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IDENTITY PRINCIPLES FOR COMMUTING HOLOMORPHIC SELF-MAPS OF THE UNIT DISC

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ABSTRACT. - Let f, g be two commuting holomorphic self-maps of the unit disc. If f and g agree at the common Wolff point up to a certain order of derivatives (no more than 3 if the Wolff point is on the unit circle), then $f \equiv g$.

1. INTRODUCTION

The non-constant holomorphic map from the unit disc Δ of $\mathbb{C}$ into $\mathbb{C}$ given by $z \mapsto \exp(-(i\frac{z+1}{z-1})^{\frac{1}{3}})$ is C^∞ up to the boundary and it has all derivatives at 1 equal to zero. In particular then there exist holomorphic mappings from the unit disc which extend smoothly to the boundary and which coincide up to any order at a given point of the boundary but which are not identically equal. The problem of finding suitable additional conditions (both of geometrical and analytical flavour) for a holomorphic map to be constant if it behaves like a constant at a boundary point has been studied by several authors (*e.g.* Bell-Lempert [BeLe], Alinhac-Baouendi-Rothschild [ABR], Alexander [Al1], [Al2], Huang-Krantz [HuKr]). Their methods however don't give conditions under which two holomorphic maps (not necessarily identically constant) must be identical if they coincide at a boundary point up to any order. Recently Burns and Krantz [BuKr] and the two last quoted authors [TaVl] gave conditions on derivatives at a boundary point for a holomorphic self-map of the unit disc to be identically equal to the identity. In particular they stated that a holomorphic self-map of Δ is the identity map if it coincides with the identity up to the third order of expansion at a boundary point.

In these notes we prove that two commuting holomorphic self-maps of Δ which have the same expansions up to the third order at their common (boundary) *Wolff point* are in fact identically equal. We will see that the order three is necessary only in a particular case (which contains the case studied by Burns and Krantz).

By Schwarz Lemma (and its boundary versions) a holomorphic self-map f of Δ has a simple dynamical behavior, *i.e.* the sequence of iterates of f, $\{f^k\}$, converges (in any topology in $\mathrm{Hol}(\Delta, \overline{\Delta})$) to a unique point, called the *Wolff point* of f. The Wolff point of f is the fixed point of f if f has one in Δ, otherwise it belongs to $\partial\Delta$. From the end of the nineteenth century it has been known that if the Wolff point of

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f is in Δ then almost all the information are contained in the first derivative of f at that point. In a certain way that is true also for boundary Wolff points. Moreover, thanks to the so-called Behan-Shields Theorem, two commuting holomorphic self-maps of Δ have the same Wolff point unless they are hyperbolic automorphisms. Then it seems to be natural to study identity principles for commuting holomorphic maps at their common Wolff point (for the sake of completeness we will also deal with the cases of interior fixed points and of automorphisms).

The techniques essentially used are based on the possibility of building *representative fractional linear models* for holomorphic self-maps of Δ. Namely, given $f \in \mathrm{Hol}(\Delta, \Delta)$, there exists a "change of coordinates" in a neighborhood of the Wolff point of f such that after this conjugation f looks like an automorphism of the right half-plane. The construction of such a model is due, in several steps and with different degrees of generality, to Valiron [Va], Pommerenke [Po], Pommerenke and Baker [BaPo], Cowen [Co1] and Bourdon and Shapiro [BoSh]. In order to handle the fractional linear models in the case of a boundary Wolff point, one of the main problems is that the "intertwining map" (*i.e.* the change of coordinates) has no regularity *a priori* at the Wolff point of f. To get the necessary regularity in order to transfer the information on the derivatives of f to the automorphism, we need some regularity of f at its Wolff point. It turns out that the regularity requested on f at its Wolff point increases according to how much f is "near" to the identity. Another problem we have to deal with is the representativeness of the model. Notwithstanding the model does always exist, sometimes it is not well representative, in the sense that the dynamical behavior of the automorphism is completely different from that of f. For instance if $f(1) = 1$, $f'(1) = 1$ and $f''(1) = 0$ then the automorphism associated to f tends to its Wolff point faster than f. To overcome this difficulty we build another model, which is no more global and linear (we call it *partial fractional linear model*) but which is representative in the sense of the dynamical behavior. Once we have these models, we can transfer the information on the derivatives of f to the parameters defining the associated automorphism. Now every holomorphic mapping which has the same model of f with the same automorphism is identically equal to f. Since mappings which commute with f and agree with it up to a certain order (depending on the model) have this property, we are then able to prove the identity principle.

The paper is organized as follows. In section 2 we introduce some notations we will use through out the paper and, after some basic preliminary results, we state our main result (Theorem 2.4). The remaining sections are more or less devoted to the proof of that result. In particular in section 3 we deal with the case of interior fixed points. In section 4 we discuss some tools for handling the fixed point free case (boundary derivatives, fractional linear models and pseudo-iteration semigroups). Section 5 concerns about the case of automorphisms. In section 6 we deal with the case of hyperbolic non-automorphism mappings (*i.e.* mappings with derivative < 1 at their Wolff points). Section 7 and section 8 are devoted to the case of parabolic non-automorphism mappings, *i.e.* maps with derivative 1 at their Wolff points. In section 9 we discuss the representativeness of models, stressing out that our previous construction can be regarded as a new fractional linear model for the parabolic case.

