Explicit Characterization of Consensus in a Distributed Estimation Problem on Chain Graphs

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An Important Problem

We address the problem of looking for time-varying vectors $\hat{\Theta}(t)$ – named Θ-estimates – that exponentially converge to the unknown constant parameter vector $\Theta \in \mathbb{R}^m$ defined by the set of linear time-varying equations:

$$y_1(t) = \phi_1^{\mathrm{T}}(t)\Theta \quad (1)$$
$$y_i(t) = \phi_i^{\mathrm{T}}(t)\Theta, \quad i = 2, \ldots, p-1 \quad (2)$$
$$y_p(t) = \phi_p^{\mathrm{T}}(t)\Theta, \quad (3)$$

where y_i are the locally measured outputs and $\phi_i(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^m$ are the local regressor vectors, $i = 1, \ldots, p$, each of them assumed to be available at the running time at each node of the graph. This general problem is referred to as the estimation problem for (1)-(3) on a graph. Now, when there are no edges connecting nodes of the graph that are far from each other, the problem passes from being full-graph-knowledge based to being partial-graph-knowledge based. This way, the burden of information that has to be communicated to the various measurement/estimation nodes of the graph might be (even largely) reduced.

Indeed, the general scenario is the one given by a multisensor network in which a parameter estimator is to be designed on the basis of space-distributed sensing (figure on the right). The set of agents (the swarm of drones) at the nodes face a local identification problem, in which they cannot consistently estimate the parameter vector (position of a target in the space) in isolation, so they have to engage in communication with their neighbours. In particular, estimate-consensus has to be achieved through a sort of penalization of the mismatch between the parameter estimates.

Original Contribution

The original contribution consists in showing that, under the weakest Θ-identification condition (5), namely the *Cooperative PE Condition* [1] (PE stands for Persistency of Excitation), a set of suitably tailored differential equations for the time-dependent vectors $\hat{\Theta}^{[i]}(t)$, all of them converging to the unknown Θ (*consensus*), can be (redundantly) designed at each node $i = 1, \ldots, p$. This is proved under the condition that nodes undirectedly connected in series (undirected chain graph) are considered, so that each estimation scheme at the node can share information – namely, its own Θ-estimate – with the neighbours only, one for node 1 and p, two for the remaining nodes.

Indeed, the derivations of this paper move along the direction of [1] and are in exact accordance with it. However, in contrast to [1], which – for general graph topologies – uses weaker contradiction arguments to prove that cooperative PE condition guarantees exponential consensus, here original proofs of convergence are able to provide an explicit characterization of the exponentially achieved consensus in terms of PE constants and Lyapunov functions.

Nevertheless, the problem of identifying time-varying parameters that are periodic with known periods can be innovatively solved as well. Adaptive tools can be directly replaced by repetitive learning tools within the same theoretical framework, where the asymptotic consensus is successfully ensured under identification mechanisms based on the information exchange between neighbours.

