THE R-MATRIX FOR (TWISTED) AFFINE QUANTUM ALGEBRAS

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To the victims of the NATO aggression against Yugoslavia.

March 24^{th}, 1999: NATO starts bombing the Federal Republic of Yugoslavia, aiming at its civil and productive structures and killing thousands of people.
May 8^{th}, 1999: NATO missiles hit the Chinese Embassy in Belgrade, destroying it and killing 3 people.

I want to dedicate this paper to the Yugoslav students, whose right to life and culture is threatened, and to the Chinese students, who raised their voice against this criminal war.

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0. INTRODUCTION.

The aim of this paper is to give an exponential multiplicative formula for the R-matrix of the general affine quantum algebra $\mathcal{U}_q$. The problem of describing R has been attacked from different viewpoints, and different kinds of formulas for R have already been given explicitly in many (general) cases.

For the finite type algebras, Rosso gave in [30] the first exponential multiplicative formula (for $\mathfrak{sl}_n$): this result was generalized in [22], where the strategy of partial R-matrices (that is the study of the connections between the coproduct and the braid group action) was introduced and developed, and the general (finite) case solved.

This exponential approach, mainly based on the description of R via the Killing form (see [31]), has been extended to $A_1^{(1)}$ in [24], and to the general non twisted affine case in [6].

Other descriptions of R in the affine case (both non twisted and twisted) are also known, by means of different techniques. Of particular relevance is the application of the theory of vertex algebras, which turns out to be a powerful instrument for studying the representations of the affine quantum algebras; in particular it is important to mention the works of Khoroshkin-Tolstoy (see [20] and [19]) which (together with [24]) are the first to face the study of R for the affine algebras, and those of Delius-Gould-Zhang ([8]) and Gandenberger-MacKay-Watts ([12]), where the R-matrix was constructed in the twisted case.

The purpose of the present paper is to complete the "exponential picture" by including the twisted affine case in this description.

To this aim a quite precise analysis of the twisted affine quantum algebras is needed: since the PBW-bases (bases of type Poincaré-Birkhoff-Witt), and the possibility to make computations on them, are a fundamental tool for working in $\mathcal{U}_q$ (and in particular for the construction of R), the most important step in this direction is producing a PBW-basis and stating its main properties by generalizing the well-known results in the finite and non twisted affine case.

While approaching this question, one easily finds out that in many key points the structure of the behaviour of the twisted affine quantum algebras is not so different (up to some minor adjustments involving no conceptual difficulty and requiring just some care in the notations; notations which are fixed in section 1) from what is already known in the non twisted case: the definition of the root vectors does not present any difference from the non twisted case (it only happens that the root system with multiplicities is slightly more complicated to describe), and the proof of most of the commutation formulas lies on a property of the root system ($*$: if α and $\delta + r\alpha$ are roots, then $r = \pm 1$) which is almost always true; when this is the case the contribution of this paper is just that of remarking the analogies and giving references for the proofs. Thus, section 2, where copies of $\mathcal{U}_q(A_1^{(1)})$ are embedded in the affine quantum algebra, has been written following [1] (to which it refers heavily). The same can be said about paragraphs §5.1 and §5.3 (which are mainly based on [1]) and about section 6 (where the results previously found are gathered together to exhibit a PBW-basis). Similarly, the first two paragraphs of section 7 (§7.1 and §7.2, with the exception of lemma 7.2.3), show and use the validity for the twisted case of the results of [6] (hence those of [22] for what concerns the real root vectors) where the coproduct and the Killing form on the root vectors are computed.

But there is a case ($A_{2n}^{(2)}$) when $\delta - 2\alpha_1$ is a root, so that the property above denoted by $*$ is no more true. This situation must be dealt with more carefully, what is done in section 3 (where the case of $A_2^{(2)}$ is discussed in details) and in section 4, where $\mathcal{U}_q(A_2^{(2)})$ is embedded in $\mathcal{U}_q(A_{2n}^{(2)})$: $A_2^{(2)}$ plays here the same role that A_1 plays in the general frame (to each vertex there is attached a copy of $\mathcal{U}_q(A_1) = \mathcal{U}_q(\mathfrak{sl}_2)$) and $A_1^{(1)}$ in the generic affine picture (to "almost each" vertex of the associated finite diagram there is attached a copy of $\mathcal{U}_q(A_1^{(1)})$). The existence of a copy of $\mathcal{U}_q(A_2^{(2)})$ in $\mathcal{U}_q(A_{2n}^{(2)})$ was already proved in [4] by means of the language of vertex operators and through the Drinfeld realization of $\mathcal{U}_q$ (see [17] and [16]); here we give a different, direct proof, which has the advantage to provide a precise description of the commutation relations that we need.

Finally, paragraphs §5.2 (this in particular requires a deeper understanding of the twisted root systems and Weyl groups), §7.4 and §7.5 are devoted to the computations which, after establishing the general frame, are needed in order to make explicit the description of R, whose final formula is given in paragraphs §7.3 and §7.6.

I would like to thank Professors Chen Yu, Hu Naihong, Lin Zongzhu, Wang Jianpan and the other organizers of the International Conference on Representation Theory, held in Shanghai (China) in June/July 1998, for giving me the opportunity to participate in this meeting which has been for me of big interest, and for offering the possibility to publish a contribution on the Conference Proceedings.

Unfortunately last spring, during the work of drawing up of the paper, the country where I live (Italy) participated in the war against the Federal Republic of Yugoslavia, which has brutally involved also the People's Republic of China: the decision of the NATO countries to isolate the scientific community of Yugosalvia is just one - yet hateful - of the "collateral effects" of this aggression. Being deeply concerned by the evident and arrogant injustice of this politics, I want to express, as a mathematician, my opposition to this choice, together with my solidariety to all those who are fighting every day to make the science survive, even in these hard times, for the present and future generations.

1. GENERAL SETTING: DEFINITIONS AND PRELIMINARIES.

§1.1. Affine Kac-Moody algebras.

In this paragraph we recall some basic generalities; for the proofs and for a deeper and more detailed investigation we refer to [18], where these notions have been introduced and studied.

Let $\mathfrak{g}$ be a Kac-Moody algebra of finite type. Then an automorphism of the Dynkin diagram of $\mathfrak{g}$ of order k induces an automorphism of $\mathfrak{g}$ of the same order (it is well known and immediate to see that k =1, 2 or 3). This automorphism, whose eigenvalues are of course of the form $e^{\frac{2\pi ir}{k}}$ with $0 \leq r < k$, is diagonalizable, that is $\mathfrak{g} = \bigoplus_{r\in\mathbb{Z}/k\mathbb{Z}} \mathfrak{g}^{(r)}$, $\mathfrak{g}^{(r)}$ being the eigenspace relative to $e^{\frac{2\pi ir}{k}}$. It is straightforward to prove that this decomposition of $\mathfrak{g}$ as a direct sum is a $\mathbb{Z}/k\mathbb{Z}$-grading, that is $[\mathfrak{g}^{(r)}, \mathfrak{g}^{(s)}] \subseteq \mathfrak{g}^{(r+s)}$ $\forall r, s \in \mathbb{Z}/k\mathbb{Z}$; in particular $\mathfrak{g}^{(0)}$ is a Lie subalgebra of $\mathfrak{g}$ (indeed a Kac-Moody algebra of finite type) and $\mathfrak{g}^{(r)}$ is a $\mathfrak{g}^{(0)}$-module.

This remark allows to give a natural structure of Lie algebra to

$$\bigoplus_{r\in\mathbb{Z}} \mathfrak{g}^{([r])} \otimes \mathbb{C}t^r;$$

it is easy to construct a non trivial central extension

$$\left(\bigoplus_{r\in\mathbb{Z}} \mathfrak{g}^{([r])} \otimes \mathbb{C}t^r\right) \oplus \mathbb{C}c$$

of this algebra via the (non degenerate) Killing form on $\mathfrak{g}$ and a natural structure of Lie algebra on

$$\hat{\mathfrak{g}}^{(k)} \doteqdot \left(\bigoplus_{r\in\mathbb{Z}} \mathfrak{g}^{([r])} \otimes \mathbb{C}t^r\right) \oplus \mathbb{C}c \oplus \mathbb{C}\partial,$$

where $\mathrm{ad}\partial = t\dfrac{d}{dt}$.

It was proved by Kac (see [18]) that $\hat{\mathfrak{g}}^{(k)}$ is a Kac-Moody algebra of affine type depending just on $\mathfrak{g}$ and k and not on the chosen Dynkin diagram automorphism (remark that $\hat{\mathfrak{g}}^{(1)}$ is the non twisted affine Kac-Moody algebra associated to $\mathfrak{g}$; in case that $k > 1$ the algebra $\hat{\mathfrak{g}}^{(k)}$ is said to be twisted, that is relative to a non trivial automorphism of the Dynkin diagram); moreover all the affine Kac-Moody algebras are isomorphic to $\hat{\mathfrak{g}}^{(k)}$ for some finite Kac-Moody algebra $\mathfrak{g}$ and some $k \in \{1, 2, 3\}$.

We also have that the Dynkin diagram Γ of $\hat{\mathfrak{g}}^{(k)}$ is an extension of the Dynkin diagram Γ_0 of $\mathfrak{g}^{(0)}$ (which is called the finite diagram - or algebra - associated to the affine one) by adding an extra vertex, which will be denoted by 0; remark that if $k = 1$ then $\mathfrak{g}^{(0)} = \mathfrak{g}$.

§1.2. Some notations, structures and general properties.

Let us fix an affine Kac-Moody algebra $\hat{\mathfrak{g}}^{(k)}$. We shall now introduce some notations.

First of all let us denote by $\tilde{n}$ the number of vertices of the Dynkin diagram of $\mathfrak{g}$; this means that $\mathfrak{g}$ is of type $X_{\tilde{n}}$ for some $X \in \{A, B, C, D, E, F, G\}$, and in this case we say that $\hat{\mathfrak{g}}^{(k)}$ is of type $X_{\tilde{n}}^{(k)}$. We denote by n the cardinality of I_0, where I_0 is the set of vertices of Γ_0 (it means that the cardinality of I, the set of vertices of Γ, is $n + 1$) and note that $n = \tilde{n} \Leftrightarrow k = 1$. In particular it is possible to identify I with the set $\{0, 1, ..., n\}$ and I_0 with $\{1, ..., n\}$ (so that $I = I_0 \cup \{0\}$). One such identification will be fixed in the next paragraph.

Furthermore, $A = (a_{ij})_{ij\in I}$ will denote the Cartan matrix of $\hat{\mathfrak{g}}^{(k)}$ and A_0 will be $(a_{ij})_{ij\in I_0}$ (the Cartan matrix of $\mathfrak{g}^{(0)}$). Remember that (I, A) determines $\hat{\mathfrak{g}}^{(k)}$ completely; remark also that (I, A) can be extended to (I_∞, A_∞) where $I_\infty \doteqdot I \cup \{\infty\}$ and $A_\infty = (a_{ij})_{ij\in I_\infty}$ with $a_{i\infty} = a_{\infty i} \doteqdot \delta_{i0}$ $\forall i \in I_\infty$.

We have the following structures associated to (I, A):

1.2.A Root lattice.

The root lattice Q and the extended root lattice Q_∞ of $\hat{\mathfrak{g}}^{(k)}$, and the finite root lattice Q_0 of $\mathfrak{g}^{(0)}$ are defined respectively by:

$$Q \doteqdot \oplus_{i\in I}\, \mathbb{Z}\alpha_i, \quad Q_\infty \doteqdot \oplus_{i\in I_\infty}\, \mathbb{Z}\alpha_i, \quad Q_0 \doteqdot \oplus_{i\in I_0}\, \mathbb{Z}\alpha_i.$$

Of course $Q_0 \subseteq Q = Q_0 \oplus \mathbb{Z}\alpha_0 \subseteq Q_\infty = Q \oplus \mathbb{Z}\alpha_\infty$. We also define Q^+ and Q_0^+ as follows:

$$Q^+ \doteqdot \sum_{i\in I} \mathbb{N}\alpha_i = \left\{ \sum_{i\in I} r_i\alpha_i \in Q | r_i \geq 0 \ \forall i \in I \right\}, \quad Q_0^+ \doteqdot Q_0 \cap Q^+.$$

It can be useful to see Q_∞, Q and Q_0 as lattices in a vector space over $\mathbb{R}$ naturally defined as $Q_{\infty,\mathbb{R}} \doteqdot \mathbb{R} \otimes_{\mathbb{Z}} Q_\infty$. Set also $Q_\mathbb{R}$ and $Q_{0,\mathbb{R}}$ to be the $\mathbb{R}$-subspaces of $Q_{\infty,\mathbb{R}}$ generated respectively by Q and Q_0.

1.2.B Cartan matrix and symmetric bilinear form.

The Cartan matrix A is symmetrizable, that is there exists a diagonal matrix $D = \mathrm{diag}(d_i | i \in I)$ such that DA is symmetric; the diagonal entries d_i's can be chosen to be coprime positive integers, and this condition determines them uniquely. Remark that if we put $D_\infty \doteqdot \mathrm{diag}(d_i | i \in I_\infty)$ with $d_\infty = d_0$ then $D_\infty A_\infty$ is symmetric.

Hence $D_\infty A_\infty$ induces a symmetric bilinear form $(\cdot|\cdot)$ on $Q_{\infty,\mathbb{R}}$ ($\mathbb{Z}$-valued on Q_∞) defined by $(\alpha_i|\alpha_j) \doteqdot d_i a_{ij}$. We have that $(\cdot|\cdot)$ is non degenerate but it is not positive definite, while $(\cdot|\cdot)\big|_{Q_0\times Q_0}$ is positive definite (that is $(\cdot|\cdot)$ induces a structure of Euclidean space on $Q_{0,\mathbb{R}}$) and $(\cdot|\cdot)\big|_{Q\times Q}$ is degenerate positive semidefinite: there exists a (unique) non zero element $\delta \in Q^+$ such that $\delta - \alpha_0 \in Q_0^+$ and $(\delta|\delta) = 0$; more precisely $(\delta|\alpha) = 0 \ \forall \alpha \in Q$. Remark that Q can be decomposed as a direct sum also as $Q = Q_0 \oplus \mathbb{Z}\delta$, and that of course $(\alpha + r\delta | \beta + s\delta) = (\alpha|\beta) \ \forall \alpha, \beta \in Q$ and $\forall r, s \in \mathbb{Z}$.

1.2.C Weight lattices.

$(\cdot|\cdot)$ induces an identification of $\mathrm{Hom}(Q_0, \mathbb{Z})$ with a subgroup $\check{P}$ of $Q_{0,\mathbb{R}}$. $\check{P}$ is, by the very definition, the lattice $\check{P} = \oplus_{i\in I_0}\mathbb{Z}\check{\omega}_i$ where the $\check{\omega}_i$'s, called fundamental weights, are defined as the elements of $Q_{0,\mathbb{R}}$ characterized by the property that $(\check{\omega}_i|\alpha_j) \doteqdot \delta_{ij} \ \forall j \in I_0$. Q_0 is obviously a sublattice of $\check{P}$, and it is worth introducing another important sublattice P of $\check{P}$ as $P \doteqdot \oplus_{i\in I_0}\, \mathbb{Z}\omega_i$, where $\forall i \in I_0 \ \omega_i \doteqdot d_i\check{\omega}_i$; it is to be noticed that $Q_0 \subseteq P \subseteq \check{P}$. The elements of $\check{P}$ are called weights and the elements of

$$\check{P}_+ \doteqdot \left\{ \sum_{i\in I_0} r_i\check{\omega}_i \in \check{P} | r_i \geq 0 \ \forall i \in I_0 \right\}$$

are called dominant weights; we also denote by P_+ the set $P_+ \doteqdot \check{P}_+ \cap P$. Remark that $[\check{P} : P] = \prod_{i\in I_0} d_i$ and $[P : Q_0] = \det(A_0)$, what can be easily seen noticing that $\forall i \in I_0$ we have $\alpha_i = \sum_{j\in I_0} a_{ji}\omega_j$.

It is also important to mention the realization of $\check{P}$ as a group of transformations of Q as follows: define $t : \check{P} \to \mathrm{Hom}(Q, Q)$ by setting $t_x(\alpha) \doteqdot \alpha - (x|\alpha)\delta$. Remark that t is an injective homomorphism of groups and that $\forall x \in \check{P} \ \ t_x$ preserves $(\cdot|\cdot)\big|_{Q\times Q}$ and fixes δ; equivalently we can say that t maps $\check{P}$ into $Aut(Q_{(\cdot|\cdot),\delta})$, where $Aut(Q_{(\cdot|\cdot),\delta})$ is the group of automorphisms of Q preserving $(\cdot|\cdot)\big|_{Q\times Q}$ and fixing δ.

By abuse of notation we shall usually denote by x also its image t_x.

1.2.D Weyl and braid group.

The Weyl group W is the subgroup of $Aut(Q_{(\cdot|\cdot),\delta})$ generated by $\{s_i|i\in I\}$ where $s_i(\alpha_j)\doteqdot\alpha_j-a_{ij}\alpha_i$. W_0, the Weyl group of $\mathfrak{g}^{(0)}$, is the subgroup of W generated by $\{s_i|i\in I_0\}$.

$Aut(\Gamma)$ is the group of automorphisms of the Dynkin diagram of $\hat{\mathfrak{g}}^{(k)}$ (which can be naturally identified to a subgroup of $Aut(Q_{(\cdot|\cdot),\delta})$: $\tau(\alpha_i)\doteqdot\alpha_{\tau(i)}$).

The subgroups of $Aut(Q_{(\cdot|\cdot),\delta})$ introduced above ($\check{P}$, W, W_0, $Aut(\Gamma)$) have the following properties:

i) $W\cap Aut(\Gamma)=\{\mathrm{id}\}$, and $\tau s_i\tau^{-1}=s_{\tau(i)}$ $\forall i\in I$ $\forall\tau\in Aut(\Gamma)$; hence

$$W\trianglelefteq\langle W,Aut(\Gamma)\rangle\ \text{ and }\ \langle W,Aut(\Gamma)\rangle=W\rtimes Aut(\Gamma);$$

of course $\langle W,G\rangle=W\rtimes G$ $\forall G\leq Aut(\Gamma)$.

ii) $W_0\cap\check{P}=\{\mathrm{id}\}$ and $wxw^{-1}=w(x)$ $\forall x\in\check{P}$ $\forall w\in W$; hence $\check{P}\trianglelefteq\langle W_0,\check{P}\rangle$, so that $\langle W_0,\check{P}\rangle=\check{P}\rtimes W_0$; as above $\langle W_0,L\rangle=L\rtimes W_0$ $\forall L\leq\check{P}$ W_0-stable.

iii) $W\leq\check{P}\rtimes W_0$; more precisely

$$W=\begin{cases}\check{Q}_0\rtimes W_0 & \text{in the non twisted case}\\ \check{P}\rtimes W_0 & \text{in case } A_{2n}^{(2)}\\ Q_0\rtimes W_0 & \text{otherwise.}\end{cases}$$

($Q_0\subseteq\check{Q}_0=\oplus_{i\in I_0}\mathbb{Z}\check{\alpha}_i\subseteq\check{P}$ where $\check{\alpha}_i=\frac{\alpha_i}{d_i}$).

iv) Let $\mathcal{T}\doteqdot Aut(\Gamma)\cap(\check{P}\rtimes W_0)$; then

$$\tilde{W}\doteqdot W\rtimes\mathcal{T}=\begin{cases}\check{P}\rtimes W_0 & \text{in the non twisted case and in case } A_{2n}^{(2)}\\ P\rtimes W_0 & \text{otherwise.}\end{cases}$$

For a precise description of $\mathcal{T}$ see [14].

The length function $l:\tilde{W}\to\mathbb{N}$ is defined by

$$l(\tilde{w})\doteqdot\min\{r\in\mathbb{N}|\exists i_1,...,i_r\in I,\tau\in\mathcal{T}\text{ s.t. }\tilde{w}=s_{i_1}\cdot...\cdot s_{i_r}\tau\}.$$

Also, the braid group $\tilde{\mathcal{B}}$ is the group generated by $\{T_{\tilde{w}}|\tilde{w}\in\tilde{W}\}$ with relations $T_{\tilde{w}}T_{\tilde{w}'}=T_{\tilde{w}\tilde{w}'}$ whenever $l(\tilde{w}\tilde{w}')=l(\tilde{w})+l(\tilde{w}')$ (see [29]). Set $T_i\doteqdot T_{s_i}$.

1.2.E Root system.

The root system $\Phi\subseteq Q$ divides in two components: $\Phi=\Phi^{\mathrm{re}}\cup\Phi^{\mathrm{im}}$ where

$$\Phi^{\mathrm{re}}\doteqdot W.\{\alpha_i|i\in I\}=\tilde{W}.\{\alpha_i|i\in I\}=\{\alpha\in\Phi|(\alpha|\alpha)>0\}=\{\text{real roots}\}$$

and

$$\Phi^{\mathrm{im}}\doteqdot\{m\delta|m\in\mathbb{Z}\setminus\{0\}\}=\{\alpha\in\Phi|(\alpha|\alpha)=0\}=\{\text{imaginary roots}\}.$$

Φ^{re} can be described in terms of $\Phi_0\doteqdot W_0.\{\alpha_i|i\in I_0\}$ (the root system of $\mathfrak{g}^{(0)}$) as follows:

$$\Phi^{\mathrm{re}}=\begin{cases}\{m\delta+\alpha|\alpha\in\Phi_0,m\in\mathbb{Z}\} & \text{in the non twisted case}\\ \{m\delta+\alpha|\alpha\in\Phi_0,m\in\mathbb{Z}\}\cup & \\ \quad\cup\{(2m+1)\delta+2\alpha|\alpha\in\Phi_0,(\alpha|\alpha)=2,m\in\mathbb{Z}\} & \text{in case } A_{2n}^{(2)}\\ \{md_\alpha\delta+\alpha|\alpha\in\Phi_0,m\in\mathbb{Z}\} & \text{otherwise,}\end{cases}$$

where $d_\alpha \doteqdot \dfrac{(\alpha|\alpha)}{2}$ (or equivalently $d_{\tilde w(\alpha_i)} = d_i\ \forall i \in I,\ \tilde w \in \tilde W$); it is worth remarking that all the subgroups of $Aut(Q_{(\cdot|\cdot),\delta})$ considered in 1.2.C and 1.2.D leave Φ stable.

The multiplicity of each real root is 1; on the other hand let us define, $\forall i \in I_0$,

$$\tilde d_i \doteqdot \begin{cases} 1 & \text{in the non twisted case or in case } A_{2n}^{(2)} \\ d_i & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde k \doteqdot \max\{\tilde d_i | i \in I_0\},$$

and set $I^m \doteqdot \{i \in I_0 | \tilde d_i | m\}$; then the multiplicity of $m\delta$ is

$$\# I^m = \begin{cases} n & \text{if } k|m \\ \frac{\tilde n - n}{k-1} & \text{if } k \not| m, \end{cases}$$

which can be expressed also by saying that the root system with multiplicities is

$$\tilde\Phi \doteqdot \Phi^{\mathrm{re}} \cup \tilde\Phi^{\mathrm{im}} \quad \text{where } \tilde\Phi^{\mathrm{im}} \doteqdot \{(m\delta, i) | m \in \mathbb{Z} \setminus \{0\}, \tilde d_i | m\}$$

(Denote by $p : \tilde\Phi \to \Phi$ the natural map).

Remark that for $m \neq 0$ we have

$$(m\delta, i) \in \tilde\Phi^{\mathrm{im}} \Leftrightarrow \tilde d_i | m \Leftrightarrow m\delta + \alpha_i \in \Phi \Leftrightarrow m\check\omega_i \in \tilde W.$$

There is also another decomposition of Φ: $\Phi = \Phi_+ \cup \Phi_-$, where $\Phi_+ \doteqdot \Phi \cap Q^+ = \{\text{positive roots}\}$ and $\Phi_- \doteqdot -\Phi_+ = \{\text{negative roots}\}$. It induces an analogous decomposition of $\tilde\Phi$ and of its real and imaginary parts, with obvious notations.

Recall also the relation between the length function and the root system: $l(\tilde w) = \#\Phi_+(\tilde w)$ where

$$\Phi_+(\tilde w) \doteqdot \{\alpha \in \Phi_+ | \tilde w^{-1}(\alpha) < 0\} :$$

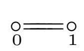
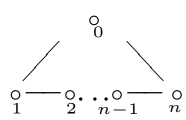
recall that if $s_{i_1} \cdot \ldots \cdot s_{i_r}$ is a reduced expression then

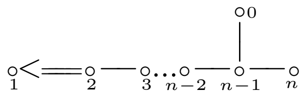
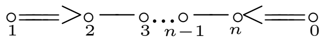
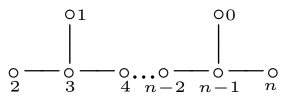
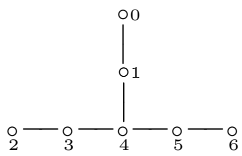
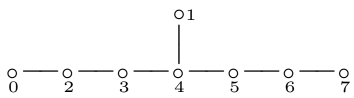
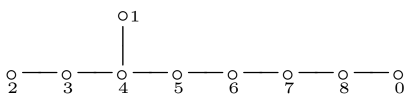
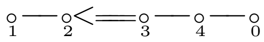
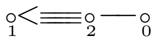
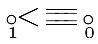
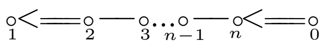
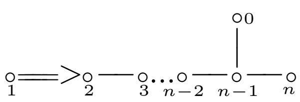
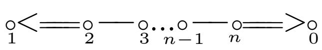
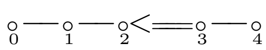
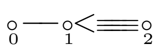
$$\Phi_+(s_{i_1} \cdot \ldots \cdot s_{i_r}) = \{s_{i_1} \cdot \ldots \cdot s_{i_{k-1}}(\alpha_{i_k}) | 1 \leq k \leq r\}.$$

Finally, $l(\tilde w \tilde w') = l(\tilde w) + l(\tilde w') \Leftrightarrow \Phi_+(\tilde w) \subseteq \Phi_+(\tilde w \tilde w')$.

§1.3. Dynkin diagrams and classification.