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2. Notations and Statement of the Main Result

In this section we state our main result. Before that we need to recall some facts on holomorphic self-maps of Δ and to introduce some notations.

Given $r \in \mathbb{R}$ we denote the shifted half-plane by $H_r \stackrel{\text{def}}{=} \{w \in \mathbb{C} \ : \ \text{Re} w > r\}$; in particular $H \stackrel{\text{def}}{=} H_0$. We recall that the *Cayley transformation* $C(z) \stackrel{\text{def}}{=} \frac{1+z}{1-z}$ is a biholomorphism between Δ and H which maps 1 to ∞. We will denote the m-th derivative of f at $z_0 \in \Delta$ by $f^{(m)}(z_0)$, the m-th iterate of f by f^m (where $f^m \stackrel{\text{def}}{=} f \circ f^{m-1}$ and $f^1 \stackrel{\text{def}}{=} f$) and the m-th power of f by $[f]^m$.

We recall that every automorphism γ of Δ is of the form

$$\gamma(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

with $a \in \Delta$ and $\theta \in \mathbb{R}$. Notice that γ extends analytically to a homeomorphism of $\overline{\Delta}$. It is easy to see that every $\gamma \in \text{Aut}(\Delta)$ different from the identity map id_Δ has at most two fixed points in $\overline{\Delta}$. More precisely γ is called *elliptic* if it has a (unique) fixed point in Δ, *parabolic* if it has a unique fixed point on $\partial\Delta$, *hyperbolic* if it has two distinct fixed point on $\partial\Delta$. It can be shown (see *e.g.* [Ab]) that if γ is a hyperbolic automorphism with fixed points $\tau_1, \tau_2 \in \partial\Delta$ then $\gamma'(\tau_1)$, $\gamma'(\tau_2)$ are two positive real numbers, different from 1 such that their product is 1. If γ is a parabolic automorphism with fixed point $\tau \in \partial\Delta$ then $\gamma'(\tau) = 1$.

Theorem 2.1 (Schwarz-Wolff). *Let $f \in \text{Hol}(\Delta, \Delta) \setminus \{\text{id}_\Delta\}$. Suppose f is not an elliptic automorphism of Δ. Then there exists a unique point $\tau \in \overline{\Delta}$ such that the sequence $\{f^k\}$ converges to τ uniformly on compact subsets of Δ. Moreover $\tau \in \Delta$ if and only it is the (only) fixed point of f.*

Definition 2.2. If $f \in \text{Hol}(\Delta, \Delta) \setminus \{\text{id}_\Delta\}$ is not an elliptic automorphism of Δ, we call *Wolff point* of f the point τ introduced by the previous theorem. If f is an elliptic automorphism of Δ then we call the Wolff point of f the unique fixed point of f.

It can be shown that if $f \in \text{Hol}(\Delta, \Delta)$ has Wolff point $\tau \in \partial\Delta$ then $\lim_{r\to 1^-} f'(r\tau)$ is a strictly positive real number less or equal to 1. Recall now the following (see [Be], [Sh]):

Theorem 2.3 (Behan-Shields). *Let $f, g \in \text{Hol}(\Delta, \Delta) \setminus \{\text{id}_\Delta\}$. If $f \circ g = g \circ f$ then f and g have the same Wolff point unless f, g are two hyperbolic automorphisms of Δ with the same fixed points.*

If $f \in \text{Hol}(\Delta, \Delta)$ and $\tau \in \partial\Delta$, we use the notation $f \in C^k(\tau)$ if $f^{(j)}$ extends continuously to $\Delta \bigcup \{\tau\}$ for $j = 1, \ldots, k$. In other words f has an expansion of the form

$$f(z) = f(\tau) + f'(\tau)(z-\tau) + \ldots + \frac{1}{k!}f^{(k)}(\tau)(z-\tau)^k + \Gamma(z),$$

for $z \in \Delta$, where $\Gamma(z) = o(|z-\tau|^k)$. Moreover, we say that $f \in C^{k+\epsilon}(\tau)$ if $f \in C^k(\tau)$ and $\Gamma(z) = O(|z-\tau|^{k+\epsilon})$.

Now we can state our main result:

Theorem 2.4. *Let $f, g \in \text{Hol}(\Delta, \Delta)$ be such that $f \circ g = g \circ f$.*

(1) *If there exist $z_0 \in \Delta$ and $k > 0$ natural number such that $f(z_0) = g(z_0) = z_0$, $f^{(m)}(z_0) = g^{(m)}(z_0) = 0$ for $1 \leq m < k$ and $f^{(k)}(z_0) = g^{(k)}(z_0) \neq 0$ then $f \equiv g$.*