Assumptions and Estimator Design

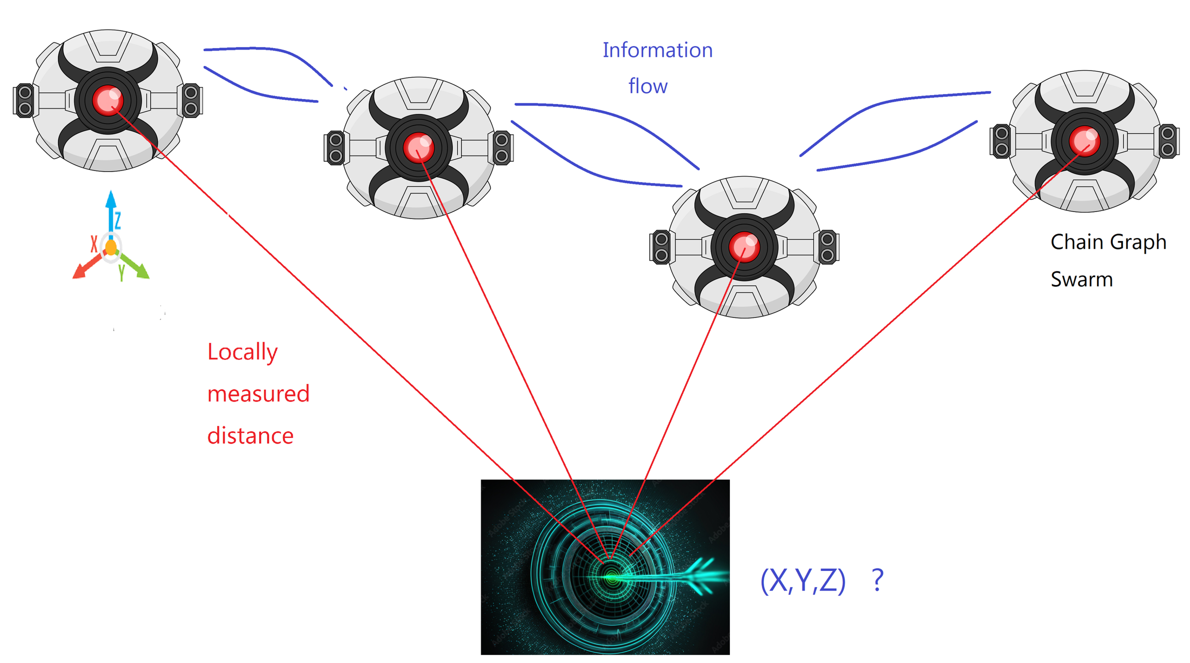
We introduce two standard assumptions.
A1. The elements of the regressor vectors are assumed to be continuous and uniformly bounded over $[0, +\infty)$ as functions of time.
A2. (*Cooperative PE Condition* [1]) The corresponding regressor matrix $\Phi^{\mathrm{T}}(\cdot) \in \mathbb{R}^{p\times m}$

$$\Phi^{\mathrm{T}}(\cdot) = \begin{bmatrix} \phi_1^{\mathrm{T}}(\cdot) \\ \cdots \\ \phi_p^{\mathrm{T}}(\cdot) \end{bmatrix} \quad (4)$$

is assumed to be persistently exciting (PE), i.e., there exist (known) positive reals c_p and T_p such that the following condition [$\mathbb{I} \in \mathbb{R}^{m\times m}$] holds:

$$\int_t^{t+T_p} \Phi(\tau)\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau \geqq c_p\mathbb{I}, \quad \forall\, t \geq 0. \quad (5)$$

Assumptions A1-A2 ensure, on the basis of [2] and references therein, that the unknown parameter vector Θ is identifiable from the entire set of available measurements, even when $p < m$ and $\Phi^{\mathrm{T}}(\cdot)$ is not a full-rank matrix.



The assumption that only partial-graph-based information is available for each node increases the complexity for the design of the differential equations for the Θ-estimate. In particular, each node i, $i = 1, \ldots, p$, has to include a (redundant) set of $\hat{\Theta}^{[i]}$-differential equations (here referred to as local estimators at the node i) based on $\phi_i(t)$ and $y_i(t)$ and the only information that can be shared between the local estimators is constituted by the Θ-estimate provided by the (one or two) closest neighbours (undirected chain graph scenario). Thus we design

$$\dot{\hat{\Theta}}^{[i]}(t) = \phi_i(t)\left(y_i(t) - \phi_i^{\mathrm{T}}(t)\hat{\Theta}^{[i]}(t)\right) - \eta_i(t). \quad (6)$$