In the following we list the affine Dynkin diagrams. The labels under the vertices fix an identification between I and $\{0, 1, \ldots, n\}$ such that I_0 corresponds to $\{1, \ldots, n\}$. For each type we also recall the coefficients r_i (for $i \in I_0$: recall that r_0 is always 1) in the expression $\delta = \sum_{i \in I} r_i \alpha_i$.

$X^{(k)}_{\tilde n = \tilde n(n)}$	n	(Γ, I)	$(r_1, \ldots, r_n)$
$A_1^{(1)}$	1		(1)
$A_n^{(1)}$	> 1		$(1, \ldots, 1)$

$B_n^{(1)}$	>2		$(2,...,2,1)$
$C_n^{(1)}$	>1		$(1,2,...,2)$
$D_n^{(1)}$	>3		$(1,1,2,...,2,1)$
$E_6^{(1)}$	6		$(2,1,2,3,2,1)$
$E_7^{(1)}$	7		$(2,2,3,4,3,2,1)$
$E_8^{(1)}$	8		$(3,2,4,6,5,4,3,2)$
$F_4^{(1)}$	4		$(2,4,3,2)$
$G_2^{(1)}$	2		$(3,2)$
$A_2^{(2)}$	1		(2)
$A_{2n}^{(2)}$	>1		$(2,...,2)$
$A_{2n-1}^{(2)}$	>2		$(1,2,...,2)$
$D_{n+1}^{(2)}$	>1		$(1,...,1)$
$E_6^{(2)}$	4		$(2,3,2,1)$
$D_4^{(3)}$	2		$(2,1)$

§1.4. The quantum algebra.

In this paragraph we recall the definition of the quantum group $\mathcal{U}_q$ associated to the affine Kac-Moody algebra $\hat{\mathfrak{g}}^{(k)}$ of type $X_{\tilde{n}}^{(k)}$, and the main structures that $\mathcal{U}_q$ can be endowed with.

Definition 1.4.1.
Let $\hat{\mathfrak{g}}^{(k)}$ be the affine Kac-Moody algebra of type $X_{\tilde{n}}^{(k)}$.

We denote by $\mathcal{U}_q = \mathcal{U}_q(\hat{\mathfrak{g}}^{(k)}) = \mathcal{U}_q(X_{\tilde{n}}^{(k)})$ the quantum algebra of $\hat{\mathfrak{g}}^{(k)}$, that is the associative $\mathbb{C}(q)$-algebra generated by

$$\{E_i, F_i, K_i^{\pm 1}, D^{\pm 1} = K_\infty^{\pm 1} | i \in I\}$$

with relations:

$$K_\lambda K_\mu = K_{\lambda+\mu} \quad \forall \lambda, \mu \in Q_\infty,$$

$$K_\lambda E_i = q^{(\lambda|\alpha_i)} E_i K_\lambda \quad \forall \lambda \in Q_\infty, \forall i \in I,$$

$$K_\lambda F_i = q^{-(\lambda|\alpha_i)} F_i K_\lambda \quad \forall \lambda \in Q_\infty, \forall i \in I,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \forall i, j \in I$$

$$\sum_{r=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^r E_j E_i^{1-a_{ij}-r} = 0 = \sum_{r=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^r F_j F_i^{1-a_{ij}-r} \quad \forall i \neq j \in I,$$

where we set:

i) $K_\lambda \doteq \prod_{i \in I_\infty} K_i^{m_i}$ if $\lambda = \sum_{i \in I_\infty} m_i \alpha_i \in Q_\infty$;

ii) $q_\alpha \doteq q^{d_\alpha}$ $\forall \alpha \in \Phi^{\mathrm{re}}$, $q_i \doteq q_{\alpha_i}$ $\forall i \in I$, $q_{(m\delta,i)} \doteq q_i$ $\forall (m,i) \in \tilde{\Phi}_+^{\mathrm{im}}$;

iii) $[m]_{q^r} \doteq \dfrac{q^{rm} - q^{-rm}}{q^r - q^{-r}}$ and $[m]_{q^r}! \doteq \prod_{s=1}^m [s]_{q^r}$ $\forall m, r \in \mathbb{Z}$;

moreover $\begin{bmatrix} r \\ s \end{bmatrix}_{q^m} = \frac{[r]_{q^m}!}{[s]_{q^m}![r-s]_{q^m}!}$ $\forall m \in \mathbb{Z}, \forall s \leq r \in \mathbb{N}$;

iv) for further use let us also define $\forall \alpha \in \tilde{\Phi}_+$, $\forall x \in \mathcal{U}_q$

$$\exp_\alpha(x) \doteq \sum_{m \geq 0} \frac{x^m}{(m)_\alpha!},$$

where $\forall \alpha \in \tilde{\Phi}_+, \forall m \in \mathbb{N}$ we have put

$$(m)_\alpha \doteq \begin{cases} \dfrac{q_\alpha^{2m} - 1}{q_\alpha^2 - 1} & \text{if } \alpha \in \Phi_+^{\mathrm{re}} \\ m & \text{if } \alpha \in \tilde{\Phi}_+^{\mathrm{im}} \end{cases} \quad \text{and} \quad (m)_\alpha! \doteq \prod_{s=1}^m (s)_\alpha.$$

Remark 1.4.2.
Recall that on $\mathcal{U}_q$ we have the following structures:

1) the Q-gradation $\mathcal{U}_q = \oplus_{\eta \in Q} \mathcal{U}_{q,\eta}$ determined by the conditions that $E_i \in \mathcal{U}_{q,\alpha_i}$ and $F_i \in \mathcal{U}_{q,-\alpha_i}$ $\forall i \in I$, $K_i^{\pm 1} \in \mathcal{U}_{q,0}$ $\forall i \in I_\infty$ and $\mathcal{U}_{q,\alpha}\mathcal{U}_{q,\beta} \subseteq \mathcal{U}_{q,\alpha+\beta}$ $\forall \alpha, \beta \in Q$;

2) the triangular decomposition: $\mathcal{U}_q \cong \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^+ \cong \mathcal{U}_q^+ \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^-$, where $\mathcal{U}_q^-$, $\mathcal{U}_q^0$ and $\mathcal{U}_q^+$ are the subalgebras of $\mathcal{U}_q$ generated respectively by $\{E_i | i \in I\}$, $\{K_i^{\pm 1} | i \in I_\infty\}$ and $\{F_i | i \in I\}$; define also $\mathcal{U}_q^{\geq 0} \doteq (\mathcal{U}_q^+, \mathcal{U}_q^0)$, $\mathcal{U}_q^{\leq 0} \doteq (\mathcal{U}_q^-, \mathcal{U}_q^0)$;

3) the $\mathbb{C}$-anti-linear anti-involution $\Omega:\mathcal{U}_q\to\mathcal{U}_q$ defined by

$$\Omega(E_i)\doteqdot F_i,\quad \Omega(F_i)\doteqdot E_i\ \forall i\in I;\quad \Omega(K_i)\doteqdot K_i^{-1}\ \forall i\in I_\infty;\quad \Omega(q)\doteqdot q^{-1};$$

4) the $\mathbb{C}(q)$-linear anti-involution $\Xi:\mathcal{U}_q\to\mathcal{U}_q$ defined by

$$\Xi(E_i)\doteqdot E_i,\quad \Xi(F_i)\doteqdot F_i\ \forall i\in I;\quad \Xi(K_i)\doteqdot K_i^{-1}\ \forall i\in I_\infty;$$

5) a braid group action commuting with Ω, verifying $\Xi T_i=T_i^{-1}\Xi\ \forall i\in I$, and preserving the gradation (that is $T_w(\mathcal{U}_{q,\eta})=\mathcal{U}_{q,w(\eta)}$); more precisely

$T_w(E_i)\in\mathcal{U}^+_{q,w(\alpha_i)}$ if $w\in W$ and $i\in I$ are such that $w(\alpha_i)\in Q^+$ (i.e. $l(ws_i)>l(w)$),

and

$$T_i(E_i)=-F_iK_i\ \ \forall i\in I,\quad T_i(E_j)=\sum_{r=0}^{-a_{ij}}(-1)^{r-a_{ij}}q_i^{-r}E_i^{(-a_{ij}-r)}E_jE_i^{(r)}\ \ \forall i\neq j\in I$$

where $\forall m\in\mathbb{N}\ E_i^{(m)}\doteqdot\frac{E_i^m}{[m]_{q_i}!}$;

6) a Hopf-structure $(\Delta,\,\epsilon,\,S)$, whose coproduct $\Delta:\mathcal{U}_q\to\mathcal{U}_q\otimes\mathcal{U}_q$ is given by

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i,$$

$$\Delta(F_i)=1\otimes F_i+F_i\otimes K_i^{-1},$$

$$\Delta(K_i)=K_i\otimes K_i;$$

Let us remark that $\Delta\circ\Omega=\sigma\circ(\Omega\otimes\Omega)\circ\Delta$ where $\sigma:\mathcal{U}_q\otimes\mathcal{U}_q\to\mathcal{U}_q\otimes\mathcal{U}_q$ is defined by $\sigma(x\otimes y)\doteqdot y\otimes x$;

7) the Killing form, that is the unique bilinear form $(\cdot,\cdot):\mathcal{U}_q^{\geq 0}\otimes\mathcal{U}_q^{\leq 0}\to\mathbb{C}(q)$ such that

i) $(x,y_1y_2)=(\Delta(x),y_1\otimes y_2)\ \forall x\in\mathcal{U}_q^{\geq 0},\ \forall y_1,y_2\in\mathcal{U}_q^{\leq 0}$;

ii) $(x_1x_2,y)=(x_2\otimes x_1,\Delta(y))\ \forall x_1,x_2\in\mathcal{U}_q^{\geq 0},\ \forall y\in\mathcal{U}_q^{\leq 0}$;

iii) $(\cdot,\cdot)\big|_{\mathcal{U}^{\geq 0}_{q,\eta}\times\mathcal{U}^{\leq 0}_{q,-\tilde\eta}}=0$ if $\eta\neq\tilde\eta$;

iv) $(K_\lambda,K_\mu)=q^{-(\lambda,\mu)}\ \forall\lambda,\mu\in Q_\infty$;

v) $(E_i,F_i)=\dfrac{1}{q_i^{-1}-q_i}\ \forall i\in I$;

$(\cdot,\cdot)$ has the following important properties:

a) $(\cdot,\cdot)\big|_{\mathcal{U}^+_{q,\eta}\times\mathcal{U}^-_{q,-\eta}}$ is non degenerate $\forall\eta\in Q_+$;

b) $(xK_\lambda,y)=(x,y)=(x,yK_\lambda)\ \forall x\in\mathcal{U}_q^+,\ \forall y\in\mathcal{U}_q^-,\ \forall\lambda\in Q_\infty$.

2. COPIES OF $A_1^{(1)}$ IN $X_{\tilde n}^{(k)}$.

§2.1. Definition of the root vectors.

In this section we shall shortly recall Beck's results for non-twisted affine quantum algebras (see [1] and [2]) and prove that his method applies word by word to the twisted case, that is we can find a copy of $\mathcal{U}_q(A_1^{(1)})$ in $\mathcal{U}_q(X_{\tilde n}^{(k)})\ \forall i\in I_0$, provided that $(X_{\tilde n}^{(k)},i)\neq(A_{2n}^{(2)},1)$.

To this aim let us first define, $\forall i \in I_0$, an element $\lambda_i \in P\check{}$ as follows:

$$\lambda_i \doteqdot \tilde{d}_i \check{\omega_i}$$

where we recall that $\tilde{d}_i = \min\{m > 0 | m\check{\omega_i} \in \tilde{W}\}$ (see 1.2.E).

The next proposition and remark is what one needs in order to define the positive root vectors.

Proposition 2.1.1
$\forall m \in \mathbb{N}$ we have $l((\lambda_1 \cdot ... \cdot \lambda_n)^m) = m \sum_{i \in I_0} l(\lambda_i)$.

Proof: See [3]. □

Remark 2.1.2.

$$\Phi_+^{\mathrm{re}} = \{m\delta - \alpha \in \Phi_+ | \alpha \in Q_0^+, m > 0\} \cup \{m\delta + \alpha \in \Phi_+ | \alpha \in Q_0^+, m \in \mathbb{N}\}.$$

Moreover

$$\{m\delta - \alpha \in \Phi_+ | \alpha \in Q_0^+, m > 0\} = \bigcup_{m>0} \Phi_+((\lambda_1 \cdot ... \cdot \lambda_n)^m),$$

$$\{m\delta + \alpha \in \Phi_+ | \alpha \in Q_0^+, m \in \mathbb{N}\} = \bigcup_{m>0} \Phi_+((\lambda_1 \cdot ... \cdot \lambda_n)^{-m}).$$

Proof: See [2]. □

The preceding results suggest to define the following sequence, by means of which it will be easy to give a suitable definition of the positive real root vectors.

Definition 2.1.3.
Let $N \doteqdot l(\lambda_1 \cdot ... \cdot \lambda_n)$ and define $\iota : \mathbb{Z} \to I_0$ to be such that

$$\begin{cases} \lambda_1 \cdot ... \cdot \lambda_n = s_{\iota_1} \cdot ... \cdot s_{\iota_N} \tau \\ \iota_{r+N} = \tau(\iota_r) \quad \forall r \in \mathbb{Z}, \end{cases}$$

where τ is a suitable element of $\mathcal{T}$ (uniquely determined by the above condition); for further determinations of ι see remark 2.1.5 and notation 4.2.5.

Notice that ι induces functions

$$\mathbb{Z} \ni r \mapsto w_r \in W \quad \text{and} \quad \mathbb{Z} \ni r \mapsto \beta_r \in \Phi_+^{\mathrm{re}}$$

by

$$w_r \doteqdot \begin{cases} s_{\iota_1} \cdot ... \cdot s_{\iota_{r-1}} \text{if } r \geq 1 \\ s_{\iota_0} \cdot ... \cdot s_{\iota_{r+1}} \text{if } r \leq 0, \end{cases} \qquad \text{and} \qquad \beta_r \doteqdot w_r(\alpha_{\iota_r}).$$

It is well known (see [2]) that β is a bijection.

Then we arrive at the definition of the root vectors:

Definition 2.1.4.
The positive real root vectors E_α (with $\alpha \in \Phi_+^{\mathrm{re}}$) are defined by

$$E_\alpha \doteqdot \begin{cases} T_{w_r}(E_{\iota_r}) & \text{if } r \geq 1 \\ T_{w_r^{-1}}^{-1}(E_{\iota_r}) = \Xi T_{w_r}(E_{\iota_r}) & \text{if } r \leq 0. \end{cases}$$

Also, positive imaginary root vectors $\tilde{E}_{(m\tilde{d}_i\delta,i)}$ (with $m>0$, $i\in I_0$) are defined by

$$\tilde{E}_{(m\tilde{d}_i\delta,i)} \doteqdot -E_{m\tilde{d}_i\delta-\alpha_i}E_i + q_i^{-2}E_iE_{m\tilde{d}_i\delta-\alpha_i}.$$

Similarly, the negative root vectors are defined by

$$F_\alpha \doteqdot \Omega(E_\alpha) \quad \text{if } \alpha\in\Phi_+^{\mathrm{re}}, \quad \tilde{F}_\alpha \doteqdot \Omega(\tilde{E}_\alpha) \quad \text{if } \alpha\in\tilde{\Phi}_+^{\mathrm{im}}.$$

Remark 2.1.5.

It is worth remarking, for later use, that ι can be chosen (and it will be chosen!) so that $\exists N_0=0<N_1<...<N_n=N$ and $\exists\tau_1,...,\tau_n\in\mathcal{T}$ such that $\forall i=1,...,n$ $\lambda_i=s_{\iota_{N_{i-1}+1}}\cdot...\cdot s_{\iota_{N_i}}\tau_i$ (notice that $\tau_1\cdot...\cdot\tau_n=\tau$).

If this is the case then

$$E_{m\tilde{d}_i\delta+\alpha_i}=T_{\lambda_i}^{-m}(E_i) \quad \forall m\in\mathbb{N},$$

$$E_{m\tilde{d}_i\delta-\alpha_i}=T_{\lambda_i}^{m}T_i^{-1}(E_i) \quad \forall m>0,$$

which depends on the fact that

1) $T_{\lambda_j}T_{\lambda_i}=T_{\lambda_i}T_{\lambda_j}$ $\forall i,j\in I_0$;

2) $T_{\lambda_j}(E_i)=E_i$ and $T_{\lambda_j}T_i=T_iT_{\lambda_j}$ $\forall i,j\in I_0$ such that $j\neq i$.

Proof: See [25] and [6]. □

Definition 2.1.6.

If $i\in I_0$ we define $\mathcal{U}_q^{(i)}$ to be the $\mathbb{C}(q^{d_i})$-subalgebra of $\mathcal{U}_q=\mathcal{U}_q(X_{\tilde{n}}^{(k)})$ generated by $\{E_i,E_{\tilde{d}_i\delta-\alpha_i},F_i,F_{\tilde{d}_i\delta-\alpha_i},K_i^{\pm1},K_{\tilde{d}_i\delta}^{\pm1}\}$.

Obviously $\mathcal{U}_q^{(i)}$ is Ω-stable and pointwise fixed by T_{λ_j} for $j\in I_0\setminus\{i\}$. Moreover it will become clear that $\mathcal{U}_q^{(i)}$ is the smallest $\mathbb{C}(q^{d_i})$-subalgebra of $\mathcal{U}_q$ containing E_i and K_i and stable under $T_i^{\pm1},T_{\lambda_i}^{\pm1}$; in particular we shall see that it contains $E_{m\tilde{d}_i\delta\pm\alpha_i}$, $F_{m\tilde{d}_i\delta\pm\alpha_i}$, $\tilde{E}_{(m\tilde{d}_i\delta,i)}$ and $\tilde{F}_{(m\tilde{d}_i\delta,i)}$ $\forall m>0$.

§2.2. The homomorphisms φ_i.

We are now ready to state the announced properties of the root vectors just defined and of the braid group action, which will allow us to construct homomorphisms $\varphi_i:\mathcal{U}_q(A_1^{(1)})\to\mathcal{U}_q^{(i)}\subseteq\mathcal{U}_q(X_{\tilde{n}}^{(k)})$ for each $i\in I_0$, under the only condition that $i\neq1$ if $X_{\tilde{n}}^{(k)}=A_{2n}^{(2)}$, condition that we shall assume in the remaining part of section 2. These homomorphisms φ_i behave "well" on the root vectors (that is, they transform root vectors in root vectors, see corollary 2.2.3), and this fact immediately implies some important consequences: in particular we shall translate the commutation relations from $\mathcal{U}_q(A_1^{(1)})$ to $\mathcal{U}_q(X_{\tilde{n}}^{(k)})$. We shall also underline the obstruction that we meet if we try to apply the same argument when $X_{\tilde{n}}^{(k)}=A_{2n}^{(2)}$ and $i=1$: this last situation will be studied in sections 3 and 4.

Lemma 2.2.1

Assume, as required, that $X_{\tilde{n}}^{(k)}\neq A_{2n}^{(2)}$ or $i\neq1$; then we have:

1) $l(\lambda_is_i\lambda_i)=2l(\lambda_i)-1$;

2) $(T_{\lambda_i}T_i^{-1})^2\in\langle T_{\lambda_j}|j\neq i\rangle$;

3) $[E_{\tilde{d}_i\delta-\alpha_i},F_i]=0$ and $E_iE_{\tilde{d}_i\delta+\alpha_i}=q_i^{-2}E_{\tilde{d}_i\delta+\alpha_i}E_i$.

4) $\forall j \in I_0 \setminus \{i\}$ we have $(T_{\lambda_i}T_i^{-1})^2 T_{\lambda_j}^{\frac{\tilde d_i a_{ij}}{\tilde d_j}} \in \langle T_{\lambda_r} | r \in I_0 \setminus \{i,j\}\rangle$; in particular $T_{\lambda_i}T_i^{-1}(E_j) = T_{\lambda_j}^{-\frac{\tilde d_i a_{ij}}{\tilde d_j}} T_i(E_j)$.

Proof: For the proof see [25] and [1]. Here we want just to point out that all the statements are based on the fact that there is no root α of the form $\alpha = m\delta + \varepsilon\alpha_i$ such that $\tilde d_i\delta + \alpha_i \prec \alpha \prec \alpha_i$.

Remark that this is no longer true in case $A_{2n}^{(2)}$ for $i = 1$: indeed in this case we have $\delta + \alpha_i \prec \delta + 2\alpha_i \prec \alpha_i$. Hence the existence of roots of the form $m\delta + 2\alpha$ with $\alpha \in \Phi_+ \cap Q_0$ (which never happens if we are not in case $A_{2n}^{(2)}$) is what makes the difference between this case and the others. $\square$

Since Beck's construction of the homomorphisms $\varphi_i : \mathcal{U}_q(A_1^{(1)}) \to \mathcal{U}_q^{(i)} \subseteq \mathcal{U}_q(X_n^{(1)})$ (and their properties) is based on the statements of lemma 2.2.1 the argument used in [1] applies exactly to the present situation, so that we have:

Proposition 2.2.2
If $X_{\tilde n}^{(k)} \neq A_{2n}^{(2)}$ or $i \neq 1$, then there exists a $\mathbb{C}$-algebra homomorphism

$$\varphi_i : \mathcal{U}_q(A_1^{(1)}) \to \mathcal{U}_q^{(i)} \subseteq \mathcal{U}_q(X_{\tilde n}^{(k)})$$

verifying the following properties:
1) $\varphi_i(K_1) = K_i$, $\varphi_i(K_1K_0) = K_{\tilde d_i\delta}$ and $\varphi_i(q) = q_i = q^{d_i}$;
2) $\varphi_i(E_1) = E_i$ and $\varphi_i(E_0) = E_{\tilde d_i\delta - \alpha_i}$;
3) $\varphi_i\Omega = \Omega\varphi_i$;
4) $\varphi_i T_1 = T_i\varphi_i$ and $\varphi_i T_\tau = T_{\lambda_i}T_i^{-1}\varphi_i$, where τ is the non trivial Dynkin diagram automorphism of $A_1^{(1)}$. $\square$

Corollary 2.2.3
Suppose again that $X_{\tilde n}^{(k)} \neq A_{2n}^{(2)}$ or $i \neq 1$ and let ν_i be the group homomorphism $\nu_i : Q(A_1^{(1)}) \to Q(X_{\tilde n}^{(k)})$ defined in the obvious way by $\nu_i(\alpha_1) \doteqdot \alpha_i$, $\nu_i(\delta) \doteqdot \tilde d_i\delta$; furthermore let the injection $\tilde\nu_i : \Phi_+(A_1^{(1)}) \to \tilde\Phi_+(X_{\tilde n}^{(k)})$ be given by:

$$\tilde\nu_i(\alpha) \doteqdot \begin{cases} \nu_i(\alpha) & \text{if } \alpha \text{ is real} \\ (\nu_i(\alpha), i) & \text{if } \alpha \text{ is imaginary;} \end{cases}$$

Then φ_i has the property that

$$\varphi_i(K_\alpha) = K_{\nu_i(\alpha)} \quad \forall \alpha \in Q(A_1^{(1)}),$$

$$\varphi_i(E_\alpha) = E_{\tilde\nu_i(\alpha)} \quad \forall \alpha \in \Phi_+(A_1^{(1)})$$

and

$$\varphi_i(\tilde E_\alpha) = \tilde E_{\tilde\nu_i(\alpha)} \quad \forall \alpha \in \Phi_+^{\mathrm{im}}(A_1^{(1)}),$$

where the vectors $E_{(m\tilde d_i\delta,i)}$ are defined by the relation

$$1 - (q_i - q_i^{-1}) \sum_{m>0} \tilde E_{(m\tilde d_i\delta,i)} u^m = \exp\left((q_i - q_i^{-1}) \sum_{m>0} E_{(m\tilde d_i\delta,i)} u^m \right).$$

□

As an immediate consequence of corollary 2.2.3, we have the following proposition.

Proposition 2.2.4

Suppose as before that $(X_{\tilde n}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$; then we have the following relations in $\mathcal{U}_q(X_{\tilde n}^{(k)})$:

1) $[\tilde E_{(r\tilde d_i\delta,i)}, \tilde E_{(s\tilde d_i\delta,i)}] = 0 = [E_{(r\tilde d_i\delta,i)}, E_{(s\tilde d_i\delta,i)}]$ $\forall r, s > 0$;

2) $T_{\lambda_j}(\tilde E_{(r\tilde d_i\delta,i)}) = \tilde E_{(r\tilde d_i\delta,i)}$ and $T_{\lambda_j}(E_{(r\tilde d_i\delta,i)}) = E_{(r\tilde d_i\delta,i)}$ $\forall r > 0, j \in I_0$;

3) $[E_{(r\tilde d_i\delta,i)}, F_{(s\tilde d_i\delta,i)}] = \delta_{rs}\dfrac{[2r]_{q_i}}{r}\dfrac{K_{r\tilde d_i\delta} - K_{r\tilde d_i\delta}^{-1}}{q_i - q_i^{-1}}$ $\forall r, s > 0$;

4) $[E_{(r\tilde d_i\delta,i)}, E_{s\tilde d_i\delta+\alpha_i}] = \dfrac{[2r]_{q_i}}{r}E_{(r+s)\tilde d_i\delta+\alpha_i}$ and

$[E_{(r\tilde d_i\delta,i)}, E_{(s+1)\tilde d_i\delta-\alpha_i}] = -\dfrac{[2r]_{q_i}}{r}E_{(r+s+1)\tilde d_i\delta-\alpha_i}$ $\forall r > 0, s \geq 0$.

Proof: The claim follows applying φ_i to the analogous relations holding in $\mathcal{U}_q(A_1^{(1)})$ (see [5] and [1]). □

3. THE CASE OF $A_2^{(2)}$.

In this section we study the case of $\mathfrak{sl}_3^{(2)} = A_2^{(2)}$ (and $\mathcal{U}_q$ will indicate $\mathcal{U}_q(\mathfrak{sl}_3^{(2)})$). In particular we want to understand some important properties of the imaginary root vectors, their commutation rules, and their behaviour under the action of the braid group.

§3.1. Real root vectors.