(2) *If g is a hyperbolic automorphism of Δ with a fixed point $\tau \in \partial\Delta$ and* $\lim_{r\to 1^-} f'(r\tau) = g'(\tau)$ *then* $f \equiv g$.
(3) *If g is a parabolic automorphism of Δ with the fixed point $\tau \in \partial\Delta$, and* $\lim_{r\to 1^-} f'(r\tau) = g'(\tau)$, $\lim_{r\to 1^-} f''(r\tau) = g''(\tau)$ *then* $f \equiv g$.
(4) *If* $g \equiv \mathrm{id}_\Delta$ *and there exists* $\tau \in \partial\Delta$ *such that* $\lim_{r\to 1^-} f(r\tau) = \tau$, $\lim_{r\to 1^-} f'(r\tau) = 1$, $\lim_{r\to 1^-} f''(r\tau) = 0$ *and* $\lim_{r\to 1^-} f'''(r\tau) = 0$ *then* $f \equiv \mathrm{id}_\Delta$.
(5) *If* $f, g \notin \mathrm{Aut}(\Delta)$ *have Wolff point* $\tau \in \partial\Delta$ *and* $\lim_{r\to 1^-} f'(r\tau) = \lim_{r\to 1^-} g'(r\tau) < 1$ *then* $f \equiv g$.
(6) *If* $f, g \notin \mathrm{Aut}(\Delta)$ *have Wolff point* $\tau \in \partial\Delta$, $f \in C^{3+\epsilon}(\tau)$, $g \in C^2(\tau)$ *and* $f'(\tau) = g'(\tau) = 1$, $f''(\tau) = g''(\tau) \neq 0$ *then* $f \equiv g$.
(7) *If* $f, g \notin \mathrm{Aut}(\Delta)$ *have Wolff point* $\tau \in \partial\Delta$, $f \in C^{5+\epsilon}(\tau)$, $g \in C^4(\tau)$ *and* $f'(\tau) = g'(\tau) = 1$, $f''(\tau) = g''(\tau) = 0$, $f'''(\tau) = g'''(\tau)$ *then* $f \equiv g$.

Remark 2.5. The above theorem deals with all possible cases. The statement *(4)*, which is a slightly improved version of the Burns-Krantz Theorem (see [BuKr]), follows also from *(7)*, but we stated it separately to make clear that there are no more cases left.

3. The Fixed Point Case

Let f and g be two commuting holomorphic self-maps of Δ. If there is a point z_0 in Δ such that $f(z_0) = z_0$ then

$$g(z_0) = g(f(z_0)) = f(g(z_0)).$$

Hence either $g(z_0) = z_0$ or f has two distinct fixed points in Δ and by Schwarz Lemma $f \equiv \mathrm{id}_\Delta$.

The aim of this section is to prove the first part (more or less already known) of Theorem 2.4:

Proposition 3.1. *Let* $f, g \in \mathrm{Hol}(\Delta, \Delta)$ *be such that* $f \circ g = g \circ f$. *If there exist* $z_0 \in \Delta$ *and* $k > 0$ *natural number such that* $f(z_0) = g(z_0) = z_0$, $f^{(m)}(z_0) = g^{(m)}(z_0) = 0$ *for* $1 \leq m < k$ *and* $f^{(k)}(z_0) = g^{(k)}(z_0) \neq 0$ *then* $f \equiv g$.

Proof. Up to conjugating f and g by a suitable automorphism of Δ, we can assume that $z_0 = 0$ without loss of generality. The Schwarz Lemma states that $|f'(0)| \leq 1$; more precisely $|f'(0)| = 1$ if and only if $f(z) = f'(0)\cdot z$. Assume first that $|f'(0)| = 1$. Since $g'(0) = f'(0)$ then $g(z) = f'(0) \cdot z$ and $f \equiv g$.

Suppose now $0 < |f'(0)| < 1$. Then, by Königs Linearization Theorem (see *e.g.* [Co1], or [CaGa]), there exists a holomorphic change of coordinates $\sigma_f \in \mathrm{Hol}(\Delta, \mathbb{C})$ such that $\sigma_f(0) = 0$ and

$$\sigma_f(f(z)) = f'(0) \cdot \sigma_f(z) \qquad \forall z \in \Delta.$$

Furthermore, if $\widetilde{\sigma}_f$ is another such a change of coordinates then $\widetilde{\sigma}_f = \lambda \cdot \sigma_f$ with $\lambda \neq 0$. Since $f \circ g = g \circ f$ then

$$\sigma_f(g(f(z))) = \sigma_f(f(g(z))) = f'(0) \cdot \sigma_f(g(z)) \qquad \forall z \in \Delta.$$

Moreover $\sigma_f(g(0)) = \sigma_f(0) = 0$. Therefore, since $g'(0) \neq 0$, $\widetilde{\sigma}_f \stackrel{\text{def}}{=} \sigma_f \circ g$ is a holomorphic change of coordinates such that $\widetilde{\sigma}_f(0) = 0$ and

$$\widetilde{\sigma}_f(f(z)) = f'(0) \cdot \widetilde{\sigma}_f(z) \qquad \forall z \in \Delta;$$

hence

$$\sigma_f \circ g = \widetilde{\sigma}_f = \lambda \cdot \sigma_f.$$

By taking the derivatives of both sides we get

$$\sigma_f'(g(z)) \cdot g'(z) = \lambda \cdot \sigma_f'(z) \qquad \forall z \in \Delta$$

and since $g(0) = 0$ and $\sigma_f'(0) \neq 0$ we conclude that $g'(0) = \lambda$, or, in other words, that

$$\widetilde{\sigma}_f = \sigma_f \circ g = g'(0) \cdot \sigma_f.$$

Therefore, since $g'(0) = f'(0)$, because of the invertibility of σ_f near 0, we actually obtain that $g \equiv f$ in Δ.

