} && 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
-a_1^{(2)} &&&& 1 &&& \\
& -a_1^{(2)} &&&& 1 && \\
&& -a_1^{(2)} &&&& 1 & \\
&&& -a_1^{(2)} &&&& 1
\end{bmatrix}
\begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ v \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
=
\begin{bmatrix}
1 &&&&&&& \\
-a_1 & 1 &&&&&& \\
a_2 & -a_1 & 1 &&&&& \\
-a_3 & a_2 & -a_1 & 1 &&&& \\
a_4 & -a_3 & a_2 & -a_1 & 1 &&& \\
-a_5 & a_4 & -a_3 & a_2 & -a_1 & 1 && \\
a_6 & -a_5 & a_4 & -a_3 & a_2 & -a_1 & 1 & \\
-a_7 & a_6 & -a_5 & a_4 & -a_3 & a_2 & -a_1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 &&&&&& \\
& 0 &&&&&& \\
& 1 & 0 &&&&& \\
&& 0 &&&&& \\
&& 1 & 0 &&&& \\
&&& 0 &&&& \\
&&& 1 & 0 &&& \\
&&&& 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&& \\
-a_1^{(1)} & 1 &&&&&& \\
a_2^{(1)} & -a_1^{(1)} & 1 &&&&& \\
-a_3^{(1)} & a_2^{(1)} & -a_1^{(1)} & 1 &&&& \\
&&&& 0 &&& \\
&&&&& 0 && \\
&&&&&& 0 & \\
&&&&&&& 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 &&&&&& \\
& 0 &&&&&& \\
& 1 & 0 &&&&& \\
&& 0 &&&&& \\
&& 1 & 0 &&&& \\
&&& 0 &&&& \\
&&& 1 & 0 &&& \\
&&&& 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&& \\
-a_1^{(2)} & 1 &&&&&& \\
&& 0 &&&&& \\
&&& 0 &&&& \\
&&&& 0 &&& \\
&&&&& 0 && \\
&&&&&& 0 & \\
&&&&&&& 0
\end{bmatrix}
\begin{bmatrix} u \\ v \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

is such that

$$
\begin{bmatrix}
1 &&&&&&& \\
a_1 & 1 &&&&&& \\
a_2 & a_1 & 1 &&&&& \\
a_3 & a_2 & a_1 & 1 &&&& \\
a_4 & a_3 & a_2 & a_1 & 1 &&& \\
a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\
a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\
a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1
\end{bmatrix}
\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix}
=
\begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ v \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

How to determine $\mathbf{z}$ such that $[L(\mathbf{a})\mathbf{z}]_i = 0$, $i = 1, 2, 4, 5, 7, 8, 10, 11, \ldots$?

$$\begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ a_1^2 - a_2 \\ x \\ y \\ z \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_3 - 2a_1a_2 + a_1^3 + x \\ a_4 - a_1a_3 + a_2a_1^2 - a_2^2 + a_1x + y \\ a_5 - a_1a_4 + a_3a_1^2 - a_2a_3 + a_2x + a_1y + z \end{bmatrix}$$

We have to choose x, y, z such that

$$a_4 - a_1a_3 + a_2a_1^2 - a_2^2 + a_1x + y = 0,$$
$$a_5 - a_1a_4 + a_3a_1^2 - a_2a_3 + a_2x + a_1y + z = 0.$$

Note that it must be $y = -a_4 + a_2^2 - a_1t$, $x = a_3 - a_1a_2 + t$, and then z is determined by

$$z = -a_5 + a_1a_4 - a_3a_1^2 + a_2a_3 - a_2a_3 + a_1a_2^2 - a_2t + a_1a_4 - a_1a_2^2 + a_1^2t = -a_5 + 2a_1a_4 - a_3a_1^2 - a_2t + a_1^2t.$$

A possible choice of t is $t = a_3$, so that in z there are not terms of type $a_i^r a_j$ with r greater than 1:

$$z = -a_5 + 2a_1a_4 - a_2a_3.$$

Exercise. What choice of t allows one to obtain the Ramanujan system from the odd (even) system ? ...