Recall that the Dynkin diagram of $A_2^{(2)}$ is

$$A_2^{(2)} \qquad \underset{1}{\circ} \langle\!\equiv\!\equiv \underset{0}{\circ}$$

and that we can describe its roots as follows: $\delta = \alpha_0 + 2\alpha_1$ and

$$\Phi_+^{\mathrm{re}} = \{m\delta + \alpha_1, (m+1)\delta - \alpha_1, (2m+1)\delta \pm 2\alpha_1 | m \in \mathbb{N}\},$$

$$\Phi_+^{\mathrm{im}} = \{(m+1)\delta | m \in \mathbb{N}\};$$

moreover $\omega_1 = \check{\omega_1} = s_0s_1$: indeed $d_1 = 1$ and

$$s_0s_1(\alpha_1) = -s_0(\alpha_1) = -(\alpha_1 + \alpha_0) = -\delta + \alpha_1 = \omega_1(\alpha_1).$$

It follows that the real root vectors are given by:

$$E_{m\delta+\alpha_1} = (T_0T_1)^{-m}(E_1),$$

$$E_{(m+1)\delta-\alpha_1} = (T_0T_1)^mT_0(E_1),$$

$$E_{(2m+1)\delta+2\alpha_1} = (T_0T_1)^{-m}T_1^{-1}(E_0),$$

$$E_{(2m+1)\delta-2\alpha_1} = (T_0T_1)^m(E_0).$$

We shall now give the following proposition which will semplify many calculations.

Proposition 3.1.1
$\forall\alpha\in\Phi_+^{\rm re}\setminus\{\alpha_0\}\quad T_0\Xi(E_\alpha)=E_{s_0(\alpha)}$.
$\forall\alpha\in\Phi_+^{\rm re}\setminus\{\alpha_1\}\quad \Xi T_1(E_\alpha)=E_{s_1(\alpha)}$.

Proof: $\Xi(E_\alpha)\in\mathcal{U}_{q,\alpha}^+$, hence $T_0\Xi(E_\alpha)\in\mathcal{U}_{q,s_0(\alpha)}$; it is then enough to prove that $T_0\Xi(E_\alpha)$ is a root vector, which is easily seen: $T_0\Xi(T_0T_1)^{-m}(E_1)=(T_0T_1)^mT_0(E_1)$ and similarly for the other root vectors, so that the first assertion is proved. From this result and from the fact that $(T_0\Xi)(\Xi T_1)=T_0T_1$ the second assertion of the proposition follows immediately. □

We shall now give some simple commutation relations between the real root vectors.

Lemma 3.1.2
$[E_{\delta-\alpha_1},F_1]=-[4]_qK_1E_0$ and $E_1E_{\delta+\alpha_1}-q^{-2}E_{\delta+\alpha_1}E_1=q^{-2}[4]_qE_{\delta+2\alpha_1}$.

Proof: The first identity is a straightforward computation, using that

$$E_{\delta-\alpha_1}=T_0(E_1)=-E_0E_1+q^{-4}E_1E_0;$$

the second relation is ΞT_1 applied to the first one. □

As a first application of lemma 3.1.2 we state the following proposition:

Proposition 3.1.3
$\{E_1,E_{\delta-\alpha_1},F_1,F_{\delta-\alpha_1},K_1^{\pm1},K_0^{\pm1}\}$ is a system of generators of $\mathcal{U}_q$ as a $\mathbb{C}(q)$-algebra.

Proof: Let $\mathcal{A}$ be the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}_q$ generated by

$$\{E_1,E_{\delta-\alpha_1},F_1,F_{\delta-\alpha_1},K_1^{\pm1},K_0^{\pm1}\}.$$

It is then enough to prove that $E_0,F_0\in\mathcal{A}$. Since $\mathcal{A}$ is Ω-stable it is enough to prove that $E_0\in\mathcal{A}$. But $E_0=-\frac{1}{[4]_q}K_1^{-1}[E_{\delta-\alpha_1},F_1]\in\mathcal{A}$. □

Corollary 3.1.4
The least subalgebra of $\mathcal{U}_q$ containing $\mathcal{U}_q^0$ and E_1 and stable under the action of T_0T_1 and $T_0\Xi$ is $\mathcal{U}_q$. □

§3.2. The imaginary root vector $\tilde{E}_\delta$.

Here we want to study some properties of $\tilde{E}_\delta$, where we recall that

$$\tilde{E}_\delta=-E_{\delta-\alpha_1}E_1+q^{-2}E_1E_{\delta-\alpha_1}.$$

Proposition 3.2.1
$T_0\Xi(\tilde{E}_\delta)=\tilde{E}_\delta$.

Proof: The claim follows immediately from the definitions and thanks to proposition 3.1.1. □

Proposition 3.2.2

The following commutation rules hold:

$$[\tilde{E}_\delta, E_1] = -[3]_q! E_{\delta+\alpha_1}, \quad [\tilde{E}_\delta, F_1] = -[3]_q! K_1 E_{\delta-\alpha_1}.$$

Proof: The proof consists in simple computations using the following ingredients: $\tilde{E}_\delta = E_0 E_1^2 - q^{-3}[2]_q E_1 E_0 E_1 + q^{-6} E_1^2 E_0$ (by the definition and the fact that $E_{\delta-\alpha_1} = -E_0 E_1 + q^{-4} E_1 E_0$),

$$E_{\delta+\alpha_1} = T_1^{-1} T_0^{-1}(E_1) = T_1^{-1}(-E_1 E_0 + q^{-4} E_0 E_1) = -K_1^{-1}[T_1^{-1}(E_0), F_1] =$$

$$= -K_1^{-1}\left[\sum_{r=0}^{4}(-1)^r q^{-r} E_1^{(r)} E_0 E_1^{(4-r)}, F_1\right] = -\sum_{r=0}^{3}(-1)^r q^{-2r} E_1^{(r)} E_0 E_1^{(3-r)}$$

and

$$[E_1^m, F_1] = [m]_q \frac{q^{1-m} K_1 - q^{m-1} K_1^{-1}}{q - q^{-1}} E_1^{m-1}.$$

□

Proposition 3.2.3
$\Xi T_1(\tilde{E}_\delta) = \tilde{E}_\delta$ (hence $T_0 T_1(\tilde{E}_\delta) = \tilde{E}_\delta$, thanks to proposition 3.2.1).

Proof: Since $\Xi T_1(E_1) = -K_1^{-1} F_1$ we have that

$$\Xi T_1(\tilde{E}_\delta) = K_1^{-1} F_1 E_{\delta+\alpha_1} - q^{-2} E_{\delta+\alpha_1} K_1^{-1} F_1 = -K_1^{-1}[E_{\delta+\alpha_1}, F_1] =$$

$$= \frac{1}{[3]_q!} K_1^{-1}[[\tilde{E}_\delta, E_1], F_1] = -\frac{1}{[3]_q!} K_1^{-1}[E_1, [[\tilde{E}_\delta, F_1]] = K_1^{-1}[E_1, K_1 E_{\delta-\alpha_1}] = \tilde{E}_\delta.$$

□

Corollary 3.2.4
Applying a power of $T_0 T_1$ to proposition 3.2.2 we get that $\forall m \in \mathbb{N}$ the following commutation relations hold:

$$[\tilde{E}_\delta, E_{m\delta+\alpha_1}] = -[3]_q! E_{(m+1)\delta+\alpha_1}, \quad [\tilde{E}_\delta, E_{(m+1)\delta-\alpha_1}] = [3]_q! E_{(m+2)\delta-\alpha_1}.$$

□

We conclude this paragraph with the commutation rules between $\tilde{E}_\delta$ and the simple root vectors E_0 and F_0.

Lemma 3.2.5
$[\tilde{E}_\delta, F_0] = -(q^2-1)[3]_q E_1^2 K_0^{-1}$ and $[\tilde{E}_\delta, E_0] = -(q^2-1)[3]_q E_{\delta-\alpha_1}^2$.

Proof: The first relation is found by straightforward calculations; the second relation is found by applying $T_0 \Xi$ to the first one. □

§3.3. The imaginary root vectors $\tilde{E}_{m\delta}$.

Here we generalize to the vectors $\tilde{E}_{m\delta} = -E_{m\delta-\alpha_1} E_1 + q^{-2} E_1 E_{m\delta-\alpha_1}$ what we know for $\tilde{E}_\delta$.

We start by proving the equivalence between some commutation rules among the imaginary root vectors and their behaviour under the action of $T_0 T_1$.

This is completely similar to what happens in the case of $A_1^{(1)}$.

Lemma 3.3.1
$\forall m > 0$ the following implications hold:

$$[\tilde{E}_\delta, \tilde{E}_{m\delta}] = 0 \Leftrightarrow T_0T_1(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta}$$

and

$$T_0T_1(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta} \Leftrightarrow T_0\Xi(\tilde{E}_{(m+1)\delta}) = \tilde{E}_{(m+1)\delta} = \Xi T_1(\tilde{E}_{(m+1)\delta}).$$

Proof: See [5]. □

The following proposition is a fundamental tool in the study of the imaginary root vectors.

Proposition 3.3.2
Let P_m be the following statement:
$i)_m$ $T_0T_1(\tilde{E}_{m\delta}) = \tilde{E}_{m\delta} = T_0\Xi(\tilde{E}_{m\delta}) = \Xi T_1(\tilde{E}_{m\delta})$;
$ii)_m$ $E_0E_{(m-1)\delta+\alpha_1} - q^{-4}E_{(m-1)\delta+\alpha_1}E_0 =$

$$= \frac{1}{[4]_q}\big((q^4 - q^2 - q^{-2})E_{\delta-\alpha_1}\tilde{E}_{(m-1)\delta} + (q^2 + q^{-2} - q^{-4})\tilde{E}_{(m-1)\delta}E_{\delta-\alpha_1} +$$

$$+q^2E_{2\delta-\alpha_1}\tilde{E}_{(m-2)\delta} - q^{-2}\tilde{E}_{(m-2)\delta}E_{2\delta-\alpha_1}\big);$$

$iii)_m$ $[\tilde{E}_{m\delta}, E_1] = (q^4 - q^{-2})\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1} + (q^2 - q^{-4})E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta} +$

$$+q^2\tilde{E}_{(m-2)\delta}E_{2\delta+\alpha_1} - q^{-2}E_{2\delta+\alpha_1}\tilde{E}_{(m-2)\delta};$$

$iv)_m$ $[\tilde{E}_{m\delta}, E_{\delta-\alpha_1}] = -(q^4 - q^{-2})E_{2\delta-\alpha_1}\tilde{E}_{(m-1)\delta} - (q^2 - q^{-4})\tilde{E}_{(m-1)\delta}E_{2\delta-\alpha_1} +$

$$-q^2E_{3\delta-\alpha_1}\tilde{E}_{(m-2)\delta} + q^{-2}\tilde{E}_{(m-2)\delta}E_{3\delta-\alpha_1};$$

$v)_m$ $[\tilde{E}_{r\delta}, \tilde{E}_{s\delta}] = 0 \quad \forall r, s \leq m$.
Then, if we set $\tilde{E}_{-\delta} = 0$ and $\tilde{E}_{0\delta} = -\dfrac{1}{q - q^{-1}}$, P_m is true for every $m > 0$.

Proof: The proof is an induction on m.
If $m = 1$ remark that: $i)_1$ is proposition 3.2.1 and proposition 3.2.3; $ii)_1$ is the expression of $E_{\delta-\alpha_1}$ in terms of E_0 and E_1; $iii)_1$ is proposition 3.2.2; $iv)_1$ is $T_0\Xi$ applied to $iii)_1$; $v)_1$ is obvious.
Let m be bigger than 1. Then:
$i)_m$ follows immediately from $v)_{m-1}$ (which in particular implies $[\tilde{E}_\delta, \tilde{E}_{(m-1)\delta}] = 0$) and from lemma 3.3.1.
$ii)_m$ is obtained by the following steps, where we shall indicate among the lines the results that we need to pass from one side to the other of the identities:

$$E_0E_{(m-1)\delta+\alpha_1} - q^{-4}E_{(m-1)\delta+\alpha_1}E_0 =$$

$$\{\text{corollary 3.2.4}\}$$

$$= -\frac{1}{[3]_q!}(E_0\tilde{E}_\delta E_{(m-2)\delta+\alpha_1} - E_0E_{(m-2)\delta+\alpha_1}\tilde{E}_\delta +$$

$$-q^{-4}\tilde{E}_\delta E_{(m-2)\delta+\alpha_1}E_0 + q^{-4}E_{(m-2)\delta+\alpha_1}\tilde{E}_\delta E_0) =$$

$$\{\text{lemma 3.2.5}\}$$

$$= -\frac{1}{[3]_q!}\big([\tilde{E}_\delta, E_0 E_{(m-2)\delta+\alpha_1} - q^{-4}E_{(m-2)\delta+\alpha_1}E_0]+$$

$$+(q^2-1)[3]_q(E^2_{\delta-\alpha_1}E_{(m-2)\delta+\alpha_1} - q^{-4}E_{(m-2)\delta+\alpha_1}E^2_{\delta-\alpha_1})\big) =$$

$$\{i)_{m-1},\ ii)_{m-1} \text{ and definition of } \tilde{E}_{(m-1)\delta}\}$$

$$= -\frac{1}{[3]_q!}\Big(\frac{1}{[4]_q}[\tilde{E}_\delta, (q^4-q^2-q^{-2})E_{\delta-\alpha_1}\tilde{E}_{(m-2)\delta} + (q^2+q^{-2}-q^{-4})\tilde{E}_{(m-2)\delta}E_{\delta-\alpha_1}+$$

$$+q^2E_{2\delta-\alpha_1}\tilde{E}_{(m-3)\delta} - q^{-2}\tilde{E}_{(m-3)\delta}E_{2\delta-\alpha_1}]+$$

$$-(q^2-1)[3]_qE_{\delta-\alpha_1}\tilde{E}_{(m-1)\delta} - (1-q^{-2})[3]_q\tilde{E}_{(m-1)\delta}E_{\delta-\alpha_1}\Big) =$$

$$\{\text{corollary 3.2.4 and } v)_{m-1}\}$$

$$= -\frac{1}{[4]_q}((q^4-q^2-q^{-2})E_{2\delta-\alpha_1}\tilde{E}_{(m-2)\delta} + (q^2+q^{-2}-q^{-4})\tilde{E}_{(m-2)\delta}E_{2\delta-\alpha_1}+$$

$$+q^2E_{3\delta-\alpha_1}\tilde{E}_{(m-3)\delta} - q^{-2}\tilde{E}_{(m-3)\delta}E_{3\delta-\alpha_1}+$$

$$-(q^2-1)(q^2+q^{-2})E_{\delta-\alpha_1}\tilde{E}_{(m-1)\delta} - (1-q^{-2})(q^2+q^{-2})\tilde{E}_{(m-1)\delta}E_{\delta-\alpha_1}) =$$

$$\{\text{reordering the summands}\}$$

$$= \frac{1}{[4]_q}((q^4-q^2-q^{-2})E_{\delta-\alpha_1}\tilde{E}_{(m-1)\delta} + (q^2+q^{-2}-q^{-4})\tilde{E}_{(m-1)\delta}E_{\delta-\alpha_1}+$$

$$+q^2E_{2\delta-\alpha_1}\tilde{E}_{(m-2)\delta} - q^{-2}\tilde{E}_{(m-2)\delta}E_{2\delta-\alpha_1})+$$

$$-\frac{1}{[4]_q}([\tilde{E}_{(m-1)\delta}, E_{\delta-\alpha_1}] + (q^4-q^{-2})E_{2\delta-\alpha_1}\tilde{E}_{(m-2)\delta} + (q^2-q^{-4})\tilde{E}_{(m-2)\delta}E_{2\delta-\alpha_1}+$$

$$+q^2E_{3\delta-\alpha_1}\tilde{E}_{(m-3)\delta} - q^{-2}\tilde{E}_{(m-3)\delta}E_{3\delta-\alpha_1}) =$$

$$\{iv)_{m-1}\}$$

$$= \frac{1}{[4]_q}((q^4-q^2-q^{-2})E_{\delta-\alpha_1}\tilde{E}_{(m-1)\delta} + (q^2+q^{-2}-q^{-4})\tilde{E}_{(m-1)\delta}E_{\delta-\alpha_1}+$$

$$+q^2E_{2\delta-\alpha_1}\tilde{E}_{(m-2)\delta} - q^{-2}\tilde{E}_{(m-2)\delta}E_{2\delta-\alpha_1}),$$

which is the claim.

$iii)_m$ is the result of the following computations, where the first identity is true thanks to $i)_m$:

$$[\tilde{E}_{m\delta}, E_1] = [-E_{(m-1)\delta-\alpha_1}E_{\delta+\alpha_1} + q^{-2}E_{\delta+\alpha_1}E_{(m-1)\delta-\alpha_1}, E_1] =$$

$$= -E_{(m-1)\delta-\alpha_1}E_{\delta+\alpha_1}E_1 + q^{-2}E_{\delta+\alpha_1}E_{(m-1)\delta-\alpha_1}E_1+$$

$$+E_1E_{(m-1)\delta-\alpha_1}E_{\delta+\alpha_1}-q^{-2}E_1E_{\delta+\alpha_1}E_{(m-1)\delta-\alpha_1}=$$

$$\{\text{lemma 3.1.2 and definition of } \tilde{E}_{(m-1)\delta}\}$$

$$=-q^2E_{(m-1)\delta-\alpha_1}E_1E_{\delta+\alpha_1}+[4]_qE_{(m-1)\delta-\alpha_1}E_{\delta+2\alpha_1}+q^{-4}E_{\delta+\alpha_1}E_1E_{(m-1)\delta-\alpha_1}+$$

$$-q^{-2}E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+E_1E_{(m-1)\delta-\alpha_1}E_{\delta+\alpha_1}-q^{-2}E_1E_{\delta+\alpha_1}E_{(m-1)\delta-\alpha_1}=$$

$$\{\text{definition of } \tilde{E}_{(m-1)\delta} \text{ and lemma 3.1.2}\}$$

$$=q^2\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1}-q^{-2}E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+$$

$$+[4]_q(E_{(m-1)\delta-\alpha_1}E_{\delta+2\alpha_1}-q^{-4}E_{\delta+2\alpha_1}E_{(m-1)\delta-\alpha_1})=$$

$$\{\Xi T_1(ii)_m) \text{ and } i)_{m-1,m-2}\}$$

$$=q^2\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1}-q^{-2}E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+(q^4-q^2-q^{-2})\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1}+$$

$$+(q^2+q^{-2}-q^{-4})E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+q^2\tilde{E}_{(m-2)\delta}E_{2\delta+\alpha_1}-q^{-2}E_{2\delta+\alpha_1}\tilde{E}_{(m-2)\delta}=$$

$$=(q^4-q^{-2})\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1}+(q^2-q^{-4})E_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+$$

$$+q^2\tilde{E}_{(m-2)\delta}E_{2\delta+\alpha_1}-q^{-2}E_{2\delta+\alpha_1}\tilde{E}_{(m-2)\delta},$$

which is $iii)_m$.

$iv)_m$ is found by applying $T_0\Xi$ to $iii)_m$.

$v)_m$: we have to prove that $[\tilde{E}_{m\delta},\tilde{E}_{r\delta}]=0\ \forall r<m$. Since $[\tilde{E}_{m\delta},\tilde{E}_{r\delta}]\in\mathcal{U}^+_{q,(m+r)\delta}$ it is enough to prove that it commutes with F_0 and F_1 (see [27]): in particular, proving that it belongs to the center of $\mathcal{U}_q$ will solve the problem; but the set of the elements of $\mathcal{U}_q$ commuting with $[\tilde{E}_{m\delta},\tilde{E}_{r\delta}]$ is a subalgebra of $\mathcal{U}_q$ obviously containing $\mathcal{U}_q^0$ and stable by the action of T_0T_1 and $T_0\Xi$ (because $[\tilde{E}_{m\delta},\tilde{E}_{r\delta}]$ is fixed by T_0T_1 and $T_0\Xi$). Hence the problem reduces (see corollary 3.1.4) to show that $[[\tilde{E}_{m\delta},\tilde{E}_{r\delta}],E_1]=0$, that is $[\tilde{E}_{m\delta},[\tilde{E}_{r\delta},E_1]]=[\tilde{E}_{r\delta},[\tilde{E}_{m\delta},E_1]]$: this will be done by induction on r:

$r=1$: proposition 3.2.2, $i)_m$, $(T_0T_1)^s(iii)_m)$ and $v)_{m-1}$ imply that

$$[\tilde{E}_{m\delta},[\tilde{E}_\delta,E_1]]=-[3]_q![\tilde{E}_{m\delta},E_{\delta+\alpha_1}]=$$

$$=-[3]_q!\big((q^2-1)[3]_q\tilde{E}_{(m-1)\delta}E_{2\delta+\alpha_1}+(1-q^{-2})[3]_qE_{2\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+$$

$$+q^2\tilde{E}_{(m-2)\delta}E_{3\delta+\alpha_1}-q^{-2}E_{3\delta+\alpha_1}\tilde{E}_{(m-2)\delta}\big)=$$

$$=[\tilde{E}_\delta,(q^2-1)[3]_q\tilde{E}_{(m-1)\delta}E_{\delta+\alpha_1}+(1-q^{-2})[3]_qE_{\delta+\alpha_1}\tilde{E}_{(m-1)\delta}+$$

$$+q^2\tilde{E}_{(m-2)\delta}E_{2\delta+\alpha_1}-q^{-2}E_{2\delta+\alpha_1}\tilde{E}_{(m-2)\delta}]=$$

$$=[\tilde{E}_\delta,[\tilde{E}_{m\delta},E_1]],$$

so that $[\tilde{E}_{m\delta},\tilde{E}_\delta]=0$.

$r > 1$: notice that if we put $x_1 \doteq q^4 - q^{-2}$, $y_1 \doteq q^2 - q^{-4}$, $x_2 \doteq q^2$, $y_2 \doteq -q^{-2}$, we have that $\forall u \leq m$, $\forall v \in \mathbb{N}$ (applying $(T_0T_1)^{-v}$ to $iii)_u$)

$$[\tilde{E}_{u\delta}, E_{v\delta+\alpha_1}] = \sum_{s=1}^{2}(x_s\tilde{E}_{(u-s)\delta}E_{(v+s)\delta+\alpha_1} + y_sE_{(v+s)\delta+\alpha_1}\tilde{E}_{(u-s)\delta});$$

hence, using the inductive hypothesis and $v)_{m-1}$, we get that

$$[\tilde{E}_{m\delta}, [\tilde{E}_{r\delta}, E_1]] = \left[\tilde{E}_{m\delta}, \sum_{s=1}^{2}(x_s\tilde{E}_{(r-s)\delta}E_{s\delta+\alpha_1} + y_sE_{s\delta+\alpha_1}\tilde{E}_{(r-s)\delta})\right] =$$

$$= \sum_{s,t=1}^{2}(x_sx_t\tilde{E}_{(r-s)\delta}\tilde{E}_{(m-t)\delta}E_{(s+t)\delta+\alpha_1} + x_sy_t\tilde{E}_{(r-s)\delta}E_{(s+t)\delta+\alpha_1}\tilde{E}_{(m-t)\delta}+$$

$$+y_sx_t\tilde{E}_{(m-t)\delta}E_{(s+t)\delta+\alpha_1}\tilde{E}_{(r-s)\delta} + y_sy_tE_{(s+t)\delta+\alpha_1}\tilde{E}_{(m-t)\delta}\tilde{E}_{(r-s)\delta}) =$$

$$= \left[\tilde{E}_{r\delta}, \sum_{t=1}^{2}x_t\tilde{E}_{(m-t)\delta}E_{t\delta+\alpha_1} + y_tE_{t\delta+\alpha_1}\tilde{E}_{(m-t)\delta}\right] = [\tilde{E}_{r\delta}, [\tilde{E}_{m\delta}, E_1]],$$

which is the claim. □

§3.4. Commutation between positive and negative real root vectors.

The preceding proposition is first of all useful in proving other commutation relations, now involving the real root vectors.

Lemma 3.4.1
$\forall r, s \in \mathbb{N}$ we have that:

$$a)_{r,s}\ [E_{r\delta+\alpha_1}, F_{s\delta+\alpha_1}] = \begin{cases} \dfrac{K_{r\delta+\alpha_1} - K_{r\delta+\alpha_1}^{-1}}{q-q^{-1}} & \text{if } r = s \\ -K_{s\delta+\alpha_1}\tilde{E}_{(r-s)\delta} & \text{if } r > s \\ -\tilde{F}_{(s-r)\delta}K_{r\delta+\alpha_1}^{-1} & \text{if } r < s; \end{cases}$$

$$b)_{r,s}\ [E_{(r+1)\delta-\alpha_1}, F_{(s+1)\delta-\alpha_1}] = \begin{cases} \dfrac{K_{(r+1)\delta-\alpha_1} - K_{(r+1)\delta-\alpha_1}^{-1}}{q-q^{-1}} & \text{if } r = s \\ K_{(s+1)\delta-\alpha_1}^{-1}\tilde{E}_{(r-s)\delta} & \text{if } r > s \\ \tilde{F}_{(s-r)\delta}K_{(r+1)\delta-\alpha_1} & \text{if } r < s. \end{cases}$$

Proof: $a)_{r,r}$ is proved by applying $(T_0T_1)^{-r}$ to the identity $[E_1, F_1] = \dfrac{K_1 - K_1^{-1}}{q-q^{-1}}$; if $r > s$ $a)_{r,s}$ follows from the identity $-E_{(r-s)\delta-\alpha_1}E_1 + q^{-2}E_1E_{(r-s)\delta-\alpha_1} = \tilde{E}_{(r-s)\delta}$ again applying to it $(T_0T_1)^{-r}$; if $r > s$ $a)_{r,s}$ is nothing but Ω applied to $a)_{s,r}$.

As for assertion $b)$, we have that $b)_{r,s} = \Xi T_1(a)_{r+1,s+1})$. □

§3.5. The imaginary root vectors $E_{m\delta}$.

Recall that the subalgebra of $\mathcal{U}_q$ generated by the $\tilde{E}_{m\delta}$'s is a commutative subalgebra pointwise fixed by T_0T_1 and $T_0\Xi$ (see proposition 3.3.2). Hence it makes sense to introduce new imaginary root vectors by the following equation:

Definition 3.5.1.