Suppose now that there exists $k > 1$ such that $f^{(m)}(0) = g^{(m)}(0) = 0$ for $m < k$ and $f^{(k)}(0) = g^{(k)}(0) \neq 0$. Then, due to Böttcher Theorem (see [Co2] or [CaGa]), there exists a local change of coordinates σ_f in a neighborhood of 0 such that $\sigma_f(0) = 0$ and

$$\sigma_f(f(z)) = [\sigma_f(z)]^k. \tag{3.1}$$

Since $f \circ g = g \circ f$ then, again by Böttcher Theorem, there exist n positive integer and ω complex number with $\omega^{k-1} = 1$ such that

$$\sigma_f(g(z)) = \omega \cdot [\sigma_f(z)]^n \tag{3.2}$$

in some neighborhood of 0 (see Theorem 3.1 in [Co2]). We can assume that $n \geq k$, otherwise we swap f and g. By taking the k-th derivative of (3.2) at 0, keeping in mind that $\sigma_f(0) = 0$ and $g^{(m)}(0) = 0$ for $m < k$, we find

$$g^{(k)}(0) \cdot \sigma_f'(0) = \omega \cdot n \cdot \ldots \cdot (n - k + 1) \cdot [\sigma_f(0)]^{n-k} \cdot \sigma_f'(0). \tag{3.3}$$

Since the left-hand side term is not 0, it follows that $n = k$. Moreover, since $g^{(k)}(0) = f^{(k)}(0)$, from (3.1) and (3.3) we have $\omega = 1$. Therefore from the local invertibility of σ_f we get $f \equiv g$. $\square$

Remark 3.2. The two maps $z \mapsto [z]^m$, $z \mapsto [z]^n$, for $m \neq n$ natural numbers, have expansions which coincide up to the $\min\{m, n\}$ order at 0 but they are different.

4. Preliminaries for Fixed Point Free Case

Given $\tau \in \partial\Delta$ and $R > 0$, the *horocycle* $E(\tau, R)$ of center τ and (hyperbolic) radius R is the disc in Δ of (euclidean) radius $R/(R+1)$ tangent to $\partial\Delta$ in τ which is defined as

$$E(\tau, R) \stackrel{\text{def}}{=} \left\{ z \in \Delta \, : \, \frac{|\tau - z|^2}{1 - |z|^2} < R \right\},$$

with the convention that $E(\tau, +\infty) = \Delta$.

For $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\tau_1, \tau_2 \in \partial\Delta$, we define $\beta_f(\tau_1, \tau_2)$ to be the following strictly positive real number (possibly $+\infty$):

$$\beta_f(\tau_1, \tau_2) \stackrel{\text{def}}{=} \sup_{z \in \Delta} \left\{ \frac{|\tau_2 - f(z)|^2}{1 - |f(z)|^2} \Bigg/ \frac{|\tau_1 - z|^2}{1 - |z|^2} \right\}.$$

The number $\beta_f(\tau_1, \tau_2)$ says how horocycles centered at τ_1 behave under the action of f, *i.e.* by definition, for any $R > 0$

$$f(E(\tau_1, R)) \subset E(\tau_2, \beta_f(\tau_1, \tau_2) R). \tag{4.1}$$

For $f \in \mathrm{Hol}(\Delta, \mathbb{C})$, $l \in \mathbb{C} \cup \{\infty\}$ is the *non-tangential limit* of f at $\tau \in \partial\Delta$ if $f(z)$ tends to l as z tends to τ in Δ within an angular sector of vertex τ and opening less than π. We summarize this definition by writing

$$\operatorname*{K-lim}_{z \to \tau} f(z) = l.$$

By Lindelöf Principle (see *e.g.* [Ab]) if $f \in \mathrm{Hol}(\Delta, \Delta)$ has radial limit l at $\tau \in \partial\Delta$, then f actually has non-tangential limit l at τ.
We recall the following fundamental theorem (see *e.g.* [Ca] or [Ab]):

Theorem 4.1 (Julia-Wolff-Carathéodory). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\tau_1, \tau_2 \in \partial\Delta$. Then*

$$\operatorname*{K-lim}_{z \to \sigma} \frac{\tau_2 - f(z)}{\tau_1 - z} = \tau_2 \overline{\tau_1} \beta_f(\tau_1, \tau_2).$$

If $\beta_f(\tau_1, \tau_2)$ is finite, then

$$\operatorname*{K-lim}_{z \to \tau_1} f(z) = \tau_2 \quad \textit{and} \quad \operatorname*{K-lim}_{z \to \tau_1} f'(z) = \tau_2 \overline{\tau_1} \beta_f(\tau_1, \tau_2).$$

In the sequel we will also use this lemma (for a simple proof see *e.g.* [Ab] or [Co1]):

Lemma 4.2 (Noshiro). *If U is a convex open subset of $\mathbb{C}$ and f is a holomorphic map on U such that $\mathrm{Re}(f'(z)) > 0$ for all $z \in U$, then f is univalent on U.*

Now we introduce the fractional linear models and their relationships with commuting holomorphic maps.

