THE CENTER OF TWISTED AFFINE QUANTUM ALGEBRAS
AT ODD ROOTS OF 1

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To the people of Yugoslavia
for the price they are paying
to the "new world order"
смрт фашизму - слобода народу

0. INTRODUCTION.

The aim of this paper is describing the center of the specialization (or, we should say, of a chosen specialization, see paragraph 1.3) of twisted affine quantum algebras at primitive odd roots of 1 (with some further very slight conditions on the order of these roots, see notation 1.3.3). This result is already known for quantum algebras of finite type (see [5]) and of untwisted affine type (see [2]); on the other hand the investigation of the structure of twisted affine quantum algebras carried out in [3] allows to complete this program with just a little more effort.

Actually the structure needed on $\mathcal{U}_q$ is very rich: for this reason section 1, where the notations are introduced, is the occasion for recalling, as shortest as possible, the instruments that will be used in the arguments presented.

In section 2 the contravariant form is used, following an argument introduced in [5] and refined in [2], in order to get a control on the "dimension" of the center: the contravariant form is studied by making use of both its general properties and its connections with the Killing form, which, on its side, was studied in details, for the twisted algebras, in [3].

In section 3 some specific computations are developed, in order to find the missing central elements, which are not simply powers of root vectors, but which are linear combinations of (non central) imaginary root vectors. Thanks to these additional elements we obtain the complete picture of the positive part of $\mathcal{Z}(\mathcal{U}_\varepsilon)$.

Finally section 4 is devoted to glueing the pieces together, in order to state the desired assertion about the center: it is the quotient of an algebra of polynomials in an infinite number of variables by a very "small" ideal, which is indeed a principal ideal generated by an element $P_Z \in \mathcal{U}_\varepsilon^0$.

out a total opposition against that war and is now fighting against its continuation in these times of so-called peace, I'm happy to bring to this Congress my, unfortunately too small, tribute and solidarity for the price that Yugoslavia paid and is still paying to the strategies of domination of the world.

1. TWISTED AFFINE QUANTUM ALGEBRAS AT ROOTS OF 1: NOTATIONS.

§1.1. *The Kac-Moody algebra.*

Twisted affine quantum algebras are the quantization of the enveloping algebra of a class of Lie algebras, namely the class of twisted affine Kac-Moody algebras.

A complete description of these KM-algebras, as well as a motivation for their denomination, can be found in [6], where they were introduced. What it is important to recall here is that a KM-algebra $\mathfrak{g}$ is a Lie algebra whose generators and relations can be expressed in terms of a matrix A, called the (generalized) Cartan matrix of $\mathfrak{g}$, and that the same information contained in A can be encoded in a diagram Γ (the Dynkin diagram of $\mathfrak{g}$).

The Dynkin diagrams associated to twisted affine KM-algebras consist in three families $(A_{2n}^{(2)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)})$ and two isolated cases $(E_6^{(2)}, D_4^{(3)})$, which are listed below (the type is in general denoted by $X_{\tilde{n}}^{(k)}$, which means that, for example, in case $E_6^{(2)}$ we have $X = E$, $\tilde{n} = 6$, $k = 2$):

$A_2^{(2)}$

$A_{2n}^{(2)}$

$A_{2n-1}^{(2)}$

$D_{n+1}^{(2)}$

$E_6^{(2)}$

$D_4^{(3)}$

The matrix $A = (a_{ij})_{ij \in I}$ can be recovered from Γ as follows:

a) I is the set of vertices of Γ;

b) $\begin{cases} a_{ii} = 2 & \forall i \in I \\ a_{ij} = -\#\{\text{edges in } \Gamma \text{ connecting } i \text{ and } j\} & \forall i \neq j \in I \text{ s.t. } \exists \text{an arrow pointing at } i \\ & \text{or there is no edge connecting } i \text{ and } j \\ a_{ij} = -1 & \text{otherwise.} \end{cases}$

Remark that the set I has been identified with $\{0, 1, ..., n\}$; the set $\{1, ..., n\} \doteq I \setminus \{0\}$ is denoted by I_0.