Let us define some new elements in the subalgebra of $\mathcal{U}_q$ generated by the $\tilde{E}_{m\delta}$'s by the following identity:

$$1-(q-q^{-1})\sum_{m>0}\tilde{E}_{m\delta}u^m=\exp\left((q-q^{-1})\sum_{m>0}E_{m\delta}u^m\right).$$

Remark that of course $\forall m>0$ $E_{m\delta}\in\mathcal{U}^+_{q,m\delta}$ and $E_{m\delta}+\tilde{E}_{m\delta}$ belongs to the subalgebra of $\mathcal{U}_q$ generated by $\{\tilde{E}_{r\delta}|r<m\}$.

Proposition 3.5.2
$\forall m>0$ we have the following commutation relations:

$$[E_{m\delta},E_1]=\frac{[2m]_q}{m}(q^{2m}+(-1)^{m-1}+q^{-2m})E_{m\delta+\alpha_1}.$$

Proof: The idea of this proof is completely similar to the proof of the analogous result for $A_1^{(1)}$ (see [1]).

Let $\tilde{S}^{\pm}_m,S^{\pm}_m:\mathcal{U}_q\to\mathcal{U}_q$ be the operators of left ($^+$) and right ($^-$) multiplication respectively by $\tilde{E}_{m\delta}$ and $E_{m\delta}$ (where we keep on indicating by $\tilde{E}_{0\delta}$ and $\tilde{E}_{-\delta}$ respectively $-\frac{1}{q-q^{-1}}$ and 0) and let us set

$$\tilde{S}^{\pm}(u)\doteq\sum_{m\geq 0}\tilde{S}^{\pm}_m u^m,\quad S^{\pm}(u)\doteq\sum_{m>0}S^{\pm}_m u^m;$$

moreover let us indicate by T the automorphism $(T_0T_1)^{-1}$: remark that $\forall m$

$$\tilde{S}^{\pm}_m T=T\tilde{S}^{\pm}_m \text{ and } S^{\pm}_m T=TS^{\pm}_m.$$

Now observe that point *iii)* of proposition 3.3.2 can be written by saying that $\forall m$

$$(\tilde{S}^+_m-\tilde{S}^-_m)(E_1)=$$
$$=\big((q^4-q^{-2})\tilde{S}^+_{m-1}T+(q^2-q^{-4})\tilde{S}^-_{m-1}T+q^2\tilde{S}^+_{m-2}T^2-q^{-2}\tilde{S}^-_{m-2}T^2\big)(E_1)$$

or, also, (multiplying by u^m and summing over m)

$$\tilde{S}^+(u)(1+q^{-2}Tu)(1-q^4Tu)(E_1)=$$
$$=\tilde{S}^-(u)(1+q^2Tu)(1-q^{-4}Tu)(E_1);$$

multiplying both sides of this identity by $-(q-q^{-1})$ and considering that

$$\log(-(q-q^{-1})\tilde{S}^{\pm}(u))=(q-q^{-1})S^{\pm}(u)$$

we get that

$$(q-q^{-1})(S^+(u)-S^-(u))(E_1)=$$
$$=(\log(1+q^2Tu)+\log(1-q^{-4}Tu)-\log(1+q^{-2}Tu)-\log(1-q^4Tu))(E_1),$$

that is

$$[E_{m\delta}, E_1] = \frac{1}{m(q-q^{-1})}((-1)^{m-1}q^{2m} - q^{-4m} + (-1)^m q^{-2m} + q^{4m})E_{m\delta+\alpha_1} =$$

$$= \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{m\delta+\alpha_1}.$$

□

The last proposition generalizes immediately to the following more general relations:

Corollary 3.5.3
Let m be bigger than 0; then $\forall r \in \mathbb{N}$ we have:
$a)_r$ $[E_{m\delta}, E_{r\delta+\alpha_1}] = \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m+r)\delta+\alpha_1}$;
$b)_r$ $[E_{m\delta}, E_{(r+1)\delta-\alpha_1}] = -\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m+r+1)\delta-\alpha_1}$;
$c)_r$ $[E_{m\delta}, F_{r\delta+\alpha_1}] =$

$$= \begin{cases} \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K_{r\delta+\alpha_1}E_{(m-r)\delta-\alpha_1} & \text{if } r < m \\ -\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K_{m\delta}F_{(r-m)\delta+\alpha_1} & \text{if } r \geq m; \end{cases}$$

$d)_r$ $[E_{m\delta}, F_{(r+1)\delta-\alpha_1}] =$

$$= \begin{cases} -\frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})E_{(m-r-1)\delta+\alpha_1}K^{-1}_{(r+1)\delta-\alpha_1} & \text{if } r < m \\ \frac{[2m]_q}{m}(q^{2m} + (-1)^{m-1} + q^{-2m})K^{-1}_{m\delta}F_{(r-m+1)\delta-\alpha_1} & \text{if } r \geq m. \end{cases}$$

Proof: $a)_r$ is $(T_0T_1)^{-r}$ applied to proposition 3.5.2; $b)_r$ is ΞT_1 applied to $a)_{r+1}$; $c)_r$ is $(T_0T_1)^{-r}\Xi T_1$ applied to $a)_0$; finally $d)_r$ is ΞT_1 applied to $c)_{r+1}$. □

Corollary 3.5.4
The imaginary root vectors form a "Heisenberg" algebra, with commutation relations given by:

$$[E_{r\delta}, F_{s\delta}] = \delta_{r,s}\frac{[2r]_q}{r}(q^{2r} + (-1)^{r-1} + q^{-2r})\frac{K_{r\delta} - K^{-1}_{r\delta}}{q-q^{-1}}.$$

Proof: The claim is a straightforward consequence of proposition 3.3.2 and corollary 3.5.3; see also [4]. □

4. $A^{(2)}_{2n}$ AND A COPY OF $A^{(2)}_2$.

§4.1. Definition of φ_1.

The aim of this section is to prove that the particular situation of $\mathfrak{sl}^{(2)}_3 = A^{(2)}_2$ studied in the preceding section plays, in some cases, the same role as $\mathfrak{sl}^{(1)}_2$ does in general. More precisely we know that if i is a vertex of the finite Dynkin diagram associated to an affine algebra $\hat{\mathfrak{g}}^{(k)}$, then, provided that $\hat{\mathfrak{g}}^{(k)}$ is not of type $A^{(2)}_{2n}$ or that i is not 1, there exists a homomorphism of $\mathbb{C}$-algebras $\varphi_i : \mathcal{U}_q(\mathfrak{sl}^{(1)}_2) \to \mathcal{U}_q(\hat{\mathfrak{g}}^{(k)})$

such that $\varphi_i(E_1) = E_i$, $\varphi_i(E_0) = E_{\tilde{d}_i\delta-\alpha_i}$, $\varphi_i(q) = q_i$ and $\Omega\varphi_i = \varphi_i\Omega$; moreover φ_i is such that $T_i\varphi_i = \varphi_i T_1$ and $T_{\lambda_i}\varphi_i = \varphi_i T_{\check{\omega_1}} (= \varphi_i T_0 T_\tau)$ (that is T_1 corresponds to T_i, $T_\tau = T_0 T_\tau T_1^{-1}$ to $T_{\lambda_i} T_i^{-1}$ and $T_0 = T_0 T_\tau^2$ to $T_{\lambda_i}^2 T_i^{-1}$) (see proposition 2.2.2).

Here we want to study the remaining cases, namely the situation where $\hat{\mathfrak{g}}^{(k)}$ is of type $A_{2n}^{(2)}$ and $i = 1$: in this case we construct a homomorphism of $\mathbb{C}$-algebras $\varphi_1 : \mathcal{U}_q(\mathfrak{sl}_3^{(2)}) \to \mathcal{U}_q^{(1)} \subseteq \mathcal{U}_q(\hat{\mathfrak{g}}^{(k)})$ such that $\varphi_1(q) = q_1 (= q)$ (φ_1 is indeed a homomorphism of $\mathbb{C}(q)$-algebras), $\varphi_1(E_1) = E_1$ and $\Omega\varphi_1 = \varphi_1\Omega$; moreover φ_1 is such that $T_1\varphi_1 = \varphi_1 T_1$ and $T_{\omega_1}\varphi_1 = \varphi_1 T_{\omega_1} (= \varphi_1 T_0 T_1)$ (that is T_1 corresponds to T_1 and T_0 to $T_{\omega_1} T_1^{-1}$). Of course now the image of E_0 can't be $E_{\delta-\alpha_1}$ (since $\delta - \alpha_1$ has the same length as α_1 while α_0 is longer): in this case we will have that $\varphi_1(E_0) = E_{\delta-2\alpha_1}$.

Definition 4.1.1.
Let $\varphi_1 : \mathcal{U}_q(A_2^{(2)}) \to \mathcal{U}_q^{(1)} \subseteq \mathcal{U}_q(A_{2n}^{(2)})$ be the homomorphism of $\mathbb{C}(q)$-algebras defined on the generators as follows:

$$\varphi_1(E_1) \doteqdot E_1, \quad \varphi_1(F_1) \doteqdot F_1, \quad \varphi_1(K_1^{\pm 1}) \doteqdot K_1^{\pm 1},$$

$$\varphi_1(E_0) \doteqdot E_{\delta-2\alpha_1}, \quad \varphi_1(F_0) \doteqdot F_{\delta-2\alpha_1}, \quad \varphi_1(K_0^{\pm 1}) \doteqdot K_{\delta-2\alpha_1}^{\pm 1}.$$

Our program is to prove first of all that φ_1 is well defined (of course if this is the case then it commutes with Ω), point to which we devote paragraph 4.3, after §4.2, whose aim is recalling and describing some (mainly combinatorial) properties of the root system and Weyl group of type $A_{2n}^{(2)}$; then, in §4.4, we shall study the relations between the braid group action and φ_1. These are the main tools which will allow us to construct a PBW-basis.

§4.2. Root system and Weyl group: some properties.

In this paragraph we shall study the properties of the root system and the Weyl group of $A_{2n}^{(2)}$ which will enable us to attack the central problem of this section, that is the research of a copy of $\mathcal{U}_q(A_2^{(2)})$ in $\mathcal{U}_q(A_{2n}^{(2)})$. What we need in particular is a more explicit description of the root vectors (in particular of some root vectors) and their behaviour under the action of the braid group, goal which can be achieved via a closer analysis of the Weyl group. More precisely our attention will concentrate on the study of a reduced expression of ω_1 (remark that $\omega_1 = \check{\omega_1}$, since $d_1 = 1$) and of some of its properties, what will help us in manipulating the root vectors.

Remark 4.2.1.
$l(\omega_1) = n(n+1)$.

Proof: It is enough to compute the cardinality of $\Phi_+(\omega_1)$:

$$\Phi_+(\omega_1) = \{\alpha \in \Phi_+ | \omega_1^{-1}(\alpha) < 0\} = \left\{\delta - \varepsilon \sum_{r=1}^{k} \alpha_r \big| 1 \le k \le n, \varepsilon = 1, 2\right\} \cup$$

$$\cup \left\{\varepsilon\delta - 2\sum_{r=1}^{k} \alpha_r - \sum_{r=k+1}^{l} \alpha_r \big| 1 \le k < l \le n, \varepsilon = 1, 2\right\},$$

so that

$$\#\Phi_+(\omega_1) = 2\left(n + \frac{n(n-1)}{2}\right) = n(n+1),$$

which is the claim. □

The element of the Weyl group which we are now going to introduce will play an important role in what follows.

Definition 4.2.2.
In the following w will denote the element of W given by:

$$w \doteq s_0 s_n s_{n-1} \cdot ... \cdot s_1.$$

Lemma 4.2.3
The action of w on Q can be described as follows:

$$w(\alpha_i) = \begin{cases} \delta - 2\sum_{r=1}^{n-1} \alpha_r & \text{if } i = 0 \\ -\delta + \alpha_1 + ... + \alpha_n & \text{if } i = 1 \\ \delta - (\alpha_2 + ... + \alpha_n) & \text{if } i = 2 \\ \alpha_{i-1} & \text{if } 3 \leq i \leq n. \end{cases}$$

Furthermore $\forall k \in \mathbb{Z}$

$$w^k(\alpha_i) = \begin{cases} \left(1 + 2\left[\frac{k}{n}\right]\right)\delta - 2\sum_{r=1}^{\overline{n-k}} \alpha_r & \text{if } i = 0 \\ \left[-\frac{k}{n}\right]\delta + \sum_{r=1}^{n+1-\overline{k}} \alpha_r & \text{if } i = 1 \\ \alpha_{\overline{i-k}} & \text{if } 2 \leq i \leq n \text{ and } \overline{i-k} \neq 1 \\ \delta - (\alpha_2 + ... + \alpha_n) & \text{if } 2 \leq i \leq n \text{ and } \overline{i-k} = 1, \end{cases}$$

where $\bar{\ } : \mathbb{Z} \to \{1, ..., n\}$ is the application such that $\overline{k} \equiv k \pmod n$ $\forall k \in \mathbb{Z}$.

Proof: It is clear that if $3 \leq i \leq n$ then

$$w(\alpha_i) = s_0 s_n \cdot ... \cdot s_1(\alpha_i) = s_0 s_n \cdot ... \cdot s_{i-1}(\alpha_i) = s_0 s_n \cdot ... \cdot s_{i+1}(\alpha_{i-1}) = \alpha_{i-1};$$

also, it is straightforward to see that $w(\alpha_1) = -\delta + (a_1 + ... + \alpha_n)$, $w(\alpha_2) = = \delta - (\alpha_2 + ... + \alpha_n)$ and $w(\alpha_0) = \alpha_0 + 2\alpha_n = \delta - 2\sum_{r=1}^{n-1} \alpha_r$, which proves the first assertion.

Remark that $w^2(\alpha_2) = \alpha_n$, so that the second assertion immediately follows for $2 \leq i \leq n$.

The fact that $w(\alpha_1 + \alpha_2) = \alpha_1$ implies that $w(\alpha_1 + ... + \alpha_r) = \alpha_1 + ... + \alpha_{r-1}$ $\forall r > 1$, which easily leads to the remaining assertions for $i = 0, 1$. □

Corollary 4.2.4
$\omega_1 = w^n$ and $(s_0 s_n \cdot ... \cdot s_1)^n$ is a reduced expression of ω_1; in particular we have that $T_{\omega_1} = T_w^n = (T_0 T_n \cdot ... \cdot T_1)^n$.

Proof: That $\omega_1 = w^n$ follows from the fact that $w^n(\alpha_i) = \alpha_i$ when $i \in I_0 \setminus \{1\}$ and $w^n(\alpha_1) = -\delta + \alpha_1$ (see lemma 4.2.3). The remaining assertion is an immediate consequence of lemma 4.2.1. □

Notation 4.2.5.
We fix ι so that $\iota_r + r \equiv 1 \pmod{n+1}$ whenever $1 \leq r \leq n(n+1)$ (this can be done thanks to corollary 4.2.4; see also remark 2.1.5).

Corollary 4.2.6
$w^{n-1}(\alpha_0) = \delta - 2\alpha_1$; in particular $E_{\delta-2\alpha_1} = T_{w^{n-1}}(E_0)$.

Proof: That $w^{n-1}(\alpha_0) = \delta - 2\alpha_1$ is a direct consequence of lemma 4.2.3. Since $T_{w^{n-1}}(E_0)$ is a root vector, it must then be $E_{\delta-2\alpha_1}$. $\square$

The lemma that we are now going to propose is devoted to the study of the reduced expressions of some suitable elements of the Weyl group. Here lies perhaps the main difference between this case and the general case that has been recalled in section 2: indeed in that case $l(\omega_i s_i \omega_i) = 2l(\omega_i) - 1$, and this implied that $(T_{\omega_i} T_i^{-1})^2$ acted as the identity on the subalgebra of $\mathcal{U}_q$ generated by $\{E_i, E_{\delta-\alpha_i}\}$ (see lemma 2.2.1 and remark 2.1.5); this is no more true here. However we can still say something.

Lemma 4.2.7
$l(s_0ws_1\omega_1) = l(s_0ws_1) + l(\omega_1)$, so that $T_{s_0ws_1w^k} = T_{s_0ws_1}T_w^k$ $\forall k \in \{0, ..., n\}$;
$l(s_0ws_1\omega_1) = l(ws_1) + l(s_1w^{-1}s_0ws_1\omega_1)$, that is $T_{s_0ws_1\omega_1} = T_{ws_1}T_{s_1w^{-1}s_0ws_1\omega_1}$;
moreover $s_0ws_1w^{n-1}(\alpha_0) = \alpha_0$ and $s_1w^{-1}s_0ws_1\omega_1s_1(\alpha_1) = \alpha_1$.

Proof: The first statement means that $\Phi_+(s_0ws_1) \subseteq \Phi_+(s_0ws_1\omega_1)$.
Now $(s_0ws_1)^{-1}(\Phi_+(s_0ws_1)) \subseteq \sum_{i=2}^n \mathbb{Z}\alpha_i$ which is pointwise fixed by ω_1, so that

$$\alpha \in \Phi_+(s_0ws_1) \Rightarrow (s_0ws_1\omega_1)^{-1}(\alpha) = (s_0ws_1)^{-1}(\alpha) < 0 \Rightarrow \alpha \in \Phi_+(s_0ws_1\omega_1).$$

The second part of the lemma is equivalent to saying that $\Phi_+(ws_1) \subseteq \Phi_+(s_0ws_1\omega_1)$. Of course

$$\Phi_+(ws_1) = \{s_0s_n \cdot ... \cdot s_{r+1}(\alpha_r) | 2 \leq r \leq n+1\} = \{\alpha_0 + \alpha_n + ... + \alpha_r | 2 \leq r \leq n+1\}$$

so that, if $2 \leq r \leq n+1$, $(s_0ws_1\omega_1)^{-1}(\alpha_0 + \alpha_n + ... + \alpha_r) =$

$$= \begin{cases} \omega_1^{-1}(\alpha_0 + 2\alpha_n + ... + 2\alpha_{r+1} + \alpha_r + ... + \alpha_2) & \text{if } r \leq n \\ \omega_1^{-1}(\alpha_0 + 2\alpha_n + ... + 2\alpha_2) & \text{if } r = n+1 \end{cases}$$

$$= \begin{cases} -\delta - (\alpha_r + ... + \alpha_2 + 2\alpha_1) < 0 & \text{if } r \leq n \\ -\delta - 2\alpha_1 < 0 & \text{if } r = n+1. \end{cases}$$

Now we have that

$$s_0ws_1w^{n-1}(\alpha_0) = s_0ws_1(\delta - 2\alpha_1) = s_0w(\delta + 2\alpha_1) = s_0(2\delta - (w(\delta - 2\alpha_1))) =$$

$$= s_0(2\delta - \omega_1(\alpha_0)) = s_0\omega_1(\delta + 2(\alpha_1 + ... + \alpha_n)) = s_0(-\alpha_0) = \alpha_0.$$

Finally we get that

$$s_1w^{-1}s_0ws_1\omega_1s_1(\alpha_1) = s_1w^{-1}s_0w(\delta + \alpha_1) =$$

$$= s_1w^{-1}(\alpha_0 + \alpha_1 + ... + \alpha_n) = s_1(-\alpha_1) = \alpha_1,$$

which concludes the proof. $\square$

Corollary 4.2.8
$E_{\delta-2\alpha_1} = T_{s_0ws_1}^{-1}(E_0)$ and $E_{\delta-\alpha_1} = T_{s_0ws_1}^{-1}T_{ws_1}(E_1)$.

Proof: Straightforward consequence of lemma 4.2.7. $\square$

§4.3. φ_1 is well defined.

Here we want to prove that φ_1 is well defined, which means that the relations between the "canonical" generators of $\mathcal{U}_q(A_2^{(2)})$ (that is E_i, F_i, K_i^{-1} for $i = 0, 1$) are preserved by φ_1 (so that $\varphi_1 : \mathcal{U}_q(A_2^{(2)}) \to \mathcal{U}_q(A_{2n}^{(2)})$ is well defined), and that $E_{\delta-2\alpha_1} \in \mathcal{U}_q^{(1)}$ (so that φ_1 effectively maps $\mathcal{U}_q(A_2^{(2)})$ in $\mathcal{U}_q^{(1)}$).

For some of these relations this result is immediate as indicated in proposition 4.3.2. The others will require some more work.

Let us recall which are the relations that we are speaking about:

Definition 4.3.1.
$\mathcal{U}_q(A_2^{(2)})$ is the $\mathbb{C}(q)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1} | i = 0, 1\}$ with the following relations:

$$[KK] \qquad K_1 K_0 = K_0 K_1;$$

$$[KE] \quad K_1 E_1 = q^2 E_1 K_1, \; K_1 E_0 = q^{-4} E_0 K_1, \; K_0 E_1 = q^{-4} E_1 K_0, \; K_0 E_0 = q^8 E_0 K_0;$$

$$[KF] \quad K_1 F_1 = q^{-2} F_1 K_1, \; K_1 F_0 = q^4 F_0 K_1, \; K_0 F_1 = q^4 F_1 K_0, \; K_0 F_0 = q^{-8} F_0 K_0;$$

$$[EF] \quad [E_1, F_1] = \frac{K_1 - K_1^{-1}}{q - q^{-1}}, \quad [E_0, F_0] = \frac{K_0 - K_0^{-1}}{q^4 - q^{-4}}, \quad [E_1, F_0] = 0 = [E_0, F_1];$$

$$[EE] \qquad \sum_{r=0}^{2} (-1)^r E_0^{(r)} E_1 E_0^{(2-r)} = 0 = \sum_{r=0}^{5} (-1)^r E_1^{(r)} E_0 E_1^{(5-r)};$$

$$[FF] \qquad \sum_{r=0}^{2} (-1)^r F_0^{(r)} F_1 F_0^{(2-r)} = 0 = \sum_{r=0}^{5} (-1)^r F_1^{(r)} F_0 F_1^{(5-r)}.$$

Proposition 4.3.2
The relations [KK], [KE], [KF] and [EF] are obviously preserved by φ_1.

Proof: The subalgebra of $\mathcal{U}_q(A_{2n}^{(2)})$ generated by $\{E_i | i \neq 1\}$ (to which $E_{\delta-2\alpha_1}$ belongs) commutes with F_1, so that $[E_{\delta-2\alpha_1}, F_1] = 0$ (and of course $[E_1, F_{\delta-2\alpha_1}] = = \Omega([E_{\delta-2\alpha_1}, F_1]) = 0$). The other relations are trivial. □

The point is now to prove the remaining relations.

Remark 4.3.3.
Once we have proved the compatibility of relations [EE] with the definition of φ_1, it is enough to apply Ω in order to find the same result for [FF]: indeed both the

left and the right hand sides of [FF] are found by applying Ω to the corresponding expressions of [EE].

Lemma 4.3.4
In $\mathcal{U}_q(A_{2n}^{(2)})$ we have that $E_{\delta-\alpha_1}E_{\delta-2\alpha_1} = q^{-4}E_{\delta-2\alpha_1}E_{\delta-\alpha_1}$.

Proof: Since $E_{\delta-2\alpha_1} = T_{w^{n-1}}(E_0)$ and $E_{\delta-\alpha_1} = T_{\omega_1}T_1^{-1}(E_1) = T_{w^n}T_1^{-1}(E_1)$ we can apply Levendorskii-Soibelman formula (see [23]): the claim follows from the fact that $(\delta-2\alpha_1|\delta-\alpha_1) = 4$, and that the only possibility for a sum of positive roots of the form $m\delta - \alpha$ (with α a positive multiple of a root of the finite system associated to $A_{2n}^{(2)}$) to be equal to $2\delta - 3\alpha_1$ is that these positive roots are $\delta - 2\alpha_1$ and $\delta - \alpha_1$. □

Lemma 4.3.5
$-E_0T_nT_{n-1}\cdot ... \cdot T_3(E_2) + q^{-4}T_nT_{n-1}\cdot ... \cdot T_3(E_2)E_0 = T_0T_nT_{n-1}\cdot ... \cdot T_3(E_2)$.

Proof: Since $T_nT_{n-1}\cdot ... \cdot T_3(E_2) \in \mathcal{U}^+_{q,\alpha_n+...+\alpha_2}$ we can write it as $\sum xE_ny$ where $x, y \in (E_2, ..., E_{n-1})$; in particular x and y commute with E_0 and are fixed points for T_0. Then we have

$$-E_0T_nT_{n-1}\cdot ... \cdot T_3(E_2) + q^{-4}T_nT_{n-1}\cdot ... \cdot T_3(E_2)E_0 = \sum x(-E_0E_n + q^{-4}E_nE_0)y =$$

$$= \sum xT_0(E_n)y = T_0\left(\sum xE_ny\right) = T_0T_nT_{n-1}\cdot ... \cdot T_3(E_2).$$

□

Corollary 4.3.6
In $\mathcal{U}_q(A_{2n}^{(2)})$ we have that $-E_{\delta-2\alpha_1}E_1 + q^{-4}E_1E_{\delta-2\alpha_1} = E_{\delta-\alpha_1}$.