Proceed analogously in order to obtain $[L(\mathbf{a})\mathbf{z}]_i = 0$, $i = 7, 8$:

$$a_7 + a_1X + Y + a_1x + y = 0, \quad a_8 \ \ldots + a_2x + a_1y + z = 0.$$

By setting $x = -X + t$, $y = -a_7 - Y - a_1t$ we satisfy the first equation, and the second equation becomes $\ldots a_2t + a_6a_1^2 - a_1^2t \ldots$. Thus choose $t = a_6$ so that the only "strange" term of the second equation, $a_6a_1^2$, is neglected, and x, y, z are determined. Idem in order to obtain $[L(\mathbf{a})\mathbf{z}]_i = 0$, $i = 10, 11$. In such case the second equation becomes $\ldots a_2t + a_9a_1^2 - a_1^2t \ldots$. Thus choose $t = a_9$ ... .

In this way we prove that

$$
\begin{bmatrix}
1 \\
a_1 & 1 \\
a_2 & a_1 & 1 \\
a_3 & a_2 & a_1 & 1 \\
a_4 & a_3 & a_2 & a_1 & 1 \\
a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & .
\end{bmatrix}
\begin{bmatrix}
1 \\
-a_1 \\
-a_2 + a_1^2 \\
2a_3 - a_1 a_2 \\
-a_4 - a_1 a_3 + a_2^2 \\
-a_5 + 2a_1 a_4 - a_2 a_3 \\
2a_6 - a_1 a_5 - a_2 a_4 + a_3^2 \\
-a_7 - a_1 a_6 + 2a_2 a_5 - a_3 a_4 \\
-a_8 + 2a_1 a_7 - a_2 a_6 - a_3 a_5 + a_4^2 \\
2a_9 - a_1 a_8 - a_2 a_7 + 2a_3 a_6 - a_4 a_5 \\
-a_{10} - a_1 a_9 + 2a_2 a_8 - a_3 a_7 - a_4 a_6 + a_5^2 \\
-a_{11} + 2a_1 a_{10} - a_2 a_9 - a_3 a_8 + 2a_4 a_7 - a_5 a_6 \\
.
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 0 \\ 0 \\ a_1^{(1)} \\ 0 \\ 0 \\ a_2^{(1)} \\ 0 \\ 0 \\ a_3^{(1)} \\ 0 \\ 0 \\ .
\end{bmatrix}
$$

$a_1^{(1)} = 3a_3 - 3a_1 a_2 + a_1^3$, $?a_2^{(1)} = 3a_6 - 3a_1 a_5 - 3a_2 a_4 + 3a_3^2 - 3a_1 a_2 a_3 + 3a_1^2 a_4 + a_2^3?$, $a_3^{(1)} = ?$.

Set $a_0 = 1$. Conjecture on the next entry of the solution:

$$2a_0 a_{12} - a_1 a_{11} - a_2 a_{10} + 2a_3 a_9 - a_4 a_8 - a_5 a_7 + a_6^2$$

Why? ...

Conjecture on the solution:

$$-\sum_{r=0}^{\lfloor \frac{i-1}{2} \rfloor} a_r a_{i-r} + \delta_{i=0 \bmod 2} a_{\frac{i}{2}}^2 + 3 \begin{cases} \sum_{s \geq \frac{3-i}{6}}^{0} a_{\frac{i-3}{2}+3s} a_{\frac{i+3}{2}-3s} & i \text{ odd} \\ \sum_{s \geq \frac{6-i}{6}}^{0} a_{\frac{i-6}{2}+3s} a_{\frac{i+6}{2}-3s} & i \text{ even} \end{cases}, \quad i = 0, 1, 2, 3, 4, 5, \ldots$$

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QUESTION Can the above $3^j \times 3^j$ matrix by vector product be computed in at most $O(3^j j)$ arithmetic operations ?

If yes, then we would have a method which solves $3^k \times 3^k$ lower triangular Toeplitz linear systems in at most $O(3^k k)$ arithmetic operations.

If no, then look for another solution $\hat{\mathbf{a}}$ of the system $L(\mathbf{a})\hat{\mathbf{a}} = [1\,0\,0\bullet 0\,0\,\cdot]^T$ such that $\{\hat{\mathbf{a}}\}_{3^j}$ is computable from $\{\mathbf{a}\}_{3^j}$ in at most $O(3^j j)$ arithmetic operations.