Attached to these data there is the notion of root system $\Phi \subseteq Q \doteq \oplus_{i \in I} \mathbf{Z}\alpha_i$, of positive and negative, real and imaginary roots, and of multiplicity of a root: $\Phi_+ \doteq \Phi \cap Q_+ = \Phi \cap (\sum_{i \in I} \mathbf{N}\alpha_i)$, $\Phi_- \doteq -\Phi_+ = \Phi \setminus \Phi_+$, $\Phi^{\text{im}} \doteq \mathbf{Z}\delta \setminus \{0\}$ with $\delta = \sum_{i \in I} r_i\alpha_i$, $r_0 = 1$ and $\sum_{j \in I} a_{ij}r_j = 0 \ \forall i \in I$, and $\Phi^{\text{re}} = \Phi \setminus \Phi^{\text{im}}$. The multiplicity of each real root is 1, while the multiplicity of $r\delta$ can be described as follows: $\forall r \neq 0$ denote by I^r the set

$I^r \doteq \begin{cases} I_0 & \text{in case } A_{2n}^{(2)} \\ \{i \in I_0 | d_i | r\} & \text{otherwise} \end{cases}$ where $\{d_i | i \in I\}$ is the set of positive integers such that $min\{d_i | i \in I\} = 1$ and $d_i a_{ij} = d_j a_{ji} \ \forall i, j \in I$ (the d_i's exist and are uniquely determined since A is symmetrizable and indecomposable, see [6]). Then the multiplicity of $r\delta$ is $\#I^r$. Thus the set of positive roots with multiplicities, denoted by $\tilde{\Phi}_+$, can be described as $\tilde{\Phi}_+ = \Phi_+^{re} \cup \tilde{\Phi}_+^{im} = \Phi_+^{re} \cup (\cup_{r>0} \{r\delta\} \times I^r)$: remark that the condition $i \in I^r$ is equivalent to $(r\delta, i) \in \tilde{\Phi}_+$ and that $\#I^r = \begin{cases} n & \text{if } k | r \\ \frac{\tilde{n} - n}{k - 1} & \text{otherwise.} \end{cases}$

It is worth noticing that $\tilde{\Phi}_+$ is an index set for a basis of $\mathfrak{n}_+ \subseteq \mathfrak{g}$ (see [6]).

§1.2. *The quantum algebra.*

In this paragraph a short account of the quantum algebra $\mathcal{U}_q$ associated to a (twisted) affine Cartan matrix A (or to the corresponding Dynkin diagram Γ) will be given, and some of the main structures will be shortly recalled.

Definition 1.2.1.

$\mathcal{U}_q$ is the $\mathbf{C}(q)$-associative unitary algebra generated by $\{E_i, F_i, K_i^{\pm 1} | i \in I\}$ with relations:

$$[K_i, K_j] = 0, \ K_i E_j = q^{d_i a_{ij}} E_j K_i, \ K_i F_j = q^{-d_i a_{ij}} F_j K_i, \ [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}} \ \forall i, j \in I$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q^{d_i}} E_i^r E_j E_i^{1-a_{ij}-r} = 0 = \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q^{d_i}} F_i^r F_j F_i^{1-a_{ij}-r} \ \forall i \neq j \in I$$

where $\forall m \geq m', r \in \mathbf{N} \ [m]_{q^r} \doteq \frac{q^{mr} - q^{-mr}}{q^r - q^{-r}}$, $[m]_{q^r}! \doteq \prod_{s=1}^m [s]_{q^r}$, $\begin{bmatrix} m \\ m' \end{bmatrix}_{q^r} \doteq \frac{[m]_{q^r}!}{[m']_{q^r}! [m-m']_{q^r}!}$.

In the following remark some fundamental structures that $\mathcal{U}_q$ can be endowed with are listed, and the references given.