Proof: The proof consists of a computation in which lemma 4.3.5 and corollary 4.2.8 play a fundamental role: if $n \geq 2$

$$-E_{\delta-2\alpha_1}E_1 + q^{-4}E_1E_{\delta-2\alpha_1} = -T^{-1}_{s_0ws_1}(E_0)E_1 + q^{-4}E_1T^{-1}_{s_0ws_1}(E_0) =$$

$$= T^{-1}_{s_0ws_1}(-E_0T_{s_0ws_1}(E_1) + q^{-4}T_{s_0ws_1}(E_1)E_0) =$$

$$= T^{-1}_{s_0ws_1}(-E_0T_{s_0ws_1s_2}(-E_2E_1 + q^{-2}E_1E_2) + q^{-4}T_{s_0ws_1s_2}(-E_2E_1 + q^{-2}E_1E_2)E_0) =$$

$$= -T^{-1}_{s_0ws_1}((-E_0T_n\cdot ... \cdot T_3(E_2) + q^{-4}T_n\cdot ... \cdot T_3(E_2)E_0)E_1 +$$

$$-q^{-2}E_1(-E_0T_n\cdot ... \cdot T_3(E_2) + q^{-4}T_n\cdot ... \cdot T_3(E_2)E_0)) =$$

$$= -T^{-1}_{s_0ws_1}(T_0T_nT_{n-1}\cdot ... \cdot T_3(E_2)E_1 - q^{-2}E_1T_0T_nT_{n-1}\cdot ... \cdot T_3(E_2)) =$$

$$= T^{-1}_{s_0ws_1}T_0T_nT_{n-1}\cdot ... \cdot T_3(-E_2E_1 + q^{-2}E_1E_2) = T^{-1}_{s_0ws_1}T_{ws_1}(E_1) = E_{\delta-\alpha_1},$$

where we have used that $[E_1, E_0] = 0$. □

Proposition 4.3.7
The following relation holds in $\mathcal{U}_q(A_{2n}^{(2)})$:

$$\sum_{r=0}^{2}(-1)^r\begin{bmatrix}2\\r\end{bmatrix}_{q^4}E^r_{\delta-2\alpha_1}E_1E^{2-r}_{\delta-2\alpha_1} = 0.$$

Proof: The proof is a simple application of lemma 4.3.4 and corollary 4.3.6. □

So we are now left to prove just the last relation [EE]. To this aim we start by recalling a very simple and basic combinatorial lemma.

Lemma 4.3.8
Let a, b be elements of a $\mathbb{C}(q)$-algebra satisfying the relation

$$\sum_{r=0}^{k}(-1)^r q^{pr}\begin{bmatrix}k\\r\end{bmatrix}_q a^r b a^{k-r}=0$$

for some natural number k and some integer p. Then a, b also satisfy the relation

$$\sum_{r=0}^{k+1}(-1)^r q^{(p+1)r}\begin{bmatrix}k+1\\r\end{bmatrix}_q a^r b a^{k+1-r}=0.$$

Proof: The proof is a straightforward computation based on the identity

$$q^{(p+1)r}\begin{bmatrix}k+1\\r\end{bmatrix}_q=q^{pr}\begin{bmatrix}k\\r\end{bmatrix}_q+q^{k+p+1}q^{p(r-1)}\begin{bmatrix}k\\r-1\end{bmatrix}_q.$$

□

Before going to the point, we study some other simple relations.

Lemma 4.3.9
The following is true in $\mathcal{U}_q(A_{2n}^{(2)})$:

$$[E_{\delta-2\alpha_1}, F_i]=0 \quad \forall i\in\{3,4,...,n\};$$

$$T_0^{-1}(E_{\delta-2\alpha_1})\in\mathcal{U}_q^+;$$

$$[T_{w^{n-2}}(E_0), E_1]=0=[T_{w^{n-2}}(E_0), F_2].$$

Proof: The proof consists in some easy remarks:
1) if $i\in\{3,4...,n\}$ then $T_{s_0ws_1}(F_i)=T_nT_{n-1}\cdot...\cdot T_{i-1}(F_i)=F_{i-1}$, which commutes with E_0, so that

$$[E_{\delta-2\alpha_1}, F_i]=T_{s_0ws_1}^{-1}([E_0, F_{i-1}])=0;$$

2) $T_0^{-1}(E_{\delta-2\alpha_1})=T_0^{-1}T_{w^{n-1}}(E_0)=T_{s_0w^{n-1}}(E_0)$ which is obviously in $\mathcal{U}_q^+$;
3) $T_{w^{n-2}}(E_0)\in(E_0, E_3, E_4, ..., E_n)$, so that it commutes with E_1 and F_2. □

The following proposition is what we still need for our goal; it concerns the last relation [EE], and the proof is adapted from the analogous situation in case $A_2^{(1)}$ (see [1]).

Proposition 4.3.10
In $\mathcal{U}_q(A_{2n}^{(2)})$ the following relation is verified:

$$\sum_{r=0}^{5}(-1)^r E_1^{(r)}E_{\delta-2\alpha_1}E_1^{(5-r)}=0.$$

Proof: Let $\mathcal{E} \doteqdot \sum_{r=0}^{5}(-1)^r E_1^{(r)} E_{\delta-2\alpha_1} E_1^{(5-r)}$; we want to prove that $\mathcal{E} = 0$. The strategy to achieve this goal is to prove that $T_i^{-1}(\mathcal{E}) \in \mathcal{U}_q^+$ $\forall i = 0, ...n$ (see [27]). Recall that $[\mathcal{E}, F_i] = 0$ implies $T_i^{-1}(\mathcal{E}) \in \mathcal{U}_q^+$.

Lemma 4.3.9 evidently implies that $[\mathcal{E}, F_i] = 0$ when $i = 3, ..., n$ and $T_0^{-1}(\mathcal{E}) \in \mathcal{U}_q^+$ (since $T_0^{-1}(E_1) \in \mathcal{U}_q^+$).

Moreover, thanks to proposition 4.3.2, $[\mathcal{E}, F_1] =$

$$= \sum_{r=1}^{5}(-1)^r \frac{q^{1-r}K_1 - q^{r-1}K_1^{-1}}{q-q^{-1}} E_1^{(r-1)} E_{\delta-2\alpha_1} E_1^{(5-r)} +$$

$$+ \sum_{r=0}^{4}(-1)^r E_1^{(r)} E_{\delta-2\alpha_1} \frac{q^{r-4}K_1 - q^{4-r}K_1^{-1}}{q-q^{-1}} E_1^{(4-r)} =$$

$$= \sum_{r=0}^{4}(-1)^r \frac{-q^{-r}K_1 + q^r K_1^{-1} + q^{-r}K_1 - q^r K_1^{-1}}{q-q^{-1}} E_1^{(r)} E_{\delta-2\alpha_1} E_1^{(4-r)} = 0.$$

So now we have just to show that $[\mathcal{E}, F_2] = 0$. To this aim we rewrite $\mathcal{E}$ (when $n > 1$, which is the case we are interested in) noticing that

$$E_{\delta-2\alpha_1} = T_{w^{n-1}}(E_0) = T_{w^{n-2}} T_0 T_n(E_0) = T_{w^{n-2}} T_n^{-1}(E_0) =$$

$$= T_{w^{n-2}}(E_0 E_n^{(2)} - q^{-2} E_n E_0 E_n + q^{-4} E_n^{(2)} E_0) =$$

$$= T_{w^{n-2}}(E_0) E_2^{(2)} - q^{-2} E_2 T_{w^{n-2}}(E_0) E_2 + q^{-4} E_2^{(2)} T_{w^{n-2}}(E_0),$$

so that $\mathcal{E}$ is given by the expression

$$\sum_{r=0}^{5}(-1)^r E_1^{(r)} (T_{w^{n-2}}(E_0) E_2^{(2)} - q^{-2} E_2 T_{w^{n-2}}(E_0) E_2 + q^{-4} E_2^{(2)} T_{w^{n-2}}(E_0)) E_1^{(5-r)}.$$

Then we are ready to conclude thanks to the last part of lemma 4.3.9:

$$[\mathcal{E}, F_2] =$$

$$= \sum_{r=0}^{5}(-1)^r E_1^{(r)} \Big(T_{w^{n-2}}(E_0) \frac{q^{-2}K_2 - q^2 K_2^{-1}}{q^2 - q^{-2}} E_2 - q^{-2} \frac{K_2 - K_2^{-1}}{q^2 - q^{-2}} T_{w^{n-2}}(E_0) E_2 +$$

$$- q^{-2} E_2 T_{w^{n-2}}(E_0) \frac{K_2 - K_2^{-1}}{q^2 - q^{-2}} + q^{-4} \frac{q^{-2}K_2 - q^2 K_2^{-1}}{q^2 - q^{-2}} E_2 T_{w^{n-2}}(E_0) \Big) E_1^{(5-r)} =$$

$$= T_{w^{n-2}}(E_0) \sum_{r=0}^{5}(-1)^r X E_1^{(r)} E_2 E_1^{(5-r)} - \sum_{r=0}^{5}(-1)^r E_1^{(r)} E_2 E_1^{(5-r)} Y T_{w^{n-2}}(E_0)$$

where

$$X = \frac{q^{2r-2}K_2 - q^{2-2r}K_2^{-1} - q^{2r-6}K_2 + q^{2-2r}K_2^{-1}}{q^2 - q^{-2}} = q^{2r-4}K_2$$

and

$$Y = \frac{q^{2r-8}K_2 - q^{4-2r}K_2^{-1} - q^{2r-12}K_2 + q^{4-2r}K_2^{-1}}{q^2 - q^{-2}} = q^{2r-10}K_2,$$

so that

$$[\mathcal{E}, F_2] = q^{-4}T_{w^{n-2}}(E_0)K_2\left(\sum_{r=0}^{5}(-1)^r q^{2r}E_1^{(r)}E_2E_1^{(5-r)}\right)+$$

$$-q^{-10}\left(\sum_{r=0}^{5}(-1)^r q^{2r}E_1^{(r)}E_2E_1^{(5-r)}\right)K_2T_{w^{n-2}}(E_0)$$

which is zero thanks to lemma 4.3.8 and to the relation, valid in $\mathcal{U}_q(A_{2n}^{(2)})$ (if $n > 1$),

$$\sum_{r=0}^{3}(-1)^r E_1^{(r)}E_2E_1^{(3-r)} = 0.$$

This completely proves the claim, that is $\mathcal{E} = 0$. □

The paragraph can now be closed with the next proposition, which states the achievement of our goal:

Proposition 4.3.11
φ_1 is well defined.

Proof: The only thing to prove is that $E_{\delta-2\alpha_1} \in \mathcal{U}_q^{(1)}$. This follows immediately from proposition 4.3.2 and corollary 4.3.6: indeed by straightforward computations

$$[E_{\delta-\alpha_1}, F_1] = -[4]_q K_1 E_{\delta-2\alpha_1}.$$

□

§4.4. Relations between φ_1 and the braid group action.

The results of paragraph 4.3 are the first step in our attempt to prove that the relations between the root vectors in $\mathcal{U}_q(A_2^{(2)})$ found in section 3 are still valid in $\mathcal{U}_q(A_{2n}^{(2)})$. To conclude this piece of program we need to analyze how the braid group acts with respect to φ_1. As we already announced, we want to prove that $T_1\varphi_1 = \varphi_1 T_1$ and $T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1 T_0$. We start from this second statement.

Proposition 4.4.1
The action of T_0 on $\mathcal{U}_q(A_2^{(2)})$ corresponds, under φ_1, to the action of $T_{\omega_1}T_1^{-1}$, that is $T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1 T_0$.

Proof: Of course it is enough to prove that $T_{\omega_1}T_1^{-1}\varphi_1 = \varphi_1 T_0$ on the generators of $\mathcal{U}_q(A_2^{(2)})$, that is on E_1, E_0, F_1, F_0, K_1, K_0. For the elements of $\mathcal{U}_q^0(A_2^{(2)})$ the claim is trivial.

Remark that $T_{\omega_1}T_1^{-1}$, φ_1 and T_0 all commute with Ω, so that we are reduced to prove that $T_{\omega_1}T_1^{-1}\varphi_1(E_1) = \varphi_1 T_0(E_1)$ and $T_{\omega_1}T_1^{-1}\varphi_1(E_0) = \varphi_1 T_0(E_0)$; this means that we want to prove on one hand that

$$\varphi_1(-E_0E_1 + q^{-4}E_1E_0) = E_{\delta-\alpha_1}$$

(which is nothing but corollary 4.3.6), and on the other hand that

$$-F_{\delta-2\alpha_1}K_{\delta-2\alpha_1}=T_{\omega_1}T_1^{-1}(E_{\delta-2\alpha_1}),$$

or equivalently (see corollaries 4.2.4, 4.2.6 and 4.2.8) that $-F_0K_0=T_{ws_1}T_{s_0ws_1}^{-1}(E_0)$; but of course $T_{ws_1}T_{s_0ws_1}^{-1}(E_0)=T_0(E_0)$, which proves the claim. □

Remark 4.4.2.
We concentrate now on the question of proving that $T_1\varphi_1=\varphi_1T_1$, or equivalently that $T_1^{-1}\varphi_1=\varphi_1T_1^{-1}$. The same considerations as before tell us that this problem reduces to the question whether $T_1^{-1}\varphi_1(E_1)=\varphi_1T_1^{-1}(E_1)$ and $T_1^{-1}\varphi_1(E_0)=\varphi_1T_1^{-1}(E_0)$: notice that the first relation is obviously true while the second one can be translated into

$$T_1^{-1}(E_{\delta-2\alpha_1})=\sum_{r=0}^{4}(-1)^r q^{-r}E_1^{(r)}E_{\delta-2\alpha_1}E_1^{(4-r)}.$$

To face this problem let us set the following definition:

Definition 4.4.3.
We introduce the elements X_0,...,X_n of $\mathcal{U}_q(A_{2n}^{(2)})$ as follows:

$$X_n\doteqdot E_0,\quad X_{s-1}\doteqdot T_s^{-1}(X_s)\ \ \forall s=1,...,n.$$

Remark that $X_1=E_{\delta-2\alpha_1}$ (corollary 4.2.8) and $X_0=T_1^{-1}(E_{\delta-2\alpha_1})$.
Our strategy will divide in two steps:
1) proving that $X_{s-1}=\sum_{r=0}^{2}(-1)^r q^{-2r}E_s^{(r)}X_sE_s^{(2-r)}$ $\forall s=2,...,n$;
2) computing X_0 thanks to the description of X_1 given in 1): this will be the claim.

Lemma 4.4.4
Let s be in $\{2,...,n-1\}$ and let $X\in\mathcal{U}_q$ be an element of $\mathcal{U}_q(A_{2n}^{(2)})$ commuting with E_s and fixed by T_s. Then

$$\sum_{r=0}^{2}(-1)^r q^{-2r}E_s^{(r)}E_{s+1}XE_{s+1}E_s^{(2-r)}=T_s^{-1}(E_{s+1}XE_{s+1}).$$

Proof: Using the Serre relation between E_s and E_{s+1} we have that

$$\sum_{r=0}^{2}(-1)^r q^{-2r}E_s^{(r)}E_{s+1}XE_{s+1}E_s^{(2-r)}=$$

$$=E_{s+1}E_sXE_{s+1}E_s-\frac{1}{q^2+q^{-2}}E_{s+1}E_sXE_sE_{s+1}-q^{-2}E_sE_{s+1}XE_{s+1}E_s+$$

$$+q^{-4}E_sE_{s+1}XE_sE_{s+1}-\frac{q^{-4}}{q^2+q^{-2}}E_{s+1}E_sXE_sE_{s+1}=$$

$$=(-E_{s+1}E_s+q^{-2}E_sE_{s+1})X(-E_{s+1}E_s+q^{-2}E_sE_{s+1})=$$

$$=T_s^{-1}(E_{s+1})XT_s^{-1}(E_{s+1})=T_s^{-1}(E_{s+1}XE_{s+1}).$$

□

Corollary 4.4.5

Let s and X be as in lemma 4.4.4, and let $Y \doteqdot \sum_{r=0}^{2}(-1)^r q^{-2r} E_{s+1}^{(r)} X E_{s+1}^{(2-r)}$; then

$$\sum_{r=0}^{2}(-1)^r q^{-2r} E_s^{(r)} Y E_s^{(2-r)} = T_s^{-1}(Y).$$

Proof: The claim is an immediate application of lemma 4.4.4. □

Corollary 4.4.6

Let s be in $\{2, ..., n\}$; then $X_{s-1} = \sum_{r=0}^{2}(-1)^r q^{-2r} E_s^{(r)} X_s E_s^{(2-r)}$.

In particular $E_{\delta-2\alpha_1} = \sum_{r=0}^{2}(-1)^r q^{-2r} E_2^{(r)} X_2 E_2^{(2-r)}$.

Proof: The statement is proved by induction on s: if $s = n$ the claim is obvious (it is the definition of $T_n^{-1}(E_0)$).

If $2 \leq s < n$ define X by $X \doteqdot X_{s+1}$; then $X \in (E_0, E_n, ..., E_{s+2})$, hence it commutes with E_s and is fixed by T_s. Moreover, by the inductive hypothesis, the element Y defined in corollary 4.4.5 is nothing but X_s. Hence we have that

$$X_{s-1} = T_s^{-1}(X_s) = T_s^{-1}(Y) = \sum_{r=0}^{2}(-1)^r q^{-2r} E_s^{(r)} Y E_s^{(2-r)} =$$

$$= \sum_{r=0}^{2}(-1)^r q^{-2r} E_s^{(r)} X_s E_s^{(2-r)}.$$

□

We have thus solved the first step. For the second one let us develop some computations.

Lemma 4.4.7

Let X be an element of $\mathcal{U}_q$ commuting with E_1. Then we have the following identities:

$$E_2 X E_2 E_1^{(4)} =$$

$$= E_2 E_1^{(2)} X E_2 E_1^{(2)} - \frac{[2]_q}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1 + \frac{1}{q^2 + q^{-2}} E_2 E_1^{(2)} X E_1^{(2)} E_2;$$

$$E_1 E_2 X E_2 E_1^{(3)} =$$

$$= E_1 E_2 E_1 X E_2 E_1^{(2)} - \frac{1}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 + \frac{1}{[3]_q} E_1 E_2 E_1 X E_1^{(2)} E_2;$$

$$E_1^{(3)} E_2 X E_2 E_1 =$$

$$= E_1^{(2)} E_2 X E_1 E_2 E_1 - \frac{1}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 + \frac{1}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1;$$

$$E_1^{(4)} E_2 X E_2 =$$

$$= E_1^{(2)} E_2 X E_1^{(2)} E_2 - \frac{[2]_q}{[3]_q} E_1 E_2 E_1 X E_1^{(2)} E_2 + \frac{1}{q^2 + q^{-2}} E_2 E_1^{(2)} X E_1^{(2)} E_2.$$

Proof: It is a matter of manipulation of the left hand sides using the relation

$$\sum_{r=0}^{3}(-1)^r E_1^{(r)} E_2 E_1^{(3-r)} = 0.$$

Corollary 4.4.8
Let X be an element of $\mathcal{U}_q$ commuting with E_1 and fixed by T_1. Then

$$\sum_{r=0}^{4}(-1)^r q^{-r} E_1^{(r)} E_2 X E_2 E_1^{(4-r)} = T_1^{-1}(E_2 X E_2).$$

Proof: The claim is a consequence of lemma 4.4.7 once that one remarks that

$$T_1^{-1}(E_2) = E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_1^{(2)} E_2;$$

indeed

$$\begin{gathered}
\sum_{r=0}^{4}(-1)^r q^{-r} E_1^{(r)} E_2 X E_2 E_1^{(4-r)} = \\
= E_2 E_1^{(2)} X E_2 E_1^{(2)} - \frac{[2]_q}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1 + \frac{1}{q^2+q^{-2}} E_2 E_1^{(2)} X E_1^{(2)} E_2 + \\
-q^{-1} E_1 E_2 E_1 X E_2 E_1^{(2)} + \frac{q^{-1}}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 - \frac{q^{-1}}{[3]_q} E_1 E_2 E_1 X E_1^{(2)} E_2 + \\
+q^{-2} E_1^{(2)} E_2 X E_2 E_1^{(2)} + \\
-q^{-3} E_1^{(2)} E_2 X E_1 E_2 E_1 + \frac{q^{-3}}{[2]_q} E_1 E_2 E_1 X E_1 E_2 E_1 - \frac{q^{-3}}{[3]_q} E_2 E_1^{(2)} X E_1 E_2 E_1 + \\
+q^{-4} E_1^{(2)} E_2 X E_1^{(2)} E_2 - \frac{q^{-4}[2]_q}{[3]_q} E_1 E_2 E_1 X E_1^{(2)} E_2 + \frac{q^{-4}}{q^2+q^{-2}} E_2 E_1^{(2)} X E_1^{(2)} E_2 = \\
= E_2 E_1^{(2)} X E_2 E_1^{(2)} - q^{-1} E_2 E_1^{(2)} X E_1 E_2 E_1 + q^{-2} E_2 E_1^{(2)} X E_1^{(2)} E_2 + \\
-q^{-1} E_1 E_2 E_1 X E_2 E_1^{(2)} + q^{-2} E_1 E_2 E_1 X E_1 E_2 E_1 - q^{-3} E_1 E_2 E_1 X E_1^{(2)} E_2 + \\
+q^{-2} E_1^{(2)} E_2 X E_2 E_1^{(2)} - q^{-3} E_1^{(2)} E_2 X E_1 E_2 E_1 + q^{-4} E_1^{(2)} E_2 X E_1^{(2)} E_2 = \\
= (E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_1^{(2)} E_2) X (E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_1^{(2)} E_2) = \\
= T_1^{-1}(E_2 X E_2).
\end{gathered}$$

□

We can now conclude:

Proposition 4.4.9
φ_1 commutes with T_1.

Proof: As already remarked, we want to prove that

$$T_1^{-1}(E_{\delta-2\alpha_1}) = \sum_{r=0}^{4}(-1)^r q^{-r} E_1^{(r)} E_{\delta-2\alpha_1} E_1^{(4-r)};$$

following the notations of definition 4.4.3 and the result of corollary 4.4.6

$$E_{\delta-2\alpha_1}=\sum_{r=0}^{2}(-1)^r q^{-2r}E_2^{(r)}X_2E_2^{(2-r)} \quad \text{with} \quad [X_2,E_1]=0 \quad \text{and} \quad T_1(X_2)=X_2;$$

then corollary 4.4.8 implies the claim. □

To sum up, we can state the following proposition, which makes the importance of the present section explicit: indeed proposition 4.4.10 allows us to read in $A_{2n}^{(2)}$ all the relations found in section 3 for $A_2^{(2)}$; at the same time it allows to define $E_{(m\delta,1)}$ $\forall m>1$.

Proposition 4.4.10

Let ν be the group homomorphism $\nu: Q(A_2^{(2)})\to Q(A_{(2n)}^{(2)})$ defined in the obvious way by $\nu(\alpha_1)\doteqdot\alpha_1$, $\nu(\delta)\doteqdot\delta$ and let the injection $\tilde\nu:\Phi_+(A_2^{(2)})\to\tilde\Phi_+(A_{(2n)}^{(2)})$ be given by:

$$\tilde\nu(\alpha)\doteqdot\begin{cases}\nu(\alpha) & \text{if } \alpha \text{ is real}\\ (\nu(\alpha),1) & \text{if } \alpha \text{ is imaginary;}\end{cases}$$

Then φ_1 has the property that

$$\varphi_1(E_\alpha)=E_{\tilde\nu(\alpha)} \quad \forall\alpha\in\Phi_+(A_2^{(2)}).$$

Moreover $\varphi_1(\tilde E_\alpha)=\tilde E_{\tilde\nu(\alpha)} \quad \forall\alpha\in\Phi_+^{\rm im}(A_2^{(2)})$. □

Proof: The claim for $\tilde E_\alpha$ when $\alpha\in\Phi_+^{\rm im}(A_2^{(2)})$ follows from the assertion on the real roots. Then $E_{(m\delta,1)}\in\mathcal{U}_q(A_{2n}^{(2)})$ can be defined $\forall m>0$ (see the analogous definition, corollary 2.2.3) and the claim for $E_{m\delta}$ is also abvious.

For the (real) roots of the form $m\delta-\alpha_1$ the claim follows from propositions 4.4.1 and 4.4.9, and from remark 2.1.5; on the other hand, since $E_{\delta-2\alpha_1}\in\mathcal{U}_q^{(1)}$, $T_{\check{\omega_i}}(E_{\delta-2\alpha_1})=E_{\delta-2\alpha_1}$ $\forall i\in I_0\setminus\{1\}$; thus

$$E_{(2m+1)\delta-2\alpha_1}=(T_{\check{\omega_1}}\cdot\ldots\cdot T_{\check{\omega_n}})^m T_w^{n-1}(E_0)=$$

$$=(T_{\check{\omega_1}}\cdot\ldots\cdot T_{\check{\omega_n}})^m(E_{\delta-2\alpha_1})=T_{\check{\omega_1}}^m(E_{\delta-2\alpha_1})=\varphi_1(T_0T_1)^m(E_0)$$

and analogously

$$E_{(2m+1)\delta+2\alpha_1}=(T_{\check{\omega_1}}\cdot\ldots\cdot T_{\check{\omega_n}})^{-m-1}T_w^{n-1}T_0(E_0)=$$

$$=T_{\check{\omega_1}}^{-m-1}(-F_{\delta-2\alpha_1}K_{\delta-2\alpha_1})=\varphi_1(T_0T_1)^{-m-1}T_0(E_0),$$

which concludes the proof. □

5. RELATIONS BETWEEN $\mathcal{U}_q^{(i)}$ AND $\mathcal{U}_q^{(j)}$ $(i\neq j)$.

In this section we shall complete the description of the commutation relations among the root vectors. The basis for this section is, as for section 2, an application of Beck's work (with something more to check in the twisted case: see §5.2); the argument works also in the case $A_{2n}^{(2)}$.

In particular the aim of this part of the work is to understand the commutation rules between the vectors from $\mathcal{U}_q^{(i)}$ and $\mathcal{U}_q^{(j)}$ when $i \neq j$.

Remark 5.1.
$\forall i, j \in I_0$ such that $a_{ij} = 0$ we have $[\mathcal{U}_q^{(i)}, \mathcal{U}_q^{(j)}] = 0$.

Proof: $[E_i, \Omega^r(E_{\tilde{d}_j\delta-\alpha_j})] = T_{\lambda_j}([E_i, \Omega^r T_j^{-1}(E_j)]) = 0$. □

Hence we are reduced to the case when $a_{ij} < 0$; recall also that, if this is the case, then either $a_{ij} = -1$ or $a_{ji} = -1$.

§5.1. First level of commutation.

In this paragraph we sketch Beck's recursive approach to the problem.