Definition 4.3. A set $V \subset \Delta$ is said to be *fundamental* for $f \in \mathrm{Hol}(\Delta, \Delta)$ if V is connected, simply connected, $f(V) \subseteq V$ and for all K compact subsets of Δ there exists a natural number n (depending on K) such that $f^n(K) \subset V$.

Definition 4.4. A triple (Ω, σ, Φ) is a *fractional linear model* for $f \in \mathrm{Hol}(\Delta, \Delta)$ if

(i) $\Omega = H$ or $\Omega = \mathbb{C}$,
(ii) $\sigma \in \mathrm{Hol}(\Delta, \Omega)$,
(iii) $\Phi(w) = \alpha w + \beta$ with $\alpha, \beta \in \mathbb{C}$,
(iv) there exists $V \subset \Delta$ fundamental for f such that $\sigma|_V$ is univalent and $\sigma(V)$ is fundamental for Φ,
(v) $\sigma \circ f = \Phi \circ \sigma$ *i.e.* the following diagram commutes

$$\begin{array}{ccc} \Delta & \xrightarrow{f} & \Delta \\ \sigma \downarrow & & \downarrow \sigma \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

The fractional linear model is said *univalent* if $V = \Delta$.

Notice that a fractional linear model for $f \in \mathrm{Hol}(\Delta, \Delta)$ is univalent if and only if f is univalent on Δ. Now we recall the following theorem on the existence of a fractional linear model (see [BaPo], [Po], [Va], [Co1]):

Theorem 4.5 (Valiron, Baker, Pommerenke, Cowen). *If $f \in \mathrm{Hol}(\Delta, \Delta)$ has Wolff point $\tau \in \partial\Delta$, then there exists a fractional linear model (Ω, σ, Φ) for f.*

Remark 4.6. Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and let (Ω, σ, Φ) be a fractional linear model for f. Our definition of fractional linear models doesn't guarantee that Φ is an *automorphism* of Ω. For instance (transfering everything in H by means of the Cayley transformation) the map $f(w) = w + a$ with a real positive number has obviously the fractional linear model (H, id_H, f), but f is not an automorphism of H. However, it also has the model $(\mathbb{C}, j, \Phi)$ where $j : H \hookrightarrow \mathbb{C}$ is the immersion and $\Phi(w) = w + a$ is an automorphism of $\mathbb{C}$. In general, embedding H in $\mathbb{C}$ if necessary, we can always find a fractional linear model for f in which Φ is an automorphism of Ω. In this case we have the following uniqueness statement due to Cowen (see [Co1]). If $(\widetilde{\Omega}, \widetilde{\sigma}, \widetilde{\Phi})$ is another fractional linear model for f, such that $\widetilde{\Phi}$ is an automorphism of $\widetilde{\Omega}$, then $\Omega = \widetilde{\Omega}$, and moreover there exists a Möbius transformation φ such that $\varphi(\Omega) = \Omega$, $\widetilde{\Phi} = \varphi \circ \Phi \circ \varphi^{-1}$ and $\widetilde{\sigma} = \varphi \circ \sigma$. In what follows we will not use this fact.

A first step to relate the expansions of f and Φ at their Wolff points is the following (see [Co1]):

Theorem 4.7 (Cowen). *Let $f \in \mathrm{Hol}(\Delta, \Delta) \setminus \mathrm{Aut}(\Delta)$ have Wolff point $\tau \in \partial\Delta$. Let (Ω, σ, Φ) be a fractional linear model for f. If $f \in C^1(\tau)$ or if for some $z_0 \in \Delta$ the sequence $\{f^k(z_0)\}$ converges to τ non-tangentially, then*

$$\operatorname*{K\text{-}lim}_{z \to \tau} f'(z) = \Phi'(\sigma(\tau)).$$

We recall now (see [BoSh] Theorem 4.12):

Theorem 4.8 (Bourdon-Shapiro). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f \in C^{3+\epsilon}(1)$. Let $f(1) = 1$, $f'(1) = 1$ and $f''(1) = a \neq 0$. Suppose f is univalent on Δ, continuous up to $\partial\Delta$ and $f(\overline{\Delta} \setminus \{1\}) \subset \Delta$. Then there exists a univalent fractional linear model (Ω, σ, Φ) for f with $\Phi(w) = w + a$. Moreover*

$$\lim_{w \to \infty} \frac{\sigma(C^{-1}(w))}{w} = 1,$$

where $C : \Delta \to H$ is the Cayley transformation which maps 1 to ∞.