Remark 1.2.2.

a) $\mathcal{U}_q = \mathcal{U}_q^- \otimes_{\mathbf{C}(q)} \mathcal{U}_q^0 \otimes_{\mathbf{C}(q)} \mathcal{U}_q^+$ where $\mathcal{U}_q^-, \mathcal{U}_q^0, \mathcal{U}_q^+$ are the subalgebras of $\mathcal{U}_q$ respectively generated by $\{F_i | i \in I\}$, $\{K_i^{\pm 1} | i \in I\}$, $\{E_i | i \in I\}$ (see [7]);

b) $\mathcal{U}_q = \oplus_{\eta \in Q} \mathcal{U}_{q,\eta}$ is the Q-gradation induced by $E_i \in \mathcal{U}_{q,\alpha_i}$, $F_i \in \mathcal{U}_{q,-\alpha_i}$, $K_i^{\pm 1} \in \mathcal{U}_{q,0}$;

c) $\Omega : \mathcal{U}_q \to \mathcal{U}_q$ is the antilinear antiinvolution such that $E_i \overset{\Omega}{\leftrightarrow} F_i$, $K_i \overset{\Omega}{\leftrightarrow} K_i^{-1}$, $q \overset{\Omega}{\mapsto} q^{-1}$;

d) the braid group $\mathcal{B}$ acts on $\mathcal{U}_q$ and its quotient W (the Weyl group) acts on Q in such a way that $T(\mathcal{U}_{q,\eta}) = \mathcal{U}_{q,w(\eta)}$ where w is the image of T in W (see [7]);

e) $W.\{\alpha_i | i \in I\} = \Phi^{re}$; this allows to construct elements $E_\alpha \in \mathcal{U}_{q,\alpha}^+ \ \forall \alpha \in \Phi_+^{re}$ (with $E_{\alpha_i} = E_i$), which are called positive real root vectors (see [7], [1], [3]);

f) by the commutation of the E_α's ($\alpha \in \Phi_+^{re}$) the imaginary root vectors $E_{(r\delta,i)} \in \mathcal{U}_{q,r\delta}^+$ with $(r\delta, i) \in \tilde{\Phi}_+^{im}$ are defined: they generate a commutative subalgebra of $\mathcal{U}_q^+$ (see [1], [3]);

g) given a total ordering $<$ of $\tilde{\Phi}_+$ set, $\forall \eta \in Q_+$, $\mathcal{P}(\eta) \doteq \{\underline{\gamma} = (\gamma_1 \leq ... \leq \gamma_m) | \sum_{s=1}^m p(\gamma_s) = \eta\}$ (where $p : \tilde{\Phi}_+ \to \Phi_+$ is the natural projection), $par(\eta) \doteq \#\mathcal{P}(\eta)$ and, $\forall x : \tilde{\Phi}_+ \ni \alpha \mapsto x_\alpha \in \mathcal{U}_q$, $\forall \underline{\gamma} = (\gamma_1, ..., \gamma_m) \in \mathcal{P}(\eta)$, $x(\underline{\gamma}) \doteq x_{\gamma_1} \cdot ... \cdot x_{\gamma_m}$; then $\{E(\underline{\gamma}) | \underline{\gamma} \in \cup_{\eta \in Q_+} \mathcal{P}(\eta)\}$ is a basis of $\mathcal{U}_q^+$, called PBW-basis (for a discussion on the admitted orderings on $\tilde{\Phi}_+$, see [3]);

h) the Killing form $(\cdot, \cdot) : \mathcal{U}_q^{\geq 0} \times \mathcal{U}_q^{\leq 0} \to \mathbf{C}(q)$ connects the algebra and coalgebra structures of $\mathcal{U}_q$ (see [9]);

i) the ("universal") contravariant form $H : \mathcal{U}_q^- \times \mathcal{U}_q^- \to \mathcal{U}_q^0$ contains information about the commutation relations (see [5]); its restriction to $\mathcal{U}_{q,-\eta}^-$ is denoted by H_η.

§1.3. *The specialization at (odd) roots of 1.*

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The aim of this paragraph is to define the specialization of $\mathcal{U}_q$ at ε when ε is a nonzero complex number. Roughly speaking, the idea is to transform the indeterminate q into a parameter which takes values in $\mathbf{C}^*$, thus obtaining, for each value ε of the parameter, an associative $\mathbf{C}$-algebra, that will be denoted by $\mathcal{U}_\varepsilon$; this can be done in different ways: in this paper one of these different methods to specialize $\mathcal{U}_q$ is chosen (see [5]), but it can be useful to recall that other specializations (see [8]) have an essentially different behaviour exactly under the aspect that we are going to consider, the center.