Lemma 5.1.1
Let $i \neq j \in I_0$. Then $\forall r, s \in \mathbb{N}$ the following relations hold:

$$[E_{r\tilde{d}_i\delta+\alpha_i}, F_{s\tilde{d}_j\delta+\alpha_j}] = 0 \quad \text{and} \quad E_{r\tilde{d}_i\delta+\alpha_i} E_{(s+1)\tilde{d}_j\delta-\alpha_j} = q_i^{a_{ij}} E_{(s+1)\tilde{d}_j\delta-\alpha_j} E_{r\tilde{d}_i\delta+\alpha_i}.$$

Proof: It is enough to apply $T_{\lambda_i}^{-r} T_{\lambda_j}^{-s}$ and $T_{\lambda_i}^{-r} T_{\lambda_j}^{s+1}$ to the relation $[E_i, F_j] = 0$. □

Lemma 5.1.2
Let $i, j \in I_0$ be such that $a_{ij} = -1$ (then $\frac{\tilde{d}_i}{\tilde{d}_j} \in \mathbb{N}$). If $r \geq 1$, we have:

$$[\tilde{E}_{(r\tilde{d}_i\delta,i)}, E_j] = \begin{cases} -E_{\tilde{d}_i\delta+\alpha_j} & \text{if } \ r = 1 \\ T_{\lambda_j}^{-\frac{\tilde{d}_i}{\tilde{d}_j}} (q_i E_j \tilde{E}_{((r-1)\tilde{d}_i\delta,i)} - q_i^{-1} \tilde{E}_{((r-1)\tilde{d}_i\delta,i)} E_j) & \text{if } \ r > 1 \end{cases}$$

and

$$[\tilde{E}_{(r\tilde{d}_i\delta,i)}, F_j] = \begin{cases} -K_j E_{\tilde{d}_i\delta-\alpha_j} & \text{if } \ r = 1 \\ K_{\tilde{d}_i\delta} T_{\lambda_j}^{\frac{\tilde{d}_i}{\tilde{d}_j}} (q_i^{-1} F_j \tilde{E}_{((r-1)\tilde{d}_i\delta,i)} - q_i \tilde{E}_{((r-1)\tilde{d}_i\delta,i)} F_j) & \text{if } \ r > 1; \end{cases}$$

similarly, if $r \geq \frac{\tilde{d}_i}{\tilde{d}_j}$, we get:

$$[\tilde{E}_{(r\tilde{d}_j\delta,j)}, E_i] = \begin{cases} [a_{ji}]_{q_j} E_{\tilde{d}_i\delta+\alpha_i} & \text{if } \ r = \frac{\tilde{d}_i}{\tilde{d}_j} \\ T_{\lambda_i}^{-1} (q_j^{-a_{ji}} E_i \tilde{E}_{((r\tilde{d}_j-\tilde{d}_i)\delta,j)} - q_j^{a_{ji}} \tilde{E}_{((r\tilde{d}_j-\tilde{d}_i)\delta,j)} E_i) & \text{if } \ r > \frac{\tilde{d}_i}{\tilde{d}_j} \end{cases}$$

and

$$[\tilde{E}_{(r\tilde{d}_j\delta,j)}, F_i] = \begin{cases} [a_{ji}]_{q_j} K_i E_{\tilde{d}_i\delta-\alpha_i} & \text{if } \ r = \frac{\tilde{d}_i}{\tilde{d}_j} \\ K_{\tilde{d}_i\delta} T_{\lambda_i} (q_j^{a_{ji}} F_i \tilde{E}_{((r\tilde{d}_j-\tilde{d}_i)\delta,j)} - q_j^{-a_{ji}} \tilde{E}_{((r\tilde{d}_j-\tilde{d}_i)\delta,j)} F_i) & \text{if } \ r > \frac{\tilde{d}_i}{\tilde{d}_j}. \end{cases}$$

Proof: The result on which this lemma is based on is that $T_{\lambda_i} T_i^{-1}(E_j) = T_{\lambda_j}^{\frac{\tilde{d}_i}{\tilde{d}_j}} T_i(E_j)$ and $T_{\lambda_i} T_i^{-1}(F_j) = T_{\lambda_j}^{\frac{\tilde{d}_i}{\tilde{d}_j}} T_i(F_j)$ (see [1] and point 4) of lemma 2.2.1): indeed notice that if $a_{ij} = -1$ then $(X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$. The fact that $\frac{\tilde{d}_i}{\tilde{d}_j} \in \mathbb{N}$ is a consequence of the following simple remark: we have either $\tilde{d}_i = 1\ \forall i \in I_0$ or $\tilde{d}_i = d_i\ \forall i \in I_0$, so that $\frac{\tilde{d}_i}{\tilde{d}_j}$ is either 1 or $\frac{d_i}{d_j} = \frac{a_{ji}}{a_{ij}} = -a_{ji}$. $\square$

§5.2. Some particular computations: low twisted cases.

What happens of $[\tilde{E}_{(r\tilde{d}_j\delta,j)}, E_i]$ and $[\tilde{E}_{(r\tilde{d}_j\delta,j)}, F_i]$ when $1 \leq r < \frac{\tilde{d}_i}{\tilde{d}_j}$ (and $i,j \in I_0$ are such that $a_{ij} = -1$) is the matter of the present paragraph; of course this situation occurs neither for the non twisted algebras, nor for $A^{(2)}_{2n}$, since in these cases we have $\tilde{k} = 1$. Hence we have now to figure out what happens in cases $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$, $E^{(2)}_6$ and $D^{(3)}_4$ (when $1 \leq r < \frac{\tilde{d}_i}{\tilde{d}_j}$).

The aim of this paragraph is to prove that the brackets above are zero.

The problem will be first reduced to compute the commutation of F_j with a suitable element lying in $\mathcal{U}^+_q$; this simple remark will immediately solve the case $D^{(2)}_{n+1}$ (see proposition 5.2.2).

In the other cases the question will be further modified in looking for an element of the Weyl group with "good" properties (see remark 5.2.3 and lemma 5.2.4); the last part of the paragraph will be devoted to a closer and more detailed investigation of $A^{(2)}_{2n-1}$, $D^{(3)}_4$ and $E^{(2)}_6$.

Remark 5.2.1.

Let $i,j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$ and let r be such that $1 \leq r < \frac{\tilde{d}_i}{\tilde{d}_j}$ (notice that $\tilde{d}_j = d_j = 1$, $\tilde{d}_i = d_i = k > 1$ and $\lambda_j = \omega_j = \check{\omega_j}$); we want to prove that $[\tilde{E}_{(r\delta,j)}, E_i] = 0$ and $[\tilde{E}_{(r\delta,j)}, F_i] = 0$.

Since, by a simple manipulation depending on $a_{ij} = -1$ (which implies that $-E_jE_i + q_i^{-1}E_iE_j = T_i^{-1}(E_j)$), and on the definition of $\tilde{E}_{(r\delta,i)}$, we see that

$$[\tilde{E}_{(r\delta,j)}, E_i] = -T_i^{-1}T_j^{-1}([T_jT_iT^r_{\omega_j}T_j^{-1}(E_j), F_j]K_j)$$

and

$$[\tilde{E}_{(r\delta,j)}, F_i] = -q_iK_{\alpha_i-\alpha_j}T_j^{-1}([T_jT_iT^r_{\omega_j}T_j^{-1}(E_j), F_j]),$$

the problem reduces to prove that $[T_jT_iT^r_{\omega_j}T_j^{-1}(E_j), F_j] = 0$.

Remark that from $l(s_js_i\omega^r_j) = l(\omega^r_j)+2$ we get $T_jT_iT^r_{\omega_j}T_j^{-1}(E_j) \in \mathcal{U}^+_{q,r\delta-s_j(\alpha_i+\alpha_j)}$. $\blacksquare$

Proposition 5.2.2

Let $X^{(k)}_{\tilde{n}} = D^{(2)}_{n+1}$ and let $i,j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$ (this means that $i = 2$, $j = 1$).

Then we have that $[\tilde{E}_{(\delta,j)}, E_i] = 0$ and $[\tilde{E}_{(\delta,j)}, F_i] = 0$.

Proof: Thanks to remark 5.2.1 we need to show that $[T_1T_2T_{\omega_1}T_1^{-1}(E_1), F_1] = 0$, where $T_1T_2T_{\omega_1}T_1^{-1}(E_1) \in \mathcal{U}^+_{q,\delta-s_1(\alpha_1+\alpha_2)}$. But, since $\delta = \sum_{r=0}^n \alpha_r$ and $a_{12} = -2$,

$$\delta - s_1(\alpha_1+\alpha_2) = \delta - (\alpha_1+\alpha_2) = \alpha_0 + \sum_{r=3}^n \alpha_r,$$

so that $T_1T_2T_{\omega_1}T_1^{-1}(E_1)$ belongs to the subalgebra of $\mathcal{U}^+_q$ generated by $\{E_r | r \neq 1\}$, which immediately implies that it commutes with F_1 (since E_r does for $r \neq 1$). $\square$

Remark 5.2.3.

Remark that for an element $x \in \mathcal{U}_q^+$ the property $[x, F_j] = 0$ is equivalent to the condition $T_j^{\pm 1}(x) \in \mathcal{U}_q^+$ (see [27]); since for any $i, j \in I_0$ and $r > 0$ we obviously have that $T_j^{-1} T_j T_i T_{\omega_j}^r T_j^{-1}(E_j) = T_i T_{\omega_j}^r T_j^{-1}(E_j) \in \mathcal{U}_q^+$, remark 5.2.1 allows us to translate the problem of understanding the behaviour of $[\tilde{E}_{(r\delta,j)}, E_i]$ and $[\tilde{E}_{(r\delta,j)}, F_i]$ (when $1 \leq r < \tilde{d}_i$ and $i, j \in I_0$ are such that $a_{ij} = -1 \neq a_{ji}$) into the following question: is $T_j^2 T_i T_{\omega_j}^r T_j^{-1}(E_j)$ an element of $\mathcal{U}_q^+$?

The strategy that we shall use to prove that the answer is "yes" is to find an element w of the Weyl group $\tilde{W}$ and an element $t \in I$ with the property that $l(s_j w s_t) = l(w) + 2$ and that $T_j^2 T_i T_{\omega_j}^r T_j^{-1}(E_j) = T_{s_j w}(E_t)$.

The next lemma illustrates more precisely what we are looking for.

Lemma 5.2.4

Fix $i \neq j \in I_0$ and $r > 0$, and let $w \in \tilde{W}$ satisfy the following conditions:

1) $l(w^{-1} s_j s_i \omega_j^r s_j) = l(s_j s_i \omega_j^r s_j) - l(w)$;

2) $w^{-1} s_j s_i \omega_j^r s_j(\alpha_j)$ is a simple root;

3) $w^{-1}(\alpha_j) > 0$.

Then $T_j^2 T_i T_{\omega_j}^r T_j^{-1}(E_j) \in \mathcal{U}_q^+$.

Proof: The first hypothesis means that $T_j T_i T_{\omega_j}^r T_j^{-1} = T_w T_{w^{-1} s_j s_i \omega_j^r s_j}$ and the third means that $T_j T_w = T_{s_j w}$.

Then we have: $T_j^2 T_i T_{\omega_j}^r T_j^{-1}(E_j) = T_{s_j w} T_{w^{-1} s_j s_i \omega_j^r s_j}(E_j) = T_{s_j w}(E_t)$, where t is such that $w^{-1} s_j s_i \omega_j^r s_j(\alpha_j) = \alpha_t$; hence the only thing to prove is that $s_j w(\alpha_t) > 0$, which is obvious since $s_j w(\alpha_t) = s_i \omega_j^r s_j(\alpha_j) = r\delta - (\alpha_i + \alpha_j)$. □

The aim of the next lemmas is now to find an element w with the properties stated in lemma 5.2.4: in order to exhibit such an element we need to understand something about ω_j.

Definition 5.2.5.

Consider the case of $A_{2n-1}^{(2)}$. Let τ be the Dynkin diagram automorphism defined by

$$\tau(t) \doteqdot \begin{cases} t & \text{if } 1 \leq t \leq n-1 \\ 0 & \text{if } t = n \\ n & \text{if } t = 0 \end{cases}$$

and define elements $y_r \in \tilde{W}$ (for $r = 1, ..., n$) as follows: $y_r \doteqdot \tau s_n s_{n-1} \cdot ... \cdot s_r$.

Lemma 5.2.6

In $A_{2n-1}^{(2)}$ the following assertions hold:

a) if $1 \leq s \leq r \leq n-1$ and $r < i \leq n$ or $i = 0$, then

$$y_{s+1} \cdot ... \cdot y_r(\alpha_i) = \begin{cases} \alpha_{i-r+s} & \text{if } i \neq 0 \\ \alpha_0 + \sum_{t=1}^{r-s}(\alpha_{n-t} + \alpha_{n-t+1}) & \text{if } i = 0; \end{cases}$$

b) $\forall r = 2, ..., n$ $\Phi_+(y_r) = \{\tau(\alpha_i + ... + \alpha_n) | r \leq i \leq n\}$;

c) if $1 \leq r < i \leq n$ we have

$$y_2 \cdot ... \cdot y_r(\tau(\alpha_i + ... + \alpha_n)) = \delta - (\alpha_1 + 2(\alpha_2 + ... + \alpha_{i-r}) + \alpha_{i-r+1} + ... + \alpha_{n-r+1});$$

d) if $1 \leq r < i \leq n$ we have

$$s_2 \omega_2^{-1} s_1 s_2 y_2 ... y_r(\tau(\alpha_i + ... + \alpha_n)) =$$

$$= \begin{cases} \alpha_2 & \text{if } i-r=1=n-r \\ -\delta-(\alpha_1+...+\alpha_{n-r+1}) & \text{if } i-r=1<n-r \\ -(\alpha_3+...+\alpha_{n-r+1}) & \text{if } i-r=2 \\ -\delta-(\alpha_1+2(\alpha_2+...+\alpha_{i-r})+\alpha_{i-r+1}+...+\alpha_{n-r+1}) & \text{if } i-r>2. \end{cases}$$

Proof: Statement a) is easily proved by induction on $r-s$.

For b) one immediately shows that $y_r^{-1}(\tau(\alpha_i+...+\alpha_n)) = s_r \cdot ... \cdot s_n(\alpha_i+...+\alpha_n) < 0$ if $r \leq i \leq n$, which means that $\{\tau(\alpha_i+...+\alpha_n) | r \leq i \leq n\} \subseteq \Phi_+(y_r)$; but $\#\Phi_+(y_r) = l(y_r) \leq n-r+1$, which proves the claim.

Assertion c) follows from a) remarking that $\tau(\alpha_i+...+\alpha_n) = \alpha_i+...+\alpha_{n-1}+\alpha_0$ and $y_2 \cdot ... \cdot y_r(\alpha_0) = \delta-(\alpha_1+2(\alpha_1+...+\alpha_{n-r})+\alpha_{n-r+1})$.

d) is just $s_2\omega_2^{-1}s_1s_2$ applied to c). $\square$

Proposition 5.2.7

Let $X_{\tilde{n}}^{(k)} = A_{2n-1}^{(2)}$ and let $i,j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$ (this means that $i=1$, $j=2$).

Then we have:

1) $l((y_2 \cdot ... \cdot y_{n-1})^{-1}s_2s_1\omega_2s_2) = l(s_2s_1\omega_2s_2) - l(y_2 \cdot ... \cdot y_{n-1})$;

2) $(y_2 \cdot ... \cdot y_{n-1})^{-1}s_2s_1\omega_2s_2(\alpha_2) = \alpha_0$;

3) $(y_2 \cdot ... \cdot y_{n-1})^{-1}(\alpha_2) > 0$.

In particular $T_2^2T_1T_{\omega_2}T_2^{-1}(E_2) \in \mathcal{U}_q^+$, so that $[\tilde{E}_{(\delta,2)}, E_1] = 0$ and $[\tilde{E}_{(\delta,2)}, F_1] = 0$.

Proof: To prove 1) it is enough to show that $\Phi_+(y_r) \subseteq \Phi_+((y_2 \cdot ... \cdot y_{r-1})^{-1}s_2s_1\omega_2s_2)$ $\forall r = 2, ..., n-1$, and this is a straightforward consequence of points b) and d) of lemma 5.2.6.

From point a) of lemma 5.2.6 we see also that $y_2 \cdot ... \cdot y_{n-1}(\alpha_0) = \delta-(\alpha_1+\alpha_2) = s_2s_1\omega_2s_2(\alpha_2)$, which is 2), and $y_2 \cdot ... \cdot y_{n-1}(\alpha_n) = \alpha_2$, which implies 3). $\square$

Proposition 5.2.8

Consider the case $D_4^{(3)}$ and let $i,j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$: this means that $i=2$ and $j=1$. Then $[\tilde{E}_{(r\delta,1)}, E_2] = 0$ and $[\tilde{E}_{(r\delta,1)}, F_2] = 0$ for $r = 1,2$.

Proof: First of all remark that $\delta = \alpha_0+2\alpha_1+\alpha_2$, hence $s_1s_2\omega_1s_1(\alpha_1) = \alpha_0$, so that lemma 5.2.4 applies with $w =$id; it follows from remark 5.2.3 that $[\tilde{E}_{(\delta,1)}, E_2] = 0$ and $[\tilde{E}_{(\delta,1)}, F_2] = 0$.

Consider now the element $s_0s_1s_2$; then

1) $l((s_0s_1s_2)^{-1}s_1s_2\omega_1^2s_1) = l(s_1s_2\omega_1^2s_1) - 3$: indeed

$$s_1\omega_1^{-2}s_2s_1(\alpha_0), \quad s_1\omega_1^{-2}s_2s_1(\alpha_0+\alpha_1) \quad \text{and} \quad s_1\omega_1^{-2}s_2s_1(3\alpha_0+3\alpha_1+\alpha_2)$$

are negative roots (they are respectively $-\delta+\alpha_1$, $-3\delta-(\alpha_1+\alpha_2)$ and $-3\delta-\alpha_2$);

2) $(s_0s_1s_2)^{-1}s_1s_2\omega_1^2s_1(\alpha_1) = \alpha_1$ which is a simple root;

3) $(s_0s_1s_2)^{-1}(\alpha_1) = \alpha_0 > 0$.

As above, remark 5.2.3 and lemma 5.2.4 apply to the present situation and imply that $[\tilde{E}_{(2\delta,1)}, E_2] = 0$ and $[\tilde{E}_{(2\delta,1)}, F_2] = 0$. $\square$

Proposition 5.2.9

Consider the case $E_6^{(2)}$ and let $i,j \in I_0$ be such that $a_{ij} = -1 \neq a_{ji}$: this means that $i=3$, $j=2$. Then $[\tilde{E}_{(\delta,2)}, E_3] = 0$ and $[\tilde{E}_{(\delta,2)}, F_3] = 0$.

Proof: Consider the element $w \doteq s_0s_1s_2s_3s_4s_2s_3s_2s_1s_0s_2s_3s_2s_1s_2s_3$; then, through direct computations very similar (even though longer!) to those used for $D_4^{(3)}$ (see the proof of proposition 5.2.8, 1), 2), 3)), one sees that:
1) $l(w^{-1}s_2s_3\omega_2s_2) = l(s_2s_3\omega_2s_2) - l(w)$;
2) $s_2s_3\omega_2s_2(\alpha_2) = w(\alpha_2)$;
3) $w^{-1}(\alpha_2)(=\alpha_0) > 0$.
The claim follows. □

§5.3. General commutation formulas.

We are now ready to discuss the commutation relations, as announced at the beginning of this section. Before giving them explicitely, let us introduce some definitions and notations.

Definition 5.3.1.
In the twisted case different from $A_{2n}^{(2)}$ let us define a matrix $A^s = (a_{ij}^s)_{ij\in I_0}$ (called the simply laced matrix associated to (I,A)) as follows:

$$a_{ij}^s \doteq \max\{a_{ij}, -1\};$$

this can be said equivalently by:

$$\begin{cases} a_{ii}^s = 2 & \\ a_{ij}^s = 0 & \text{if } a_{ij} = 0 \\ a_{ij}^s = -1 & \text{if } a_{ij} < 0. \end{cases}$$

Remark that (I_0, A^s) is a Kac-Moody datum of type A_n.

Define also $b_{ij} \doteq \begin{cases} 1 & \text{if } a_{ij} \geq -1 \\ d_j & \text{otherwise.} \end{cases}$

Theorem 5.3.2
Let $i,j \in I_0$ and $r,s > 0$ be such that $\tilde{d}_i|r$ and $\tilde{d}_j|s$; then we have:

(1) $$[E_{(r\delta,i)}, E_{(s-\tilde{d}_j)\delta+\alpha_j}] = \begin{cases} x_{ijr}E_{(s-\tilde{d}_j+r)\delta+\alpha_j} & \text{if } \tilde{d}_j|r \\ 0 & \text{otherwise;} \end{cases}$$

(2) $$[E_{(r\delta,i)}, E_{s\delta-\alpha_j}] = \begin{cases} -x_{ijr}E_{(s+r)\delta-\alpha_j} & \text{if } \tilde{d}_j|r \\ 0 & \text{otherwise;} \end{cases}$$

(3) $$[E_{(r\delta,i)}, F_{(s-\tilde{d}_j)\delta+\alpha_j}] = \begin{cases} x_{ijr}K_{(s-\tilde{d}_j)\delta+\alpha_j}E_{(r-s+\tilde{d}_j)\delta-\alpha_j} & \text{if } \tilde{d}_j|r \text{ and } r \geq s \\ -x_{ijr}K_{r\delta}F_{(s-r-\tilde{d}_j)\delta+\alpha_j} & \text{if } \tilde{d}_j|r \text{ and } r < s \\ 0 & \text{otherwise;} \end{cases}$$

(4) $$[E_{(r\delta,i)}, F_{s\delta-\alpha_j}] = \begin{cases} -x_{ijr}E_{(r-s)\delta+\alpha_j}K_{\alpha_j-s\delta} & \text{if } \tilde{d}_j|r \text{ and } r \geq s \\ x_{ijr}F_{(s-r)\delta-\alpha_j}K_{r\delta}^{-1} & \text{if } \tilde{d}_j|r \text{ and } r < s \\ 0 & \text{otherwise;} \end{cases}$$

$$(5) \qquad [E_{(r\delta,i)}, E_{(s\delta,j)}] = 0;$$

$$(6) \qquad [E_{(r\delta,i)}, F_{(s\delta,j)}] = \delta_{rs} x_{ijr} \frac{K_{r\delta} - K_{r\delta}^{-1}}{q_j - q_j^{-1}};$$

where

$$x_{ijr} = \begin{cases} (o(i)o(j))^r \frac{[ra_{ij}]_{q_i}}{r} & \text{in the non twisted case and} \\ & \quad \text{in case } A_{2n}^{(2)} \text{ with } (i,j) \neq (1,1) \\ \frac{[2r]_q}{r}(q^{2r} + (-1)^{r-1} + q^{-2r}) & \text{in case } A_{2n}^{(2)} \text{ with } i = j = 1 \\ (o(i)o(j))^{\frac{r}{\tilde{d}_i b_{ij}}} \frac{\tilde{d}_i b_{ij} [ra_{ij}^s]_q}{r[d_i]_q} & \text{otherwise} \end{cases}$$

and $o : I_0 \to \{\pm 1\}$ is such that $a_{ij} < 0 \Rightarrow o(i)o(j) = -1$.

Proof: For $i = j$, the claim is nothing but proposition 2.2.4 and φ_1 applied to 3.5.3 and 3.5.4.

For $a_{ij} = 0$, the claim follows from remark 5.1.

For $a_{ij} < 0$, (1) and (2) are found as an application of lemma 5.1.2 and of the results of paragraph 5.2 by a standard argument: see [1] and proposition 3.5.2; (3) and (4) are found applying a suitable power of T_{λ_j} respectively to (2) and (1); (5) follows immediately from (1) and (2) while (6) follows from (3) and (4) (using proposition 2.2.4,2)). □

6. A PBW-BASIS.

An important and useful consequence of the commutation relations given in the preceding section is the construction of a PBW-basis for the general quantum algebra of affine type. The aim of this section is thus to exhibit such a basis using a convex ordering of the root system, to recall its principal properties (such as the Levendorskii-Soibelman formula) and to introduce some notations.

§6.1. A convex ordering.

Definition 6.1.1.
We denote by $\preceq$ the total ordering of $\tilde{\Phi}_+$ defined by:

$$\forall r, s \in \mathbb{Z} : \quad \beta_r \preceq \beta_s \Leftrightarrow \begin{cases} s \leq r \leq 0 & \text{or} \\ r \leq 0 < s & \text{or} \\ 1 \leq s \leq r, \end{cases}$$

$$\forall r \in \mathbb{Z}, \forall \alpha \in \tilde{\Phi}_+^{\text{im}} : \quad \beta_r \preceq \alpha \Leftrightarrow r \leq 0,$$

$$\forall (r\delta, i), (s\delta, j) \in \tilde{\Phi}_+^{\text{im}} : \quad (r\delta, i) \preceq (s\delta, j) \Leftrightarrow \begin{cases} r > s & \text{or} \\ r = s \text{ and } i \leq j. \end{cases}$$

Remark 6.1.2.
$\preceq$ is a convex ordering of $\tilde{\Phi}_+$ (see [2]).

Notation 6.1.3.
$\forall\eta\in Q$ define $\mathcal{P}(\eta)$ as follows:

$$\mathcal{P}(\eta)\doteqdot\{\underline{\gamma}=(\gamma_1,...,\gamma_r)\in\bigcup_{r\in\mathbb{N}}\tilde{\Phi}_+^r|\gamma_1\preceq...\preceq\gamma_r,\sum_{u=1}^r p(\gamma_u)=\eta\};$$

define also $\mathcal{P}$ to be $\mathcal{P}\doteqdot\bigcup_{\eta\in Q}\mathcal{P}(\eta)$.

We denote by $\tilde{p}:\bigcup_{r\in\mathbb{N}}\tilde{\Phi}_+^r\to\mathcal{P}$ the natural (reordering) map; moreover $\forall\alpha\in\tilde{\Phi}_+$ $\mu^\alpha:\mathcal{P}\to\mathbb{N}$ ($\underline{\gamma}\mapsto\mu^\alpha_{\underline{\gamma}}$) indicates the occurrence of α in $\underline{\gamma}$, that is if $\underline{\gamma}=(\gamma_1,...,\gamma_r)$ then $\mu^\alpha_{\underline{\gamma}}\doteqdot\#\{u=1,...,r|\gamma_u=\alpha\}$.