Remark 4.9. Strictly speaking in [BoSh] it is not proven that the model given by *Theorem 4.12* is a fractional linear model according to our definition, since it is not shown that $\sigma(\Delta)$ is fundamental for the automorphism Φ. However, using the estimates on the shape of $\sigma(\Delta)$ given there, it is possible to see that $\sigma(\Delta)$ is fundamental for Φ. Here we give a sketch of how to do that. Transfer everything to the half-plane H. Then *Theorem 4.12* in [BoSh] gives us the following expression:

$$\sigma(w) = w + h(w) \tag{4.2}$$

where h is holomorphic on H and $\lim\limits_{w \to \infty} h(w)/w = 0$. Then

$$\lim_{y \to \pm\infty} \mathrm{Im}(\sigma(iy)) = \pm\infty \quad \text{and} \quad \lim_{y \to \pm\infty} \frac{\mathrm{Re}(\sigma(iy))}{\mathrm{Im}(\sigma(iy))} = 0,$$

which prevents $\sigma(H)$ to have oblique asymptotes. When $\mathrm{Re}(a) > 0$ this implies that $\sigma(H)$ is fundamental for Φ in $\mathbb{C}$. In the case $\mathrm{Re}(a) = 0$, assuming $a = \alpha i$ with $\alpha > 0$, *Theorem 4.12* in [BoSh] gives us, for w in the upper half part of H, the following representation:

$$\sigma(w) = w + i\frac{b}{\alpha}\log(1 + w) + B(w) \tag{4.3}$$

where $b \geq 0$ and B is a bounded continuous function on $\overline{H} \cup \{\infty\}$, holomorphic in H (if $\alpha < 0$ there is a similar expression in the lower part of H, and then one can proceed similarly). A straightforward calculation gives

$$\mathrm{Re}(\sigma(iy)) = -\frac{b}{\alpha}\arg(1 + iy) + \mathrm{Re}(B(iy)).$$

Then $\lim_{y \to +\infty} \mathrm{Re}(\sigma(iy)) = M$ for $M \in \mathbb{R}$. We can suppose $M = 0$, up to subtract M from σ. If $\inf_{w \in H} \mathrm{Re}(\sigma(w)) < 0$, then there would exist $y_0 \in \mathbb{R}$ such that $\mathrm{Re}(\sigma(iy_0)) < 0$. But $\sigma(H)$ is invariant for the translation $w \to w + i\alpha$ then $\lim_{n \to +\infty} \mathrm{Re}(\sigma(i(n\alpha + y_0))) < 0$. Therefore $\inf_{w \in H} \mathrm{Re}(\sigma(w)) = 0$. So $\sigma(H) \subset H$ and $\sigma(H)$ has ∂H as vertical asymptote. Hence $\sigma(H)$ is fundamental for Φ in H.

Definition 4.10. Let $f, g \in \mathrm{Hol}(\Delta, \Delta)$ and let (Ω, σ, Φ) be a fractional linear model for f. We say that g belongs to the *pseudo-iteration semigroup of* f, (shortly $g \in \mathrm{PIS}(f)$), if there exists a Möbius transformation Ψ such that $\Psi \circ \Phi = \Phi \circ \Psi$ and $\sigma \circ g = \Psi \circ \sigma$.

Remark 4.11. If $f \neq \mathrm{id}_\Delta$ and $g \in \mathrm{PIS}(f)$, from our definition of fractional linear model, then Ψ is actually an affine transformation since it commutes with the affine transformation Φ.

Remark 4.12. If $g \in \mathrm{PIS}(f)$ then (Ω, σ, Ψ) is generally not a fractional linear model for g. An example is as follows. Consider the conformal map σ from H to $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$. Let $\Phi(z) := z + i$, $\Psi(z) := z + 1$ and $f(w) := \sigma^{-1} \circ \Phi \circ \sigma(w)$, $g(w) := \sigma^{-1} \circ \Psi \circ \sigma(w)$. Then it is clear that (H, σ, Φ) is a univalent fractional linear model for f, f and g commute, $g \in \mathrm{PIS}(f)$ but (H, σ, Ψ) is not a fractional linear model for g since $\sigma(H)$ is not fundamental for Ψ.

The pseudo-iteration semigroup and commuting holomorphic self-maps of Δ are related by the following:

Theorem 4.13 (Cowen)**.** *Let $f, g \in \mathrm{Hol}(\Delta, \Delta)$ with common Wolff point $\tau \in \partial\Delta$. If $f \circ g = g \circ f$ then $f, g \in \mathrm{PIS}(f \circ g)$. Moreover, if $\lim_{r \to 1} f'(r\tau) < 1$ then $g \in \mathrm{PIS}(f)$.*

For our purpose we need also the following:

Proposition 4.14. *Let $f, g \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f \in C^{3+\epsilon}(1)$ and $g \in C^2(1)$. Let $f(1) = g(1) = 1$, $f'(1) = g'(1) = 1$ and $f''(1) = g''(1) = a \neq 0$. Suppose f, g are univalent on Δ, continuous up to $\partial\Delta$ and map $\overline{\Delta} \setminus \{1\}$ into Δ. Suppose $f \circ g = g \circ f$. Then $g \in \mathrm{PIS}(f)$.*

Proof. Transfer everything to H. Let (Ω, σ, Φ) be the fractional linear model for f given by Theorem 4.8. The map $\Psi : \Omega \to \Omega$ given by:

$$\Psi : w \mapsto \Phi^{-n} \circ \sigma \circ g \circ \sigma^{-1} \circ \Phi^n(w),$$

where n is big enough in order to assure $\Phi^n(w) \in \sigma(\Delta)$, is well defined (see *p. 689* in [Co2]). Then the condition $g \in \mathrm{PIS}(f)$ is equivalent to $\Psi \in \mathrm{Aut}(\mathbb{C})$. Since Φ, σ, g are univalent so is Ψ. We are left to show that Ψ is surjective. This follows easily whenever we prove that $\sigma(g(H))$ is fundamental for Ψ in H. Since $g \in C^2(\infty)$, then $g(w) = w + a + \Gamma(w)$ with $\Gamma(w) \to 0$ as $w \to \infty$. Using this expression and arguing as we did in Remark 4.9 it is easy to see that $\sigma(g(H))$ is fundamental for Φ. □

Remark 4.15. In the previous proposition it is possible to release some hypothesis on the regularity of f and g assuming for instance that $f^n(0)$ converges to 1 non-tangentially. However we are not interested in such results here.