Definition 1.3.1.

Let $\mathcal{U}_q$ be a quantum algebra of type $X_{\tilde{n}}^{(k)}$ and let $\mathcal{A}$ be the subalgebra of $\mathbf{C}(q)$ generated over $\mathbf{C}$ by $\{q, q^{-1}, (q^m - q^{-m})^{-1} | 1 \le m \le k\}$, i.e. $\mathcal{A} \doteq \mathbf{C}[q, q^{-1}, (q^m - q^{-m})^{-1} | 1 \le m \le k]$; the $\mathcal{A}$-subalgebra $\mathcal{U}_\mathcal{A}$ of $\mathcal{U}_q$ generated by $\{E_i, F_i, K_i^{\pm 1} | i \in I\}$ is called the integer form of $\mathcal{U}_q$. Remark that if $\varepsilon \in \mathbf{C}^*$ is such that $\varepsilon^{2m} \ne 0 \; \forall m = 1, ..., k$ than $q - \varepsilon$ is not invertible in $\mathcal{U}_\mathcal{A}$. In this case we call specialization of $\mathcal{U}_q$ at ε, and denote it by $\mathcal{U}_\varepsilon$, the quotient $\mathcal{U}_\varepsilon \doteq \mathcal{U}_\mathcal{A}/(q - \varepsilon)$.

Remark 1.3.2.

It is possible to avoid the restrictions on ε introduced for the construction above, but since we are actually interested in further restrictions it is useless, for the purpose of this paper, to look for an unnecessary generality.

Notation 1.3.3.

From now on l will denote an odd integer bigger than k, and ε a primitive l^{th} root of 1. Remark that such an ε satisfies the restrictions required in definition 1.3.1.

Proposition 1.3.4.

$\mathcal{U}_\varepsilon$ inherits from $\mathcal{U}_q$ most of its structures:

a) the triangular decomposition: $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon^- \otimes_\mathbf{C} \mathcal{U}_\varepsilon^0 \otimes_\mathbf{C} \mathcal{U}_\varepsilon^+$;

b) the Q-gradation: $\mathcal{U}_\varepsilon = \oplus_{\eta \in Q} \mathcal{U}_{\varepsilon,\eta}$;

c) the antilinear antiinvolution $\Omega : \mathcal{U}_\varepsilon \to \mathcal{U}_\varepsilon$ such that $\mathcal{U}_\varepsilon^+ \overset{\Omega}{\leftrightarrow} \mathcal{U}_\varepsilon^-$;

d) The structure of $\mathcal{U}_\varepsilon^0$: $\mathcal{U}_\varepsilon^0 = \mathbf{C}[K_i^{\pm 1} | i \in I]$; $\forall \lambda = \sum_{i \in I} m_i \alpha_i \in Q$, $K_\lambda \doteq \prod_{i \in I} K_i^{m_i}$;

e) the braid group action, the root vectors and their basis-properties (PBW basis).

Moreover recall that $(q - \varepsilon) | [m]_{q^r}$ (that is $[m]_{q^r} = 0$ in $\mathcal{U}_\varepsilon$) $\Leftrightarrow l | mr$ and $l \nmid r$, and that $(q - \varepsilon)^2$ never divides $[m]_{q^r}$ (if $m \ne 0$); given an element a in an $\mathcal{A}$-algebra, $mult_\varepsilon a$ denotes the multiplicity of ε in a (that is $(q - \varepsilon)^{mult_\varepsilon a} || a$).

Proof: See [4], paragraph 1.D1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

2. AN UPPER BOUND FOR $\dim(\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_{\varepsilon,\eta}^+)$.

The strategy followed to describe the center of the algebras that we are dealing with passes through the investigation of its positive part, which is the goal of this and the next section; it consists in exhibiting a family of central elements, proving that they constitute a set of generators and showing that there are no relations among them.

Exhibiting a first set of central elements is not difficult: it can be done thanks to the results on the commutation relations between the root vectors and to the commutation properties of the elements K_λ's. Analogously, it is thanks to the PBW basis (see [3], section 6) that we can deduce that there are no relations among these central root vectors.