Definition 6.1.4.
By abuse of notation we denote by $\preceq$ also the induced lessicographical ordering of $\mathcal{P}$: if $\underline{\gamma}=(\gamma_1,...,\gamma_r)$ and $\underline{\gamma}'=(\gamma_1',...,\gamma_s')$ we set

$$\underline{\gamma}\preceq\underline{\gamma}'\Leftrightarrow \text{if}\quad l=\min\{u=1,...,r|\gamma_u\neq\gamma_u'\}\quad\text{then}\quad\gamma_l\prec\gamma_l'.$$

§6.2. PBW-basis and L-S formula.

Notation 6.2.1.
Let $x:\tilde{\Phi}_+\to\mathcal{U}_q$ ($\alpha\mapsto x_\alpha$) be any function. Then $\forall\underline{\gamma}=(\gamma_1,...,\gamma_r)\in\mathcal{P}$ we set $x(\underline{\gamma})\doteqdot x_{\gamma_1}\cdot...\cdot x_{\gamma_r}$.
We are now ready to state the main theorem of this section.

Theorem 6.2.2
The set $\{E(\underline{\gamma})|\underline{\gamma}\in\mathcal{P}\}$ is a $\mathbb{C}(q)$-basis of $\mathcal{U}_q^+$; more precisely $\forall\eta\in Q$ $\{E(\underline{\gamma})|\underline{\gamma}\in\mathcal{P}(\eta)\}$ is a $\mathbb{C}(q)$-basis of $\mathcal{U}_{q,\eta}^+$, while more generally $\{E(\underline{\gamma})K_\lambda F(\underline{\gamma}')|\lambda\in Q,\underline{\gamma},\underline{\gamma}'\in\mathcal{P}\}$ is a $\mathbb{C}(q)$-basis of $\mathcal{U}_q$.

Proof: The argument used in [26] applies here word for word. □

The following theorem is the Levendorskii-Soibelman formula:

Theorem 6.2.3
1) $\forall\alpha,\beta\in\tilde{\Phi}_+$, $\alpha\succ\beta\Rightarrow E_\alpha E_\beta-q^{(p(\alpha)|p(\beta))}E_\beta E_\alpha=\sum_{\underline{\gamma}\succ(\beta,\alpha)}a_{\underline{\gamma}}E(\underline{\gamma})$;

2) $\forall\underline{\gamma},\underline{\gamma}'\in\mathcal{P}$, $\underline{\gamma}\succ\underline{\gamma}'\Rightarrow E(\underline{\gamma})E(\underline{\gamma}')-q^{r(\underline{\gamma};\underline{\gamma}')}E(\underline{\gamma}')E(\underline{\gamma})=\sum_{\underline{\tilde{\gamma}}\succ\tilde{p}(\underline{\gamma},\underline{\gamma}')}a_{\underline{\tilde{\gamma}}}E(\underline{\tilde{\gamma}})$, where $(\underline{\gamma},\underline{\gamma}')\mapsto r(\underline{\gamma};\underline{\gamma}')$ is the only function compatible with 1).

Proof: See [23] and [1]. □

§6.3. Some subspaces of $\mathcal{U}_q^+$.

In this paragraph we first list some notations and then give a characterization of some subspaces of $\mathcal{U}_q^+$.

Notation 6.3.1.
$\forall r\in\mathbb{Z}\cup\{\pm\infty,\pm\tilde{\infty},\text{im}\}$ we define a a subset $\tilde{\Phi}_+(r)$ of $\tilde{\Phi}_+$:

$$\tilde{\Phi}_+(r)=\begin{cases}\{\beta_s|1\leq s<r\} & \text{if } r\geq 1\\ \{\beta_s|0\geq s>r\} & \text{if } r\leq 0\\ \tilde{\Phi}_+^{\text{im}} & \text{if } r=\text{im}\\ \bigcup_{s>0}\tilde{\Phi}_+(\pm s) & \text{if } r=\pm\infty\\ \tilde{\Phi}_+^{\text{im}}\cup\tilde{\Phi}_+(\pm\infty) & \text{if } r=\pm\tilde{\infty};\end{cases}$$

moreover we define $\mathcal{U}_q^+(r)$ and $\mathcal{U}_q^{\geq 0}(r)$ by:

$$\mathcal{U}_q^+(r) \doteq \mathrm{span}_{\mathbb{C}(q)}\{E(\underline{\gamma})|\underline{\gamma} = (\gamma_1, ..., \gamma_s) \in \mathcal{P}, \gamma_u \in \tilde{\Phi}_+(r) \ \forall u = 1, ..., s\},$$

$\mathcal{U}_q^{\geq 0}(r) \doteq \mathcal{U}_q^0 \mathcal{U}_q^+(r)$.

In order to describe $\mathcal{U}_q^+(r)$ we need the following lemma.

Lemma 6.3.2
$\forall \alpha \in \tilde{\Phi}_+^{\mathrm{im}}$ we have that

$$T_{w_r}^{-1}(E_\alpha) \in \mathcal{U}_q^+ \ \ \forall r \geq 1, \quad T_{w_r^{-1}}(E_\alpha) \in \mathcal{U}_q^+ \ \ \forall r \leq 0.$$

Proof: Following [2], we have to prove that $\forall j \in I_0$ if $\lambda_j = \tau_j s_{j_1} \cdot ... \cdot s_{j_d}$ then $T_{j_r} \cdot ... \cdot T_{j_d}(E_\alpha) \in \mathcal{U}_q^+ \ \forall r = 1, ..., d$; let $\alpha = (m\tilde{d}_i\delta, i)$ with $i \in I_0$, $m > 0$.

If $i \neq j$ the claim is obvious; if $i = j$ and $(X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$, see [2].

Finally we need to study the case when $(X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1)$: first of all remark that $j_d = 1$ and that $\forall m > 0$

$$\varphi_1 \Xi(E_{m\delta - \alpha_1}) = \varphi_1 \Xi(T_0 T_1)^m T_1^{-1}(E_1) = \varphi_1 (T_0^{-1} T_1^{-1})^m T_1(E_1) =$$

$$= \varphi_1 T_1 (T_0 T_1)^{-m}(E_1) = T_1 T_{\omega_1}^{-m}(E_1)$$

which belongs to $\mathcal{U}_q^+$; then

$$T_1(E_{(m\delta,1)}) = \varphi_1 T_1(E_{m\delta}) = \varphi_1 \Xi(E_{m\delta}) =$$

$$= -E_1 T_1 T_{\omega_1}^{-m}(E_1) + q^{-2} T_1 T_{\omega_1}^{-m}(E_1) E_1 \in \mathcal{U}_q^+.$$

The claim now follows remarking that $l(s_{j_1} \cdot ... \cdot s_{j_{d-1}} s_1) = d$, that $m > 0$ and that $T_{j_r} \cdot ... \cdot T_{j_{d-1}} T_1 T_{\omega_1}^{-1} = (T_{j_1} \cdot ... \cdot T_{j_{r-1}})^{-1}$. □

Proposition 6.3.3
$\forall r \in \mathbb{Z} \cup \{\pm\infty, \pm\tilde{\infty}, \mathrm{im}\}$ $\mathcal{U}_q^+(r)$ is a subalgebra of $\mathcal{U}_q^+$.
Moreover

$$\mathcal{U}_q^+(\pm\infty) = \{x \in \mathcal{U}_q^+ | \exists m > 0 \text{ s.t. } T_{w_{\pm m}^{\pm 1}}^{\mp 1}(x) \in \mathcal{U}_q^{\leq 0}\} =$$

$$= \{x \in \mathcal{U}_q^+ | T_{w_{\pm m}^{\pm 1}}^{\mp 1}(x) \in \mathcal{U}_q^{\leq 0} \ \forall m >> 0\},$$

$$\mathcal{U}_q^+(\pm\tilde{\infty}) = \{x \in \mathcal{U}_q^+ | T_{w_{\mp m}^{\mp 1}}^{\pm 1}(x) \in \mathcal{U}_q^+ \ \forall m > 0\},$$

$$\mathcal{U}_q^+(\mathrm{im}) = \mathcal{U}_q^+(\tilde{\infty}) \cap \mathcal{U}_q^+(-\tilde{\infty}) = \{x \in \mathcal{U}_q^+ | T_{\lambda_i}(x) = x \ \forall i \in I_0\}.$$

Proof: The first assertion follows from L-S formula and from the convexity of $\preceq$; for the second it is enough to apply lemma 6.3.2 (see [2] and [6]), while the third follows immediately from the preceding ones, remarking that $T_{\lambda_i}(E_\alpha) = E_\alpha \ \forall i \in I_0$ $\forall \alpha \in \tilde{\Phi}_+^{\mathrm{im}}$. □

7. THE R-MATRIX.

Here it will be shown, applying to the present situation the strategy and the results used for the non twisted quantum algebras (see [24] and [6]), how it is possible to exhibit an explicit multiplicative formula for the R-matrix (denoted by R) in the twisted case.

Hence most of this section will be devoted to follow the fundamental steps which lead to the description of the R-matrix for the non twisted quantum algebras, to recall the main properties used to establish them, and to remark how, thanks to the results developed in the preceding parts of this paper, the same properties (and consequently the same results) are still valid in this more general setting: we shall refer to [22] and [6] for the proofs. Also, some minor differences occurring in the twisted case (mainly in case $A_{2n}^{(2)}$) will be discussed, and the computations needed to complete the argument performed.

§7.1. General strategy for the construction of the R-matrix.

First of all we recall a very general result connecting the R-matrix to the Killing form (see [31]).

Theorem 7.1.1

Let $C_\eta \in \mathcal{U}_{q,\eta}^+ \otimes \mathcal{U}_{q,-\eta}^-$ (with $\eta \in Q^+$) be the canonical element of $(\cdot,\cdot)\big|_{\mathcal{U}_{q,\eta}^+ \times \mathcal{U}_{q,-\eta}^-}$. Then

$$R = \Big(\sum_{\eta \in Q^+} C_\eta\Big) q^{-t_\infty},$$

where q^{-t_∞} is the diagonal operator on $V \otimes \tilde{V}$ (V and $\tilde{V}$ $\mathcal{U}_q$-modules) which acts as $q^{-(\eta|\tilde{\eta})}$ on $V_\eta \otimes \tilde{V}_{\tilde{\eta}}$. □

Then we recall the behaviour of the real root vectors with respect to the coproduct and to the Killing form.

Proposition 7.1.2

If $\alpha = \beta_r \in \Phi_+^{\mathrm{re}}$ then $\Delta(E_\alpha) - (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha) \in \begin{cases} \mathcal{U}_q^{\geq 0}(r) \otimes \mathcal{U}_q^+ & \text{if } r \geq 1 \\ \mathcal{U}_q^{\geq 0} \otimes \mathcal{U}_q^+(r) & \text{if } r \leq 0. \end{cases}$

Moreover let $\underline{\gamma} = (\gamma_1, ..., \gamma_r)$, $\underline{\gamma}' = (\gamma_1', ..., \gamma_s') \in \mathcal{P}$ be such that $\gamma_u, \gamma_v' \in \tilde{\Phi}_+(\infty)$ $\forall u, v$ or $\gamma_u, \gamma_v' \in \tilde{\Phi}_+(-\infty)$ $\forall u, v$; then

$$(E(\underline{\gamma}), F(\underline{\gamma}')) = \delta_{\underline{\gamma},\underline{\gamma}'} \prod_{\alpha \in \tilde{\Phi}_+} \frac{(\mu_{\underline{\gamma}}^\alpha)_\alpha!}{(q_\alpha^{-1} - q_\alpha)^{\mu_{\underline{\gamma}}^\alpha}}.$$

Proof: The proof given in [22] and [6] depends only on the action of the braid group on $\mathcal{U}_q$ and on its connection with the coproduct Δ (that is on the construction of "partial R-matrices" and on their properties), hence it applies word for word to the present situation. □

Before going to the next proposition we shall introduce a notation and give a simple lemma.

Notation 7.1.3.

We denote by $\det_r X_{\tilde{n}}^{(k)}$ the determinant of the matrix

$$\Big(\frac{r[d_i]_q}{\tilde{d}_i [r]_q} x_{ijr}\Big)_{ij \in I^r} = \frac{r}{[r]_q} \mathrm{diag}\Big(\frac{[d_i]_q}{\tilde{d}_i}\Big| i \in I^r\Big) (x_{ijr})_{ij \in I^r}$$

for the quantum algebra of type $X^{(k)}_{\tilde{n}}$.

Lemma 7.1.4
$\forall r > 0$ we have $\det_r X^{(k)}_{\tilde{n}} \neq 0$ (equivalently $\det(x_{ijr})_{ij\in I^r} \neq 0$).
More precisely

$$\det_r X^{(1)}_n = \begin{cases} [n+1]_{q^r} & \text{in case } A^{(1)}_n \\ [2]^{n-1}_{q^r}[2]_{q^{r(2n-1)}} & \text{in case } B^{(1)}_n \\ [2]_{q^r}[2]_{q^{r(n+1)}} & \text{in case } C^{(1)}_n \\ [2]_{q^r}[2]_{q^{r(n-1)}} & \text{in case } D^{(1)}_n \\ \frac{[3]_{q^r}[2]_{q^{6r}}}{[2]_{q^{2r}}} & \text{in cases } E^{(1)}_6 \text{ and } G^{(1)}_2 \\ \frac{[2]_{q^r}[2]_{q^{9r}}}{[2]_{q^{3r}}} & \text{in case } E^{(1)}_7 \\ [2]_{q^r}[2]_{q^{7r}} - [3]_{q^r} = \frac{[2]_{q^r}[2]_{q^{15r}}}{[2]_{q^{3r}}[2]_{q^{5r}}} & \text{in case } E^{(1)}_8 \\ \frac{[2]^2_{q^r}[2]_{q^{9r}}}{[2]_{q^{3r}}} & \text{in case } F^{(1)}_4 \\ [2]^n_{q^r}[2n+1]_{q^r} & \text{in case } A^{(2)}_{2n} \text{ if } 2 \not| r \\ [2]^{n-1}_{q^r}[2]_{q^{r(2n+1)}} & \text{in case } A^{(2)}_{2n} \text{ if } 2|r \\ [\frac{\tilde{n}-n}{k-1}+1]_{q^r} & \text{if } \tilde{k} \not| r \\ [2]_{q^{rn}} & \text{in cases } A^{(2)}_{2n-1} \text{ and } D^{(2)}_{n+1} \text{ if } 2|r \\ \frac{[2]_{q^{6r}}}{[2]_{q^{2r}}} & \text{in case } E^{(2)}_6 \text{ if } 2|r \\ \frac{[2]_{q^{3r}}}{[2]_{q^r}} & \text{in case } D^{(3)}_4 \text{ if } 3|r. \end{cases}$$

Proof: The proof consists in simple computations: in the non twisted case we have that $\det_r X^{(k)}_{\tilde{n}} = \det([d_i a_{ij}]_{q^r})_{ij\in I_0}$: see [7] and [6]; the result when $\tilde{k} \not| r$ follows from the remark that $\det_r X^{(k)}_{\tilde{n}} = \det([a^s_{ij}]_{q^r})_{ij\in I^r}$ (and from the result for $A^{(1)}_n$). The remaining assertions follow considering that

$$\det_r A^{(2)}_{2n} = (q^{2r} + (-1)^{r-1} + q^{-2r})[2]^n_{q^r}\det_{2r}A^{(1)}_{n-1} - [2]^n_{q^r}\det_{2r}A^{(1)}_{n-2},$$

$$1 < k|r,\ X^{(k)}_{\tilde{n}} = A^{(2)}_{2n-1}, D^{(k)}_{\tilde{n}} \Rightarrow \det_r X^{(k)}_{\tilde{n}} = [2]_{q^r}\det_r A^{(1)}_{n-1} - k\det_r A^{(1)}_{n-2},$$

$$\det_r E^{(2)}_6 = [2]_{q^r}\det_r D^{(2)}_3 - \det_r A^{(1)}_2.$$

$\square$

Proposition 7.1.5
1) Let $\alpha \in \tilde{\Phi}^{\mathrm{im}}_+$; then $\Delta(E_\alpha) - (E_\alpha \otimes 1 + K_{p(\alpha)} \otimes E_\alpha) \in \mathcal{U}^{\geq 0}_q(\infty) \otimes \mathcal{U}^+_q(-\infty)$;

2) Let $\alpha \in \tilde{\Phi}_+$ and $\underline{\gamma} \in \mathcal{P}$ be such that $(E_\alpha, F(\underline{\gamma})) \neq 0$ or $(E(\underline{\gamma}), F_\alpha) \neq 0$; then $\underline{\gamma} = (\beta)$ with $p(\alpha) = p(\beta)$.

3) Let $\{\bar{E}_\alpha | \alpha \in \tilde{\Phi}_+\}$ be elements of $\mathcal{U}^+_q$ satisfying the following requirements:
i) $\bar{E}_\alpha = E_\alpha\ \forall \alpha \in \Phi^{\mathrm{re}}_+$;

ii) $\forall r>0$ $\{\bar{E}_{(r\delta,i)}|i\in I^r\}$ and $\{E_{(r\delta,i)}|i\in I^r\}$ are bases of the same $\mathbb{C}(q)$-vector subspace of $\mathcal{U}_q$;

iii) $(\bar{E}_{(r\delta,i)},\bar{F}_{(r\delta,j)})=0$ if $i\neq j$ (where $\bar{F}_\alpha\doteqdot\Omega(\bar{E}_\alpha)$);

then $\forall\underline{\gamma},\underline{\gamma}'\in\mathcal{P}$ we have that

$$(\bar{E}(\underline{\gamma}),\bar{F}(\underline{\gamma}'))=\delta_{\underline{\gamma},\underline{\gamma}'}\prod_{\alpha\in\Phi_+^{\rm re}}\frac{(\mu_{\underline{\gamma}}^\alpha)_\alpha!}{(q_\alpha^{-1}-q_\alpha)^{\mu_{\underline{\gamma}}^\alpha}}\prod_{\alpha\in\tilde{\Phi}_+^{\rm im}}\mu_{\underline{\gamma}}^\alpha!(\bar{E}_\alpha,\bar{F}_\alpha)^{\mu_{\underline{\gamma}}^\alpha}.$$

Proof: For the proof see [6]. Indeed the argument used in the non twisted case is based on the following properties, which are still valid in the twisted case:

a) $T_{\lambda_1\cdot\ldots\cdot\lambda_n}(E_{\beta_r})=E_{\beta_{r+N}}$ if $r>0$ or $r\leq -N$ and $T_{\lambda_1\cdot\ldots\cdot\lambda_n}(E_\alpha)=E_\alpha$ if $\alpha\in\tilde{\Phi}_+^{\rm im}$; more precisely see proposition 6.3.3;

b) the matrix $(x_{ijr})_{ij\in I^r}$ is non degenerate (see lemma 7.1.4);

c) $\prec$ is a convex ordering of $\tilde{\Phi}_+$. □

Corollary 7.1.6

Let $\bar{E}_\alpha$, $\bar{F}_\alpha$ $(\alpha\in\tilde{\Phi}_+)$ be as in proposition 7.1.5; then

$$R=\Big(\prod_{\alpha\in\tilde{\Phi}_+}\exp_\alpha\Big(\frac{\bar{E}_\alpha\otimes\bar{F}_\alpha}{(\bar{E}_\alpha,\bar{F}_\alpha)}\Big)\Big)q^{-t_\infty}=\Big(\prod_{\alpha\in\tilde{\Phi}_+}\exp_\alpha\Big(\frac{q_\alpha^{-1}-q_\alpha}{c_\alpha}\bar{E}_\alpha\otimes\bar{F}_\alpha\Big)\Big)q^{-t_\infty}$$

where $c_\alpha=(q_\alpha^{-1}-q_\alpha)(\bar{E}_\alpha,\bar{F}_\alpha)=\begin{cases}1 & \text{if } \alpha\in\Phi_+^{\rm re}\\ (q_\alpha^{-1}-q_\alpha)(\bar{E}_\alpha,\bar{F}_\alpha) & \text{if } \alpha\in\tilde{\Phi}_+^{\rm im}.\end{cases}$

Proof: See [22] and [6]. □

Remark 7.1.7.

Remark that $\forall r>0$

$$\prod_{i\in I^r}\exp_{(r\delta,i)}\frac{\bar{E}_{(r\delta,i)}\otimes\bar{F}_{(r\delta,i)}}{(\bar{E}_{(r\delta,i)},\bar{F}_{(r\delta,i)})}=\exp\Big(\sum_{i\in I^r}\frac{\bar{E}_{(r\delta,i)}\otimes\bar{F}_{(r\delta,i)}}{(\bar{E}_{(r\delta,i)},\bar{F}_{(r\delta,i)})}\Big).$$

Hence corollary 7.1.6 can be rewritten as

$$R=\prod_{m\leq 0}\exp_{\beta_m}((q_{\beta_m}^{-1}-q_{\beta_m})E_{\beta_m}\otimes F_{\beta_m})\cdot\prod_{r>0}\exp\tilde{C}_r\cdot\prod_{m\geq 1}\exp_{\beta_m}((q_{\beta_m}^{-1}-q_{\beta_m})E_{\beta_m}\otimes F_{\beta_m})q^{-t_\infty}$$

where $\tilde{C}_r=\sum_{i\in I^r}\frac{\bar{E}_{(r\delta,i)}\otimes\bar{F}_{(r\delta,i)}}{(\bar{E}_{(r\delta,i)},\bar{F}_{(r\delta,i)})}$ is the canonical element of the restriction of $(\cdot,\cdot)$ to

$$\mathrm{span}_{\mathbb{C}(q)}\{E_{(r\delta,i)}|i\in I^r\}\times\mathrm{span}_{\mathbb{C}(q)}\{F_{(r\delta,i)}|i\in I^r\},$$

and both the first and the third products in the right hand side are performed in decreasing order. □

§7.2. The Killing form on the imaginary root vectors.

We have seen how to reduce the explicit description of R to the construction of $\tilde{C}_m$ or equivalently to the construction of elements $\bar{E}_\alpha$ with the properties described in proposition 7.1.5, and to the computation of $(\bar{E}_\alpha,\bar{F}_\alpha)$ for $\alpha\in\tilde{\Phi}_+^{\rm im}$. Of course for both approaches the problem is that of understanding $(\cdot,\cdot)$. Thus we start from the computation of $(E_{(r\delta,i)},F_{(r\delta,j)})$, following the path indicated in [6] and remarking that in case $A_{2n}^{(2)}$ it has to be slightly modified: the problem arises, here too (see lemma 2.2.1), from the fact that in this case $\delta-2\alpha_1$ is a root.

Definition 7.2.1.
Let $i\in I_0$; we shall denote by π_i the natural projection $\pi_i:\mathcal{U}_q\otimes\mathcal{U}_q^{\geq 0}\to\mathcal{U}_q\otimes\mathbb{C}(q)E_i$ (with respect to the Q-gradation).

If $r>0$ and $i,j\in I^r$ we define c_r^{ij} by the following property:

$$\pi_j(\Delta(E_{(r\delta,i)}))=c_r^{ij}E_{r\delta-\alpha_j}K_j\otimes E_j$$

(c_r^{ij} is well defined since $\pi_j(\Delta(E_{(r\delta,i)}))$ is indeed a multiple of $E_{r\delta-\alpha_j}K_j\otimes E_j$: see [6], using proposition 7.1.5, 1) and the convexity of $\prec$).

Lemma 7.2.2
Let $r>0$ and $i,j\in I^r$; then $(E_{(r\delta,i)},F_{(r\delta,j)})=-q_j^2\frac{c_r^{ij}}{(q_j^{-1}-q_j)^2}$.

Proof: The proof is based on proposition 7.1.5, 2), on 1.4.2, 7,i), and on proposition 7.1.2 (see [6]). □

Lemma 7.2.3
Let $r>0$ and $i,j\in I^r$; then $c_r^{ij}=q_j^{-2}(q_j-q_j^{-1})x_{ijr}$.

Proof: The proof consists in showing that $c_r^{ij}=\frac{c_1^{jj}}{x_{jj1}}x_{ijr}$ and in computing $\frac{c_1^{jj}}{x_{jj1}}$.

For $c_r^{ij}=\frac{c_1^{jj}}{x_{jj1}}x_{ijr}$ see [6].

For $\frac{c_1^{jj}}{x_{jj1}}=q_j^{-2}(q_j-q_j^{-1})$ see [6], when we are not in the case $A_{2n}^{(2)}$ with $j=1$.

So it remains to study c_1^{11} just in case $A_{2n}^{(2)}$: remark that $\pi_1(\Delta(E_{\delta-2\alpha_1}))=0$ (because $\delta-3\alpha_1\notin\tilde\Phi$); then, using corollary 4.3.6, we have that

$$\pi_1(\Delta(E_{\delta-\alpha_1}))=-E_{\delta-2\alpha_1}K_1\otimes E_1+q^{-4}K_1E_{\delta-2\alpha_1}\otimes E_1=-(1-q^{-8})E_{\delta-2\alpha_1}K_1\otimes E_1;$$

it follows that

$$\pi_1(\Delta(E_{(\delta,1)}))=\pi_1(\Delta(E_{\delta-\alpha_1}))(E_1\otimes 1)-q^{-2}(E_1\otimes 1)\pi_1(\Delta(E_{\delta-\alpha_1}))+$$

$$+E_{\delta-\alpha_1}K_1\otimes E_1-q^{-2}K_1E_{\delta-\alpha_1}\otimes E_1=(q^2+1-q^{-4}-q^{-6})E_{\delta-\alpha_1}K_1\otimes E_1,$$

which means that $c_1^{11}=q^2+1-q^{-4}-q^{-6}=q_1^{-2}(q_1-q_1^{-1})x_{111}$. □

Corollary 7.2.4
Let $r>0$ and let x (respectively y) belong to the linear span of $\{E_{(r\delta,i)}|i\in I^r\}$ (respectively $\{F_{(r\delta,i)}|i\in I^r\}$).

Then $[x,y]=-(x,y)(K_{r\delta}-K_{r\delta}^{-1})$ and $(\Omega(y),\Omega(x))=-\Omega((x,y))$.

Proof: The claim follows from lemmas 7.2.2 and 7.2.3, from theorem 5.3.2, from the bilinearity of $(\cdot,\cdot)$ and $[\cdot,\cdot]$ and from the fact that Ω is an antiautomorphism. □

§7.3. A "canonical" form for the R-matrix.