5. The Automorphism Case

In this section we are going to prove the cases *(2)*, *(3)* and *(4)* of Theorem 2.4. We start with the following result, interesting for its own (see also [TaVl]):

Theorem 5.1. *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and let $\varrho : [0,1[\to \Delta$ be a continuous curve that tends to $\tau \in \partial\Delta$ non-tangentially. If*

$$\lim_{t\to 1^-} \frac{f(\varrho(t)) - \varrho(t)}{(\varrho(t)-\tau)^3} = l \tag{5.1}$$

for some $l \in \mathbb{C}$ then $\tau^2 l$ is a non-positive real number and f is the identity map if and only if $l = 0$. Moreover if $f \in C^3(\tau)$ then $f(\tau) = \tau$, $f'(\tau) = 1$, $f''(\tau) = 0$ and $f'''(\tau) = 6l$ if and only if equation (5.1) holds.

Proof. If $f \equiv \mathrm{id}_\Delta$ then obviously $l = 0$. Assume now that f is not the identity map. We define the holomorphic map

$$h(z) \stackrel{\text{def}}{=} -\varphi^{-1}(\varphi(f(z)) - \varphi(z))$$

where $\varphi(z) = \frac{\tau+z}{\tau-z}$ is a biholomorphism of Δ onto the right half-plane H. Now, from equation (5.1)

$$\lim_{t\to 1^-} \frac{\tau - f(\varrho(t))}{\tau - \varrho(t)} = 1 + \lim_{t\to 1^-} \frac{\varrho(t) - f(\varrho(t))}{\tau - \varrho(t)} = 1. \tag{5.2}$$

Theorem 4.1 then implies that $\operatorname*{K-lim}_{z\to\tau} f(z) = \tau$ and $\operatorname*{K-lim}_{z\to\tau} f'(z) = 1$. Moreover from equation (4.1) we have

$$\mathrm{Re}(\varphi(f(z))) = \frac{1-|f(z)|^2}{|\tau - f(z)|^2} \geq \frac{1-|z|^2}{|\tau - z|^2} = \mathrm{Re}(\varphi(z)) \qquad \forall z \in \Delta. \tag{5.3}$$

Then h maps Δ into $\overline{\Delta}$. By the maximum principle, if there is a point $z_0 \in \Delta$ such that $h(z_0) \in \partial\Delta$ then h is identically equal to a constant. Since

$$h(z) = -\varphi^{-1}(2\tau \frac{f(z)-z}{(z-\tau)^2} \cdot \frac{\tau - z}{\tau - f(z)}) \qquad \forall z \in \Delta$$

then, from (5.1), the limit of h as $z \to \tau$ along ϱ is $-\varphi^{-1}(0) = \tau$ (notice that the term $\frac{\tau - z}{\tau - f(z)}$ tends to 1 by Theorem 4.1). Therefore $h \equiv \tau$, that is f is the identity. This contradicts our assumption, so $h \in \mathrm{Hol}(\Delta, \Delta)$.

After some easy calculations we find that, for any $z \in \Delta$

$$\frac{\tau - h(z)}{\tau - z} = \frac{-4\tau^2 \frac{f(z)-z}{(z-\tau)^3}}{\frac{\tau - f(z)}{\tau - z} + 2\tau \frac{f(z)-z}{(z-\tau)^2}}.$$

Passing to the limits as $z \to \tau$ along ϱ in both sides of the above equation, by (5.1) and (5.2), by applying Theorem 4.1 to the map h, we obtain

$$\beta_h(\tau,\tau) = -4\tau^2 l.$$

Then

$$\tau^2 l = -\frac{1}{4}\beta_h(\tau,\tau) < 0.$$

The last statement follows directly from the previous arguments. □

Proposition 5.2. *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and $g \in \mathrm{Aut}(\Delta)$ be such that $f \circ g = g \circ f$.*

(i) *If g is a hyperbolic automorphism with a fixed point $\tau \in \partial\Delta$ and $\lim_{r\to 1^-} f'(r\tau) = g'(\tau)$ then $f \equiv g$.*

(ii) *If g is a parabolic automorphism with the fixed point $\tau \in \partial\Delta$ and $\lim_{r\to 1^-} f'(r\tau) = g'(\tau)$, $\lim_{r\to 1^-} f''(r\tau) = g''(\tau)$ then $f \equiv g$.*

(iii) *If* $g \equiv \mathrm{id}_\Delta$ *and there exists* $\tau \in \partial\Delta$ *such that* $\lim\limits_{r\to 1^-} f(r\tau) = \tau$, $\lim\limits_{r\to 1^-} f'(r\tau) = 1$, $\lim\limits_{r\to 1^-} f''(r\tau) = 0$, $\lim\limits_{r\to 1^-} f'''(r\tau) = 0$ *then* $f \equiv g$.