Thus the starting point of this section is an estimate of the dimension of the center, or better of the homogeneous components of its positive part, which could allow us to say

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that the number of independent central elements cannot be too big: this estimate comes from the comparison of $\dim(\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}^+_{\varepsilon,\eta})$ with the multiplicity of ε in $\det H_\eta$.

Proposition 2.1.

Suppose given $\{x_\alpha | \alpha \in \tilde{\Phi}_+\}$, with $x_\alpha \in \mathcal{U}^+_{\mathcal{A},\alpha}$, such that:

a) $\forall \eta \in Q_+$ $\{x(\underline{\gamma}) | \underline{\gamma} \in \mathcal{P}(\eta)\}$ is a basis of $\mathcal{U}^+_{\varepsilon,\eta}$;

b) $\exists f : J \to \mathbf{Z}_+$ (with $J \subseteq \tilde{\Phi}_+$) such that $x_\alpha^{f(\alpha)} \in \mathcal{Z}(\mathcal{U}_\varepsilon)$ $\forall \alpha \in J$;

c) $\forall \eta \in Q_+$ $mult_\varepsilon \det H_\eta \leq \sum_{\alpha \in J, m>0} par(\eta - mf(\alpha)p(\alpha))$;

then $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}^+_\varepsilon$ is the algebra of polynomials in $\{x_\alpha^{f(\alpha)} | \alpha \in J\}$.

Proof: See [2] (corollary 3.2.4 and proposition 3.2.5) or equivalently [4] (proposition 3.2.12). $\qquad \square$

The aim is now to find elements x_α's satisfying the conditions of proposition 2.1.

Lemma 2.2.

a) $\forall \alpha \in \tilde{\Phi}^{\mathrm{re}}_+$ the element $E_\alpha^{l_\alpha}$ is central in $\mathcal{U}_\varepsilon$, where $l_\alpha \doteq \frac{l}{g.c.d.(l,d_\alpha)}$ and $d_\alpha \doteq d_i$ where $i \in I$ is such that $\exists w \in W$ with $\alpha = w(\alpha_i)$;

b) $\forall i \in I_0$, $\forall r > 0$ the root vector $E_{(lr\delta,i)}$ is central in $\mathcal{U}_\varepsilon$.

Proof: Part a) is the immediate generalization to the real root vectors of a classical result by Kac (see [5]). For part b) see the commutation formulas in [3] (theorem 5.3.2). $\qquad \square$

Remark 2.3.

A comparison between proposition 2.1 and lemma 2.2 suggests to look for a set J containing $J' \doteq \Phi^{\mathrm{re}}_+ \cup \{(lr\delta,i) | i \in I_0, r > 0\}$ and for a function $f : J \to \mathbf{Z}_+$ such that $f(\alpha) = l_\alpha$ $\forall \alpha \in \Phi^{\mathrm{re}}_+$ and $f((lr\delta,i)) = 1$ $\forall i \in I_0, r > 0$; of course following this suggestion we shall have $x_\alpha = E_\alpha$ $\forall \alpha \in J'$.

Proposition 2.4.

$\forall \eta \in Q_+$ the multiplicity of ε in $\det H_\eta$ is less than or equal to

$$\sum_{\substack{\alpha \in \Phi^{\mathrm{re}}_+ \\ m>0}} par(\eta - ml_\alpha\alpha) + \sum_{r,m>0} mult_\varepsilon(\det \mathcal{H}^r) par(\eta - mr\delta)$$

where $\mathcal{H}^r$ is the matrix defined by $\mathcal{H}^r \doteq (E_{(r\delta,i)}, F_{(r\delta,j)})_{ij \in I^r}$ (and $F_\alpha \doteq \Omega(E_\alpha)$ $\forall \alpha \in \tilde{\Phi}_+$).