Remark 7.1.7 reduces the problem of describing R to that of finding $\tilde C_r$ $\forall r>0$. The goal of the present paragraph, which can be achieved thanks to the results of §7.2, is to describe $\tilde C_r$ in terms of $\{E_{(r\delta,i)},F_{(r\delta,i)}|i\in I^r\}$.

The following is a simple lemma of linear algebra.

Lemma 7.3.1

Let $r > 0$ and let $\tilde{C}_r = \sum_{i,j\in I^r} y^r_{ij} E_{(r\delta,i)} \otimes F_{(r\delta,j)}$. Then if $y^r \doteqdot (y^r_{ij})_{ij\in I^r}$ and $H^r \doteqdot (E_{(r\delta,i)}, F_{(r\delta,j)})_{ij\in I^r}$ we have $y^r = ({}^tH^r)^{-1}$. □

Proposition 7.3.2

In the notation of lemma 7.3.1, if we put

$$z^r_{ij} = \frac{(o(i)o(j))^{\frac{r}{k'_r}} \tilde{d}_i [r]_q}{r[d_i]_q [d_j]_q (q^{-1} - q)} y^r_{ij} \quad \text{with} \quad k'_r \doteqdot \begin{cases} k & \text{if } k|r,\ X = D, E \\ 1 & \text{otherwise,} \end{cases}$$

we can describe z^r_{ij} as follows: of course $z^r_{ij}\tilde{d}_j = z^r_{ji}\tilde{d}_i$ (because H^r is symmetric); if $i \leq j$ then

$$A^{(1)}_n \qquad z^r_{ij} = \frac{[i]_{q^r}[n-j+1]_{q^r}}{[n+1]_{q^r}}$$

$$B^{(1)}_n \qquad z^r_{ij} = \begin{cases} \frac{[n-j+1]_{q^{2r}}}{[2]_{q^{r(2n-1)}}} & \text{if } i = 1 \\ \frac{[2]_{q^{r(2i-3)}}[n-j+1]_{q^{2r}}}{[2]_{q^r}[2]_{q^{r(2n-1)}}} & \text{otherwise} \end{cases}$$

$$C^{(1)}_n \qquad z^r_{ij} = \begin{cases} \frac{[n]_{q^r}}{[2]_{q^r}[2]_{q^{r(n+1)}}} & \text{if } i = j = 1 \\ \frac{[n-j+1]_{q^r}}{[2]_{q^{r(n+1)}}} & \text{if } j > 1 = i \\ \frac{[2]_{q^{ri}}[n-j+1]_{q^r}}{[2]_{q^{r(n+1)}}} & \text{otherwise} \end{cases}$$

$$D^{(1)}_n \qquad z^r_{ij} = \begin{cases} \frac{[n]_{q^r}}{[2]_{q^r}[2]_{q^{r(n-1)}}} & \text{if } i = j \leq 2 \\ \frac{[n-2]_{q^r}}{[2]_{q^r}[2]_{q^{r(n-1)}}} & \text{if } i = 1, j = 2 \\ \frac{[n-j+1]_{q^r}}{[2]_{q^{r(n-1)}}} & \text{if } i \leq 2 < j \\ \frac{[2]_{q^{r(i-2)}}[n-j+1]_{q^r}}{[2]_{q^{r(n-1)}}} & \text{otherwise} \end{cases}$$

$E_n^{(1)}$ $\quad \det_r E_n^{(1)} z_{ij}^r = \begin{cases} [n]_{q^r} & \text{if } \ i=j=1 \\ [j-1]_{q^r}[n-3]_{q^r} & \text{if } \ i=1<j\le 3 \\ [2]_{q^r}[2]_{q^{r(n-2)}} & \text{if } \ i=j=2 \\ [i-1]_{q^r}[n-1]_{q^r} & \text{if } \ 2\le i\le 3=j \\ [3]_{q^r}[n-j+1]_{q^r} & \text{if } \ i=1, j>3 \\ [2]_{q^r}[i-1]_{q^r}[n-j+1]_{q^r} & \text{if } \ 2\le i\le 4\le j \\ [5]_{q^r}[n-j+1]_{q^r} & \text{if } \ i=5 \\ [2]_{q^r}[2]_{q^{4r}}[n-j+1]_{q^r} & \text{if } \ i=6 \\ \frac{[3]_{q^r}[2]_{q^{6r}}[n-j+1]_{q^r}}{[2]_{q^{2r}}} & \text{if } \ i=7 \\ \frac{[2]_{q^r}[2]_{q^{9r}}}{[2]_{q^{3r}}} & \text{if } \ i=j=8 \end{cases}$

$F_4^{(1)}$ $\quad z_{ij}^r = \begin{cases} \frac{[2]_{q^{3r}}[2]_{q^{5r}}}{[2]_{q^{9r}}} & \text{if } \ i=j=1 \\ \frac{[2]_{q^{3r}}[i]_{q^r}[5-j]_{q^{2r}}}{[2]_{q^{9r}}} & \text{if } \ i\le 2\le j \\ \frac{[2]_{q^{3r}}[3]_{q^r}[5-j]_{q^{2r}}}{[2]_{q^{9r}}[2]_{q^r}} & \text{if } \ i=3 \\ \frac{[2]_{q^{3r}}[2]_{q^{4r}}}{[2]_{q^{9r}}[2]_{q^r}} & \text{if } \ i=j=4 \end{cases}$

$G_2^{(1)}$ $\quad (z_{ij}^r)_{ij\in I_0} = \frac{[2]_{q^{2r}}}{[2]_{q^{6r}}[3]_{q^r}} \begin{pmatrix} [6]_{q^r} & [3]_{q^r} \\ [3]_{q^r} & [2]_{q^r} \end{pmatrix}$

$A_{2n}^{(2)}$ $\quad z_{ij}^r = \begin{cases} \frac{[2i-1]_{q^r}[n-j+1]_{q^{2r}}}{[2]_{q^r}[2n+1]_{q^r}} & \text{if } 2\not| r \\ \frac{[2]_{q^{r(2i-1)}}[n-j+1]_{q^{2r}}}{[2]_{q^r}[2]_{q^{r(2n+1)}}} & \text{if } 2|r \end{cases}$.

$A_{2n-1}^{(2)}$ $\quad z_{ij}^r = \begin{cases} \frac{[i-1]_{q^r}[n-j+1]_{q^r}}{[n]_{q^r}} & \text{if } \ k\not| r \\ \frac{[n]_{q^r}}{[2]_{q^{rn}}} & \text{if } \ k|r, i=j=1 \\ \frac{(-1)^{\frac{r}{2}}2[n-j+1]_{q^r}}{[2]_{q^{rn}}} & \text{if } \ k|r, i=1<j \\ \frac{[2]_{q^{r(i-1)}}[n-j+1]_{q^r}}{[2]_{q^{rn}}} & \text{otherwise} \end{cases}$

$D_{n+1}^{(2)}$ $\quad z_{ij}^r = \begin{cases} \frac{1}{[2]_{q^r}} & \text{if } \ k\not| r \\ \frac{[n-j+1]_{q^r}}{[2]_{q^{rn}}} & \text{if } \ k|r, i=1 \\ \frac{[2]_{q^{r(i-1)}}[n-j+1]_{q^r}}{[2]_{q^{rn}}} & \text{otherwise} \end{cases}$

$E_6^{(2)}$ $\quad z_{ij}^r = \begin{cases} \frac{[i]_{q^r}[3-j]_{q^r}}{[3]_{q^r}} & \text{if } \ k\not| r \\ \frac{[2]_{q^{2r}}[2]_{q^{3r}}}{[2]_{q^{6r}}} & \text{if } \ k|r, i=j\in\{1,4\} \\ \frac{(-1)^{\frac{r}{2}}[2]_{q^{2r}}[5-j]_{q^r}}{[2]_{q^{6r}}} & \text{if } \ k|r, i=1<j \\ \frac{[2]_{q^{2r}}[i]_{q^r}[5-j]_{q^r}}{[2]_{q^{6r}}} & \text{otherwise} \end{cases}$

$D_4^{(3)}$ $\quad z_{ij}^r = \begin{cases} \frac{1}{[2]_{q^r}} & \text{if } \ k\not| r \\ \frac{[2]_{q^r}[i]_{q^r}[3-j]_{q^r}}{[2]_{q^{3r}}} & \text{if } \ k|r \end{cases}$

Proof: Thanks to lemma 7.3.1, the proof consists in the simple exercise of inverting the matrix $(x_{ijr})_{ij\in I^r}$. $\square$

Theorem 7.3.3
With the notations of proposition 7.3.2, we have that $Rq^{t_\infty} =$

$$=\prod_{r\le 0}\exp_{\beta_r}(q_{\beta_r}^{-1}-q_{\beta_r})E_{\beta_r}\otimes F_{\beta_r}\prod_{r>0}\exp(\sum_{ij\in I_r} y_{ij}^r E_{(r\delta,i)}\otimes F_{(r\delta,j)})\prod_{r>0}\exp_{\beta_r}(q_{\beta_r}^{-1}-q_{\beta_r})E_{\beta_r}\otimes F_{\beta_r}.$$

Proof: The claim is an immediate consequence of remark 7.1.7, of lemma 7.3.1 and of proposition 7.3.2. $\square$

§7.4. Linear triangular transformation: the $\bar{E}_\alpha$'s.

It can be useful to determine explicitly the expression of R in the form given in corollary 7.1.6, that is to describe R in terms of "dual" bases. To this aim we shall look for elements $\bar{E}_\alpha$ with $\alpha\in\tilde{\Phi}_+$ satisfying the requirements of proposition 7.1.5.

Lemma 7.4.1
For $r>0$ let $\bar{E}_{(r\delta,i)}$ $(i\in I^r)$ be elements of $\mathcal{U}_{q,r\delta}^+$ such that:
1) $\forall i\in I^r$ $\bar{E}_{(r\delta,i)}$ is a linear combination of $\{E_{(r\delta,j)}|j\in I^r, j\ge i\}$;
2) $\forall i<j\in I^r$ $(\bar{E}_{(r\delta,i)},F_{(r\delta,j)})=0$.
Then, if $\bar{F}_{(r\delta,i)}\doteqdot\Omega(\bar{E}_{(r\delta,i)})$, we have that $(\bar{E}_{(r\delta,i)},\bar{F}_{(r\delta,j)})=0$ $\forall i\ne j\in I^r$.

Proof: The assertion is clear for $i<j$ once that one notices that $\bar{F}_{(r\delta,j)}$ lies in the linear span of $\{F_{(r\delta,l)}|l\in I^r, l\ge j\}$; for $i>j$, it follows from corollary 7.2.4. $\square$

Remark 7.4.2.
Lemma 7.4.1 allows us to reduce the construction of $\bar{E}_{(r\delta,i)}$ to the solution of the following system of linear equations:

$$(*)_{r,i}\qquad \sum_{l\ge i}A_{il}^{(r)}x_{ljr}=0\quad\text{for } j\in I^r \text{ with } i<j$$

under the condition $A_{ii}^{(r)}\ne 0$, where $\bar{E}_{(r\delta,i)}=\sum_{l\ge i}A_{il}^{(r)}E_{(r\delta,l)}$.
Notice that in the non twisted case

$$x_{ljr}=(o(l)o(j))^r\frac{[ra_{lj}]_{q_l}}{r}=(o(l)o(j))^r\frac{[r]_q[d_la_{lj}]_{q^r}}{r[d_l]_q}$$

so that $(*)_{r,i}$ can be written as

$$(*)_{r,i}^{(1)}\qquad \sum_{l\ge i}\frac{A_{il}^{(r)}o(l)^r}{[d_l]_q}[d_la_{lj}]_{q^r}=0\quad\text{for } j\in I_0 \text{ with } i<j.$$

This system has been solved in [7], and those solutions will be used here in order to simplify the computations in the general case.

Proposition 7.4.3
Consider the non twisted case $(k=1)$ and let $r>0$, $i\in I_0$.

A solution of $(*)_{r,i}^{(1)}$ is given by:

$$\frac{A_{ij}^{(r)}o(j)^r}{[d_j]_q}=\begin{cases}[n-j+1]_{q^{2r}} & \text{in case } B_n^{(1)}\text{and}\\ & \text{if } (i,j)\neq(1,1) \text{ in case } F_4^{(1)}\\ [n-j+1]_{q^r}[2]_{q^r} & \text{if } j>i=1 \text{ in case } C_n^{(1)}\text{and}\\ & \text{if } j-1>i=1 \text{ in case } D_n^{(1)}\\ [n-2]_{q^r} & \text{if } i=1, j=2 \text{ in case } D_n^{(1)}\\ [j-1]_{q^r}[n-3]_{q^r} & \text{if } 4>j>i=1 \text{ in case } E_n^{(1)}\\ [n-j+1]_{q^r}[3]_{q^r} & \text{if } j\geq 4, i=1 \text{ in case } E_n^{(1)}\\ [2]_{q^{r(n+1)}} & \text{if } i=j=1 \text{ in cases } F_4^{(1)} \text{ and } G_2^{(1)}\\ [n-j+1]_{q^r} & \text{otherwise,}\end{cases}$$

where $j \in I_0$ is such that $i \leq j$.

Proof: See [7]. □

We can now pass to the study of the twisted case. We shall consider three different situations: the case $A_{2n}^{(2)}$, the twisted case different from $A_{2n}^{(2)}$ with $k \not| r$ and the twisted case different from $A_{2n}^{(2)}$ with $k|r$.

Remark 7.4.4.

Remark that $x_{ijr}(A_{2n}^{(2)}) = x_{ijr}(B_n^{(1)})$ if $(i,j) \neq (1,1)$ and that in the equations $(*)_{r,i}$ x_{11r} never appears; then $(*)_{r,i}(A_{2n}^{(2)}) = (*)_{r,i}(B_n^{(1)})$; in particular the solutions of $(*)_{r,i}$ in case $A_{2n}^{(2)}$ are the same as the solutions of $(*)_{r,i}$ in case $B_n^{(1)}$.

Remark 7.4.5.

Let $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$ and $r > 0$ be such that $k \not| r$. Then $\forall i \in I^r$ (that is $\forall i \in I$ such that $d_i = 1$) the system $(*)_{r,i}(X_{\tilde{n}}^{(k)}) = (*)_{r,i}\left(A_{\frac{\tilde{n}-n}{k-1}}^{(1)}\right)$ (with a shift of indices in case $A_{2n-1}^{(2)}$, that is $A_{ij}^{(r)}(A_{2n-1}^{(2)}) = A_{i-1j-1}^{(r)}\left(A_{\frac{\tilde{n}-n}{k-1}}^{(1)}\right) = A_{i-1j-1}^{(r)}(A_{n-1}^{(1)})$ $\forall i,j \in I^r$).

Remark 7.4.6.

In the twisted case different from $A_{2n}^{(2)}$, remarking that $\tilde{d}_i b_{ij} = \tilde{d}_j b_{ji}$ when $a_{ij} \neq 0$, the system $(*)_{r,i}$ becomes the following (under the condition that $k|r$):

$$(*)_{r,i}^{\text{tw}} \qquad \sum_{l\geq i}\frac{A_{il}^{(r)}}{[d_l]_q}(o(j)o(l))^{\frac{r}{\tilde{d}_j b_{jl}}}b_{jl}[a_{lj}^s]_{q^r}=0 \quad \text{for } j\in I_0 \text{ with } i<j.$$

Notice that if $i < j \in I_0$ are such that $b_{jl} = 1$ $\forall l \geq i$ such that $a_{lj} \neq 0$, then $\tilde{d}_j = k'_r$ and each equation above can be written as

$$(*)_{r,i,j}^{k|r} \qquad \sum_{l\geq i}\frac{A_{il}^{(r)}o(l)^{\frac{r}{k'_r}}}{[d_l]_q}[a_{lj}^s]_{q^r}=0 \quad \text{for } j\in I_0 \text{ with } i<j.$$

This always happens when $X_{\tilde{n}}^{(k)} = D_{\tilde{n}}^{(k)}$ $(k = 2,3)$; on the other hand when $X_{\tilde{n}}^{(k)} = A_{2n-1}^{(2)}$ or $E_6^{(2)}$ we have that $b_{jl} \neq 1 \Rightarrow j = 2$, which implies that if $i \neq 1$ then

$(*)^{\text{tw}}_{r,i} = \{(*)^{k|r}_{r,i,j} | j > i\}$. Moreover we see also that

$$(*)^{\text{tw}}_{r,1} = \{(*)^{k|r}_{r,1,j} | j > 2\} \cup \Big\{ \sum_{l \in I_0} \frac{A^{(r)}_{1l}}{[d_l]_q} (o(2)o(l))^{\frac{r}{d_2 b_{2l}}} b_{2l} [a^s_{l2}]_{q^r} = 0 \Big\}.$$

Remark also that $A^{(r)}_{11}$ does not appear in the equation $(*)^{k|r}_{r,i,j}$ when $(i,j) \neq (1,2)$, so that the only condition on $A^{(r)}_{11}$ is the one arising from the last equation, i.e. $\frac{A^{(r)}_{11}}{[d_1]_q}(o(2)o(1))^{\frac{r}{d_2 b_{21}}} b_{21} = \sum_{l>1} \frac{A^{(r)}_{1l}}{[d_l]_q}(o(2)o(l))^{\frac{r}{d_2 b_{2l}}} b_{2l}[a^s_{l2}]_{q^r}$, where the $A^{(r)}_{ij}$'s with $(i,j) \neq (1,1)$ are solutions of the system $\{(*)^{k|r}_{r,i,j} | i < j \text{ and } (i,j) \neq (1,2)\}$.

Proposition 7.4.7
Let $r > 0$ and $i \in I^r$. A solution of $(*)_{r,i}$ is given by:

$$\frac{A^{(r)}_{ij} o(j)^{\frac{r}{k'_r}}}{[d_j]_q} = \begin{cases} [n-j+1]_{q^{2r}} & \text{in cases } B^{(1)}_n \text{ and } A^{(2)}_{2n}, \text{ and} \\ & \text{if } (i,j) \neq (1,1) \text{ in case } F^{(1)}_4 \\ [n-j+1]_{q^r}[2]_{q^r} & \text{if } j > i = 1 \text{ in case } C^{(1)}_n \text{ and} \\ & \text{if } j-1 > i = 1 \text{ in case } D^{(1)}_n \\ [n-2]_{q^r} & \text{if } i = 1, j = 2 \text{ in case } D^{(1)}_n \\ [j-1]_{q^r}[n-3]_{q^r} & \text{if } 4 > j > i = 1 \text{ in case } E^{(1)}_n \\ [n-j+1]_{q^r}[3]_{q^r} & \text{if } j \geq 4, i = 1 \text{ in case } E^{(1)}_n \\ [2]_{q^{5r}} & \text{if } i = j = 1 \text{ in case } F^{(1)}_4 \\ [3-j]_{q^{3r}} & \text{if } i = 1 \text{ in case } G^{(1)}_2 \\ [\frac{\tilde{n}-n}{k-1} - j + 1]_{q^r} & \text{if } k \not| r \text{ in cases } D^{(k)}_{\tilde{n}} \text{ and } E^{(k=2)}_6 \\ \frac{(-1)^{\frac{r}{2}}}{2}[n]_{q^r} & \text{if } 2|r \text{ and } i = j = 1 \text{ in case } A^{(2)}_{2n-1} \\ (-1)^{\frac{r}{2}}[2]_{q^{3r}} & \text{if } 2|r \text{ and } i = j = 1 \text{ in case } E^{(2)}_6 \\ [n-j+1]_{q^r} & \text{otherwise,} \end{cases}$$

where $j \in I^r$ is such that $i \leq j$.

Proof: The only statements which do not follow immediately from proposition 7.4.3 and remarks 7.4.4, 7.4.5 and 7.4.6 are those concerning the case when $X^{(2)}_{\tilde{n}} = A^{(2)}_{2n-1}$ or $E^{(2)}_6$, $2|r$ and $i = j = 1$, and it is a matter of straightforward computations. □

§7.5. The Killing form on the new imaginary root vectors.

The next (and last) step is computing $(\bar{E}_\alpha, \bar{F}_\alpha)$ when α is an imaginary root.

Lemma 7.5.1
Let $r > 0$ and $i \in I^r$. Then

$$(\bar{E}_{(r\delta,i)}, \bar{F}_{(r\delta,i)}) = -A^{(r)}_{ii} \sum_{j \in I^r : j \geq i} A^{(r)}_{ij} \frac{x_{jir}}{q_i - q_i^{-1}} = -\frac{A^{(r)}_{ii}}{q - q^{-1}} \sum_{j \in I^r : j \geq i} \frac{A^{(r)}_{ij}}{[d_j]_q} x_{ijr}.$$

Proof: The definition of $\bar{E}_\alpha$ implies that $(\bar{E}_{(r\delta,i)}, \bar{F}_{(r\delta,i)}) = A_{ii}^{(r)}(\bar{E}_{(r\delta,i)}, F_{(r\delta,i)})$, because $(\bar{E}_{(r\delta,i)}, \bar{F}_{(r\delta,i)} - A_{ii}^{(r)}F_{(r\delta,i)}) = 0$. But

$$(\bar{E}_{(r\delta,i)}, F_{(r\delta,i)}) = \sum_{j\in I^r : j\geq i} A_{ij}^{(r)}(E_{(r\delta,j)}, F_{(r\delta,i)}) = -\sum_{j\in I^r : j\geq i} A_{ij}^{(r)} \frac{x_{jir}}{q_i - q_i^{-1}}.$$

□

Proposition 7.5.2
Let $r > 0$, $i \in I^r$. Then we have

$$\frac{o(i)^{\frac{r}{k_r'}}(q^{-1}-q)r[d_i]_q}{\tilde{d}_i[r]_q A_{ii}^{(r)}}(\bar{E}_{(r\delta,i)}, \bar{F}_{(r\delta,i)}) =$$

$$= \begin{cases} [2]_{q^{r(2n-1)}} & \text{if } i=1 \text{ in case } B_n^{(1)} \\ [n-i+2]_{q^{2r}}[2]_{q^r} & \text{if } i\neq 1 \text{ in cases } B_n^{(1)} \text{and } A_{2n}^{(2)} \text{and} \\ & \text{if } i>2 \text{ in case } F_{n=4}^{(1)} \\ [2]_{q^r}[2]_{q^{r(n+1)}} & \text{if } i=1 \text{ in case } C_n^{(1)} \\ [2]_{q^r}[2]_{q^{r(n-1)}} & \text{if } i=1 \text{ in case } D_n^{(1)} \\ \frac{[9-n]_{q^r}[2]_{q^{3r(n-4)}}}{[2]_{q^{r(n-4)}}} & \text{if } i=1 \text{ in case } E_n^{(1)} \text{ with } n=6,7 \\ \frac{[2]_{q^r}[2]_{q^{15r}}}{[2]_{q^{3r}}[2]_{q^{5r}}} & \text{if } i=1 \text{ in case } E_8^{(1)} \\ [2]_{q^{5r}} & \text{if } i=2 \text{ in case } F_4^{(1)} \\ \frac{[2]_{q^{9r}}}{[2]_{q^{3r}}} & \text{if } i=1 \text{ in case } F_4^{(1)} \\ [6]_{q^r} & \text{if } i=2 \text{ in case } G_2^{(1)} \\ \frac{[2]_{q^{6r}}}{[2]_{q^{2r}}} & \text{if } i=1 \text{ in case } G_2^{(1)} \\ [2]_{q^{r(2n+1)}} & \text{if } i=1 \text{ and } 2|r \text{ in case } A_{2n}^{(2)} \\ [2]_{q^r}[2n+1]_{q^r} & \text{if } i=1 \text{ and } 2\not|r \text{ in case } A_{2n}^{(2)} \\ [\frac{\tilde{n}-n}{k-1}-i+2]_{q^r} & \text{if } k\not|r \text{ in cases } D_{\tilde{n}}^{(k)} \text{ and } E_{\tilde{n}=6}^{k=2} \\ \frac{(-1)^{\frac{r}{2}}[2]_{q^{rn}}}{2} & \text{if } i=1 \text{ and } 2|r \text{ in case } A_{2n-1}^{(2)} \\ [2]_{q^{rn}} & \text{if } i=1 \text{ and } k=2|r \text{ in case } D_{\tilde{n}}^{(k)} \\ \frac{[2]_{q^{3r}}}{[2]_{q^r}} & \text{if } i=1 \text{ and } k=3|r \text{ in case } D_{\tilde{n}}^{(k)} \\ [2]_{q^{3r}} & \text{if } i=2 \text{ and } 2|r \text{ in case } E_6^{(2)} \\ (-1)^{\frac{r}{2}}\frac{[2]_{q^{6r}}}{[2]_{q^{2r}}} & \text{if } i=1 \text{ and } 2|r \text{ in case } E_6^{(2)} \\ [n-i+2]_{q^r} & \text{otherwise.} \end{cases}$$

Proof: For the non twisted case see [7]. Then also the cases $A_{2n}^{(2)}$ with $i\neq 1$, $X_{\tilde{n}}^{(k)}$ with $k\not|r$, $A_{2n-1}^{(2)}$ and $D_{\tilde{n}}^{(k)}$ with $k\neq 1|r$ and $i\neq 1$, and $E_6^{(2)}$ with $2|r$ and $i\neq 1,2$ are clear. □

§7.6. The expression of R in terms of the $\bar{E}_\alpha$'s.

We can now collect the results obtained till now and get an explicit formula for R.

Theorem 7.6.1

$$R = \Big(\prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \Big(\frac{q_\alpha^{-1} - q_\alpha}{c_\alpha} \bar{E}_\alpha \otimes \bar{F}_\alpha \Big) \Big) q^{-t_\infty}$$

where $c_\alpha = (q_\alpha^{-1} - q_\alpha)(\bar{E}_\alpha, \bar{F}_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Phi_+^{\mathrm{re}} \\ (q_\alpha^{-1} - q_\alpha)(\bar{E}_\alpha, \bar{F}_\alpha) & \text{if } \alpha \in \tilde{\Phi}_+^{\mathrm{im}}, \end{cases}$ and c_α is explicitly determined from proposition 7.5.2.

Proof: The claim is a straightforward consequence of the preceding results: corollary 7.1.6 and proposition 7.5.2. $\square$

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