Taking into account that any internal i-th estimation scheme ($i = 2, \ldots, p-1$) can use only the information coming from the $(i-1)$-th and the $(i+1)$-th estimation schemes, whereas the 1-st and the p-th estimation schemes can use only the information provided by the 2-nd and the $(p-1)$-th, respectively, we determine

$$\begin{aligned} \eta_1 &= \left(\hat{\Theta}^{[1]} - \hat{\Theta}^{[2]}\right) \\ \eta_i &= \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i-1]}\right) + \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i+1]}\right), i = 2, \ldots, p-1 \\ \eta_p &= \left(\hat{\Theta}^{[p]} - \hat{\Theta}^{[p-1]}\right). \end{aligned} \quad (7)$$

Theoretical Results

Defining the estimation errors $\tilde{\Theta}^{[i]} = \Theta - \hat{\Theta}^{[i]}$, the error system reads

$$\begin{bmatrix} \dot{\tilde{\Theta}}^{[1]}(t) \\ \dot{\tilde{\Theta}}^{[2]}(t) \\ \dot{\tilde{\Theta}}^{[3]}(t) \\ \cdots \\ \dot{\tilde{\Theta}}^{[p]}(t) \end{bmatrix} = -\left(\Lambda(t) + T\right) \begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \tilde{\Theta}^{[2]}(t) \\ \tilde{\Theta}^{[3]}(t) \\ \cdots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} \quad (8)$$

where $\Lambda(t) + T$ is a tridiagonal block matrix in $\mathbb{R}^{pm\times pm}$ with

$$\Lambda(t) = \mathrm{diag}[\phi_1(t)\phi_1^{\mathrm{T}}(t), \ldots, \phi_p(t)\phi_p^{\mathrm{T}}(t)] \quad (9)$$

and

$$T = \begin{bmatrix} \mathbb{I} & -\mathbb{I} & \mathbb{O} & \ldots & \ldots & \ldots & \mathbb{O} \\ -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \ldots & \ldots & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \ldots & \mathbb{O} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -\mathbb{I} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \ldots & -\mathbb{I} & \mathbb{I} \end{bmatrix}. \quad (10)$$

First, if $G(t) = \Lambda(t) + T$ is persistently exciting and the elements of $\Gamma(t)$, in the expression $G(t) = \Gamma(t)\Gamma^{\mathrm{T}}(t)$, are continuous and uniformly bounded over $[0, +\infty)$ then Theorem 1 in [2] can be applied to the overall system (8) as in the lemma below.

Lemma: *Assume that $\Phi(t)$, besides Assumption A1, satisfies the following hypothesis:*
A3. the entries of $\Phi(t)$ are analytical functions of time t.
Consider system (8) and assume that there exist (known) positive reals c_{pG} and T_{pG} such that the condition:

$$\int_t^{t+T_{pG}} G(\tau)\mathrm{d}\tau \geqq c_{pG}\mathbb{I}, \quad \forall\, t \geq 0 \quad (11)$$

holds. Then the n-dimensional extended error vector $\tilde{\Theta}^{[e]}(t) = [\tilde{\Theta}^{[1]\mathrm{T}}(t), \ldots, \tilde{\Theta}^{[p]\mathrm{T}}(t)]^{\mathrm{T}}$ ($n = pm$) globally exponentially converges to zero.

Meaningfully, the theorem below shows how the weakest and least restrictive condition (5) actually implies condition (11) and thus, used in conjunction with Theorem 1, provides the proof that the solution to system (8), under Assumptions A1-A3, globally exponentially converges to zero.

Theorem: *Under Assumption A2 [namely, condition (5)], there exist explicitly computable positive reals c_{pG} and T_{pG} such that the condition (11) holds true.*

References

[1] W. Chen et al. “Distributed cooperative adaptive identification and control for a group of continuous-time systems with a cooperative PE condition via consensus”. In: *IEEE Transactions on Automatic Control* 59 (2014), pp. 91–106.

[2] C.M. Verrelli and P. Tomei. “Non-anticipating Lyapunov functions for persistently excited nonlinear systems”. In: *IEEE Transactions on Automatic Control* 65 (2020), pp. 2634–2639.