Proof: For general affine quantum algebras we have that the highest coefficient of $\det H_\eta$ is, up to an invertible element of $\mathbf{C}[q,q^{-1}]$,

$$\prod_{\substack{\alpha \in \Phi^{\mathrm{re}}_+ \\ m>0}} \left(\frac{[m]_{q^{d_\alpha}}}{q^{d_\alpha} - q^{-d_\alpha}}\right)^{par(\eta - m\alpha)} \cdot \left(\frac{\prod_{(r\delta,i) \in \tilde{\Phi}^{\mathrm{im}}_+}(E_{(r\delta,i)}, \bar{F}_{(r\delta,i)})}{\prod_{(r\delta,i) \in \tilde{\Phi}^{\mathrm{im}}_+} A^{(r)}_{ii}}\right)^{\sum_{m>0} par(\eta - mr\delta)}$$

where $\bar{F}_{(r\delta,i)} - A^{(r)}_{ii} F_{(r\delta,i)}$ lies in the linear span of $\{F_{(r\delta,j)} | j > i\}$ and $(E_{(r\delta,i)}, \bar{F}_{(r\delta,j)}) = 0$ $\forall i > j$. Indeed this is nothing but a reformulation of theorem 2.5.4 of [2] (which in this generality does not depend on the peculiar characteristics of the untwisted algebras, but is valid for all the affine cases), remarking the connection between the Killing form and the contravariant form: if x and y belong to the linear span of the imaginary root vectors, $x \in \mathcal{U}^+_{q,r\delta}$, $y \in \mathcal{U}^-_{q,-r\delta}$, then $[x,y] = -(x,y)(K_{r\delta} - K^{-1}_{r\delta})$ (see corollary 7.2.4 of [3]).

On the other hand $\forall r > 0$ $\frac{\prod_{i \in I^r}(E_{(r\delta,i)}, \bar{F}_{(r\delta,i)})}{\prod_{i \in I^r} A^{(r)}_{ii}}$ is evidently $\det \mathcal{H}^r$ because the matrix of passage from $\{\bar{F}_{(r\delta,i)} | i \in I^r\}$ to $\{F_{(r\delta,i)} | i \in I^r\}$ is triangular. The claim then follows remarking that the multiplicity of ε in $\det H_\eta$ is less than or equal to the multiplicity of ε in the highest coefficient of $\det H_\eta$. $\qquad \square$

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Remark 2.5.

Comparing proposition 2.1, remark 2.3 and proposition 2.4, we see that for the real part they coincide. So we can concentrate our attention on the imaginary root vectors, going into the details of the twisted cases.

Lemma 2.6.

In the twisted algebras the multiplicity of ε in $\det\mathcal{H}^r$ is given by:

$$mult_\varepsilon(\det\mathcal{H}^r) = \begin{cases} \#I_0 & \text{if } l|r \\ 1 & \text{if } l \nmid r,\ 2 \nmid r, \text{ and } l|(2n+1)r \text{ in case } A_{2n}^{(2)} \text{ or} \\ & \qquad l|(\tilde{n}-n+1)r \text{ in cases } A_{2n-1}^{(2)} \text{ and } E_6^{(2)} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We have that, up to elements which give no contribution to the multiplicity of ε,

$$\det\mathcal{H}^r = [r]_q^{\#I^r} \cdot \begin{cases} [2n+1]_{q^r} & \text{if } 2 \nmid r \text{ in case } A_{2n}^{(2)} \\ [\frac{\tilde{n}-n}{k-1}+1]_{q^r} & \text{if } k \nmid r \text{ in the other twisted cases} \\ 1 & \text{otherwise:} \end{cases}$$

see [3] (theorem 5.3.2, notation 7.1.3, lemma 7.1.4, corollary 7.2.4). The claim follows immediately remarking that in the case $D_{\tilde{n}}^{(k)}$ we have $\frac{\tilde{n}-n}{k-1}=1$. $\qquad\square$

Remark 2.7.

From the comparison between proposition 2.1, remark 2.3 and lemma 2.6 we can see that they agree for what concerns the roots $(r\delta,i)$ with $l|r$. But we see also that $J \neq J'$: more precisely, in order to be able to apply proposition 2.1 we must have $J = J' \cup J''$, where J'' must be of the form

$$J'' = \begin{cases} \{(r\delta,i_*)|2 \nmid r, l \nmid r, l|(2n+1)r\} & \text{in case } A_{2n}^{(2)} \\ \{(r\delta,i_*)|2 \nmid r, l \nmid r, l|(\tilde{n}-n+1)r\} & \text{in cases } A_{2n-1}^{(2)} \text{ and } E_6^{(2)} \\ \emptyset & \text{in cases } D_{n+1}^{(2)} \text{ and } D_4^{(3)}; \end{cases}$$

