Outliers-Free Optimal Spline Spaces: Characterization via the Commutator and Joint Diagonalization

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Preliminary results — under review.
Proofs and derivations are included in the Master's Thesis

Motivation and context

The 1D Laplace eigenvalue problem (Dirichlet/Neumann/Mixed) is a classical benchmark for spline Galerkin methods. Conventional spline spaces may produce *spurious outliers* in the discrete spectrum, affecting accuracy and stability. Outliers-free optimal spline spaces $S_{p,i}$, introduced in recent literature, remove this effect.

Kolmogorov n-width and optimal spaces

Let $(X, \|\cdot\|)$ be a normed space, $A \subset X$, and let $X_n \subset X$ be an n-dimensional subspace. The distance of A from X_n is

$$E(A, X_n) := \sup_{u \in A} \inf_{v \in X_n} \|u - v\|.$$

The *Kolmogorov n-width* of A relative to X is

$$d_n(A) := \inf_{X_n} E(A, X_n).$$

A subspace X_n is said to be *optimal for A* if

$$d_n(A) = E(A, X_n).$$

In the context of isogeometric analysis, taking A as the range of a compact self-adjoint integral operator associated with the Laplace problem, the optimal subspaces $S_{p,i}$ realizing $d_n(A)$ are the **optimal spline spaces**. These spaces, first introduced for approximation-theoretic reasons, were later found to produce **outliers-free** discretizations of eigenvalue problems.

What was known and what is new

Known: Optimal spline spaces $S_{p,i}$ and their canonical bases E_i^p (Manni–Sande–Speleers, 2022 and E.Di Vona, 2019) were already known to be **outliers-free**.

New (this thesis): The property of being outliers-free is shown to be *intrinsic to the space*.

For any admissible basis B of $S_{p,i}$ satisfying the natural constraints,

$$[M_B, K_B] = 0 \implies B = Q\, E_i^p, \quad Q \in O(n).$$

Thus, the outliers-free condition does not depend on the specific basis, but on the space itself.

Explicit optimal spline spaces and knots

For degree $p \in \mathbb{N}$ and dimension n, the uniform partitions $\tau_{p,i}$ reflect the boundary conditions:

Dirichlet-type (even derivatives zero):

$$S_{p,0} = \{s \in S_{p,\tau_{p,0}} : \ \partial^\alpha s(0) = \partial^\alpha s(1) = 0, \ \ 0 \le \alpha \le p, \ \alpha \text{ even}\},$$

$$\tau_{p,0} = \begin{cases} (0, \frac{1}{n+1}, \dots, \frac{n}{n+1}, 1), & p \text{ odd}, \\ (0, \frac{1/2}{n+1}, \frac{3/2}{n+1}, \dots, \frac{n+1/2}{n+1}, 1), & p \text{ even}. \end{cases}$$

Neumann-type (odd derivatives zero):

$$S_{p,1} = \{s \in S_{p,\tau_{p,1}} : \ \partial^\alpha s(0) = \partial^\alpha s(1) = 0, \ \ 0 \le \alpha \le p, \ \alpha \text{ odd}\},$$

$$\tau_{p,1} = \begin{cases} (0, \frac{1/2}{n}, \frac{3/2}{n}, \dots, \frac{n-1/2}{n}, 1), & p \text{ odd}, \\ (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1), & p \text{ even}. \end{cases}$$

Mixed (even at x=0, odd at x=1):

$$S_{p,2} = \{s \in S_{p,\tau_{p,2}} : \ \partial^{\alpha_0} s(0) = 0 \ , \ \partial^{\alpha_1} s(1) = 0 \ , \ \ 0 \le \alpha_0, \alpha_1 \le p \ , \alpha_0 \text{ even}, \alpha_1 \text{ odd}\},$$

$$\tau_{p,2} = \begin{cases} (0, \frac{2}{2n+1}, \frac{4}{2n+1}, \dots, \frac{2n}{2n+1}, 1), & p \text{ odd}, \\ (0, \frac{1}{2n+1}, \frac{3}{2n+1}, \dots, \frac{2n-1}{2n+1}, 1), & p \text{ even}. \end{cases}$$

Main Theorems

Thm. 3.1.10 ($S_{p,0}$), Thm. 3.2.11 ($S_{p,1}$), Thm. 3.3.10 ($S_{p,2}$)

If a basis B of $S_{p,i}$ satisfies the natural constraints (locality, normalization, partition of unity, etc.) and $[M_B, K_B] = 0$, then

$$B = Q\, E_i^p \quad \text{with } Q \in O(n).$$

Hence, the commutator condition $[M_B, K_B] = 0$ **characterizes all outliers-free bases** of $S_{p,i}$.