and i_* must be an element of I^r; at the same time we must have $f(\alpha) = 1\ \forall\alpha \in J''$. This means that we must look for a nonzero central element in the span of $\{E_{(r\delta,i)}|i \in I^r\}$ when $2 \nmid r, l \nmid r, l|(2n+1)r$ in case $A_{2n}^{(2)}$ and when $2 \nmid r, l \nmid r, l|(\tilde{n}-n+1)r$ in cases $A_{2n}^{(2)}$ and $E_6^{(2)}$. The results of this section prove that once that we have found these central elements, we have described $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+$.

3. THE POSITIVE PART OF THE CENTER: $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+$.

The present section is devoted to conclude the program illustrated in section 2, that is to go into the details in order to exhibit the central vectors that we are still missing.

Remark 3.1.

Let r be as described at the end of remark 2.7; then we are looking for an index $i_* \in I^r$ and for an element $E_{(r\delta,i_*)}^* = \sum_{i\in I^r} A_i^{(r)} E_{(r\delta,i)}$ such that:

a) $E_{(r\delta,i_*)}^*$ is linearly independent of $\{E_{(r\delta,i)}|i \in I^r \setminus \{i_*\}\}$ in $\mathcal{U}_\varepsilon$;

b) $E_{(r\delta,i_*)}^* \in \mathcal{Z}(\mathcal{U}_\varepsilon)$.

The above conditions a) and b) can be translated into the following ones:

a') $A_{i_*}^{(r)} \neq 0$ in $\mathcal{U}_\varepsilon$;

b') $(E_{(r\delta,i_*)}^*, F_{(r\delta,i)}) = 0$ in $\mathcal{U}_\varepsilon\ \forall i \in I^r$.

Indeed that a) and a') are equivalent is obvious; the equivalence between b) and b') is a straightforward consequence of the commutation relations involving an imaginary root

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vector (see [3], theorem 5.3.2) and of the already mentioned connection between the bracket and the Killing form (see the proof of proposition 2.4). But the computations needed to find elements satisfying a') and b') have already been carried out in [3], so that now we have just to recall the result.

Proposition 3.2.

Let $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$, $2 \nmid r$, $l \nmid r$, $l|(2n+1)r$, and let

$$E_{(r\delta,i_*)}^* = [n]_{q^{2r}} E_{(m\delta,1)} - \sum_{i \in I_0\setminus\{1\}} (-1)^r [2]_q [n-i+1]_{q^r} E_{(m\delta,i)};$$

then $E_{(r\delta,i_*)}^*$ is central in $\mathcal{U}_\varepsilon$; morover $i_* = n$ satisfies the requirement of a').

Proof: See [3], lemma 7.4.1, remark 7.4.2 and propositions 7.4.7 and 7.5.2. As for the assertion on i_* it is enough to remark that the hypotheses imply that $l \nmid 2rn$. $\qquad \square$

Proposition 3.3.

Let $X_{\tilde{n}}^{(k)} = A_{2n-1}^{(2)}$ or $E_6^{(2)}$, $2 \nmid r$, $l \nmid r$, $l|(\tilde{n}-n+1)r$, and let

$$E_{(r\delta,i_*)}^* = \sum_{i \in I^r} (-1)^r [\nu-i+1]_{q^r} \quad \text{where} \quad \nu \doteq \begin{cases} \tilde{n}-n+1 = n & \text{in case } A_{2n-1}^{(2)} \\ \tilde{n}-n = 2 & \text{in case } E_6^{(2)}; \end{cases}$$

then $E_{(r\delta,i_*)}^*$ is central in $\mathcal{U}_\varepsilon$; morover $i_* = \nu$ satisfies the requirement of a').

Proof: The references given for proposition 3.2 are still valid, but it is worth remarking that in these cases the commutation relations involving an imaginary root vector are exactly those of $A_{\tilde{n}-n}^{(1)}$ (see [3], remark 7.4.5), so one can also refer to [2], proposition 3.3.7. The assertion about i_* is trivial. $\qquad \square$

In conclusion the results found above lead to the explicit description of $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+$.

Corollary 3.4.

$\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+$ is a **C**-algebra of polynomials in an infinite set of variables; more precisely

$$\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+ = \mathbf{C}[E_\alpha^{l_\alpha}, E_{(lr\delta,i)}, E_\beta^* | \alpha \in \Phi_+^{\mathrm{re}}, r > 0, i \in I_0, \beta \in J'']$$

where J'' is as defined in remark 2.7, with $i_* = n$ if $X_{\tilde{n}}^{(k)} = A_{\tilde{n}}^{(2)}$, $i_* = 2$ in case $E_6^{(2)}$, and the elements E_β^*'s are the ones described in propositions 3.2 and 3.3.

Proof: This is the natural conclusion of proposition 2.1, lemma 2.2, proposition 2.4, lemma 2.6 and propositions 3.2 and 3.3. $\qquad \square$

4. THE CENTER.

Here we pass from the results obtained till now to the description of the whole center: $\mathcal{Z}(\mathcal{U}_\varepsilon)$ will finally turn out to be, "essentially", an algebra of polynomials (of course in an infinite number of variables), with just one relation, regarding its null part. Since we already know $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+$ (then, by symmetry, we also know $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^-$), we are just left with the task of describing $\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^0$ (which is trivial) and of understanding how the structure of $\mathcal{Z}(\mathcal{U}_\varepsilon)$ can be directly found out from that of its positive, negative and null parts.

Lemma 4.1.

$\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^0$ is the subalgebra of $\mathcal{U}_\varepsilon^0$ generated by $\{K_i^{l_i}, K_\delta | i \in I\}$, where $l_i \doteq l_{\alpha_i}$. Remark that while $\{K_i^{l_i} | i \in I\}$ is a set of algebraically independent elements, there is a relation between them and K_δ: namely, $\prod_{i \in I}(K_i^{l_i})^{\frac{lr_i}{l_i}} = K_\delta^l$.

Proof: The claim is obvious. $\qquad \square$

Proposition 4.2.
$\mathcal{Z}(\mathcal{U}_\varepsilon) = (\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^-) \otimes (\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^0) \otimes (\mathcal{Z}(\mathcal{U}_\varepsilon) \cap \mathcal{U}_\varepsilon^+)$.
Proof: See [2], theorem 3.4.3, or equivalently [4], theorem 3.5.6. Remark that even though the given references relate to the untwisted affine setting, the proof of the general assertion that we are dealing with never makes use of the particular form of the untwisted type algebras but it depends only on the existence of two strings of real root vectors and on some properties (of the imaginary root vectors and of some commutation rules) that are common to all the affine situations; the claim is then valid also in the twisted cases. On the other hand it is worth noticing that this result is not true for the quantum algebras of finite type (these are the quantization of the enveloping algebras of simple finite dimensional complex Lie algebras), where there are also the Casimir elements, that is central elements which cannot be decomposed as algebraic combinations of "positive", "negative" and "null" central elements. □

Theorem 4.3.
Let $\mathcal{U}_\varepsilon$ be the specialization at ε of an affine quantum algebra of twisted type $X_{\tilde{n}}^{(k)}$, with $\varepsilon \in \mathbf{C}$ primitive l^{th} of 1 and l odd integer bigger than k. Then the center of $\mathcal{U}_\varepsilon$ is

$$\mathcal{Z}(\mathcal{U}_\varepsilon) = [E_\alpha^{l_\alpha}, E_{(lr\delta,i)}, E_\beta^*, F_\alpha^{l_\alpha}, F_{(lr\delta,i)}, F_\beta^*, K_j^{l_j}, K_\delta | \alpha \in \Phi_+^{\text{re}}, r > 0, i \in I_0, j \in I, \beta \in J'']/(P_Z)$$

where J'' and E_β^* are those of corollary 3.4, $F_\beta^* \doteq \Omega(E_\beta^*)$ and $P_Z \doteq K_\delta^l - \prod_{i \in I}(K_i^{l_i})^{\frac{lr_i}{T_i}}$.
Proof: The theorem is the straightforward consequence of corollary 3.4, lemma 4.1 and proposition 4.2. □

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