Unified proof sketch

Let $B = R_B\, E_i^p$ be any basis of $S_{p,i}$ and define $W := R_B^\top R_B$. Then

$$M_B = R_B M_{E_i^p} R_B^\top, \quad K_B = R_B K_{E_i^p} R_B^\top.$$

The commutator condition becomes

$$M_{E_i^p} W K_{E_i^p} - K_{E_i^p} W M_{E_i^p} = 0. \qquad (\star)$$

Because $M_{E_i^p}$ and $K_{E_i^p}$ are simultaneously diagonalizable by an orthogonal U, $(\star)$ implies W must be diagonal in that basis. Thus $W = cI$. The normalization constraint forces $c = 1$, hence $R_B \in O(n)$ and $B = Q\, E_i^p$.

Interpretation: the commutator and natural constraints completely determine the structure of admissible bases.

Spectral meaning: no outliers

When $[M, K] = 0$, M and K share orthogonal eigenvectors:

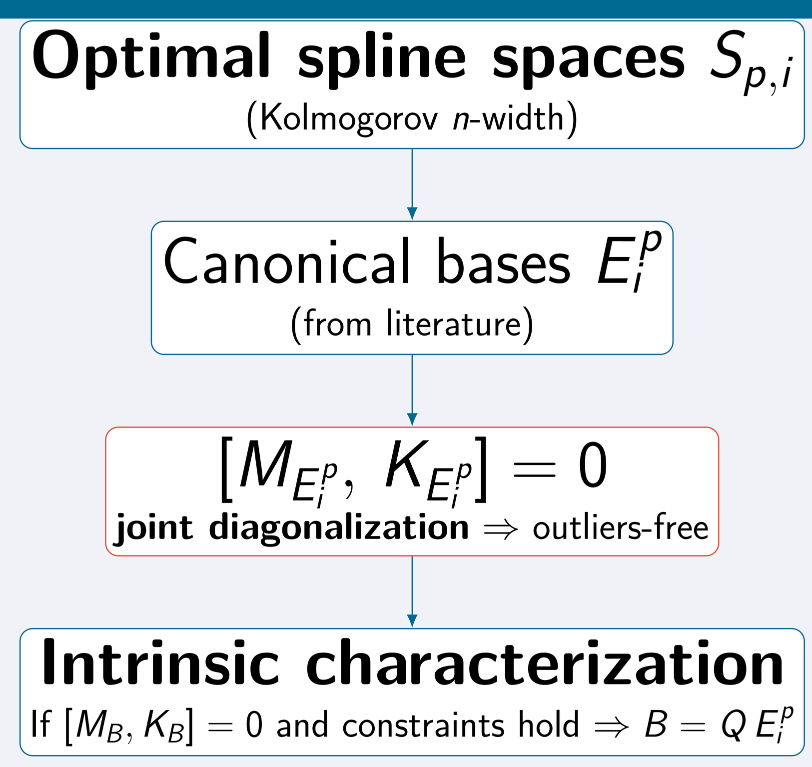
$$U^\top M U = \Lambda_M, \qquad U^\top K U = \Lambda_K,$$

yielding

$$\lambda_j = \frac{(\Lambda_K)_{jj}}{(\Lambda_M)_{jj}}.$$

The generalized eigenproblem decouples, producing exactly the expected spectrum without spurious outliers.

Concept map (spaces, bases, commutator, characterization)



Implications and outlook

- Outliers-free spectra are an intrinsic feature of optimal spline spaces.
- Uniqueness (up to orthogonal change) of admissible discrete bases.
- Foundations for higher-dimensional generalizations and polyharmonic operators.

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