Remark 5.3. There is a theorem due to Heins (see [He]) which states that in the hypothesis of case *(i)* f is actually a hyperbolic automorphism: we will give a new simpler proof of it based on Theorem 2.3. For the case *(ii)*, notice that since f commutes with g then actually $\lim\limits_{r\to 1^-} f'(r\tau) = 1$ (see [Co2]). The case *(iii)* is a slightly improved version of Burns-Krantz Theorem (see [BuKr]).

Proof of Proposition 5.2. We can assume that $\tau = 1$ up to conjugation in $\mathrm{Aut}(\Delta)$. First let g be a hyperbolic automorphism. We can suppose 1 to be the Wolff point of g (otherwise 1 is the Wolff point of g^{-1}). Now, since $f \circ g = g \circ f$ we have

$$f \circ g^{-1} = g^{-1} \circ g \circ f \circ g^{-1} = g^{-1} \circ f.$$

Then f commutes with g and g^{-1}. By Theorem 2.3 it follows that f is a hyperbolic automorphism with the same fixed points of g. Then $f \circ g^{-1}$ is a hyperbolic automorphism (for it has two fixed points on $\partial\Delta$) such that $(f \circ g^{-1})'(1) = 1$, *i.e.* $f \circ g^{-1} = \mathrm{id}_\Delta$.

If g is a parabolic automorphism then by Theorem 2.3, f has Wolff point 1. Transfer everything in the right half-plane H by means of the Cayley transformation C. Then $\widetilde{g} = C \circ g \circ C^{-1} \in \mathrm{Aut}(H)$ and

$$\widetilde{g}(w) = w + ib \quad \forall w \in H \quad \text{with } ib = g''(1).$$

Let $\widetilde{f} = C \circ f \circ C^{-1}$. Notice that, since $\lim\limits_{r\to 1^-} f'(r) = g'(1) = 1$, for r real near to 1, if we set $w_r \stackrel{\text{def}}{=} \frac{1+r}{1-r}$, then

$$\widetilde{f}(w_r) = w_r + f''(1) + \dots$$

where $f''(1) \stackrel{\text{def}}{=} \lim\limits_{r\to 1^-} f''(r)$.

Moreover, by Theorem 4.1

$$\mathrm{Re}(\widetilde{f}(w)) \geq \mathrm{Re}(w) \quad \forall w \in H$$

and the holomorphic map $h(w) \stackrel{\text{def}}{=} \widetilde{f}(w) - w$ maps H in $\overline{H}$. If there is $w_0 \in H$ such that $h(w_0) = ic \in \partial H$ then, by the maximum principle, h is identically constant. Therefore, $\widetilde{f}(w) = w + ic$ that is f is a parabolic automorphism too. In this case, since $ib = g''(1) = f''(1) = ic$ then $f \equiv g$. Assume now that $h(H) \subset H$. Let Γ be the group generated by the translation $\widetilde{g}$. Then H/Γ is biholomorphic to $\Delta \setminus \{0\}$ and the covering map $\pi : H \to \Delta \setminus \{0\}$ is

$$\pi(w) = \exp(-\frac{2\pi w}{|b|})$$

Since also $\widetilde{f}$ and $\widetilde{g}$ commute, then $h \circ \widetilde{g} = h$. So it is well defined the map $\widetilde{h} : \Delta \setminus \{0\} \to H$ such that $h = \widetilde{h} \circ \pi$ and for all $w \in H$

$$\widetilde{f}(w) - w = h(w) = \widetilde{h}(\exp(-\frac{2\pi w}{|b|})).$$

Now $\widetilde{h}$ is holomorphic and H is biholomorphic to the bounded domain Δ then the singularity of $\widetilde{h}$ in 0 can be eliminated and actually $\widetilde{h}$ is holomorphic in Δ. Then

$$f''(1) = \lim_{r\to 1^-}(\widetilde{f}(w_r) - w_r) = \lim_{r\to 1^-}\widetilde{h}(\exp(-\frac{2\pi w_r}{|b|})) = \widetilde{h}(0) \in H$$

Hence we have the contradiction $ib = g''(1) = f''(1) \in H$.

If g is the identity then, by applying Theorem 5.1, we have also that $f \equiv g$. □

Remark 5.4. Notice that if $f \in \mathrm{Hol}(\Delta,\Delta)$ and $g \in \mathrm{Aut}(\Delta)$ agree at a point on $\partial\Delta$ up to the third order then we can apply Theorem 5.1 to the map $f \circ g^{-1}$ and we find that $f \equiv g$ without assuming that f and g commute. On the other hand the following two holomorphic self maps of H coincide up to the N-th order at their common Wolff point ∞, but they are not identical:

$$f(w) = w + N + 1 + \frac{1}{1+w} + \ldots + \frac{1}{(1+w)^N},$$

$$g(w) = f(w) + \frac{1}{2(1+w)^{N+1}}.$$

Of course they don't commute.

6. The Hyperbolic Case

In this section we prove the case *(5)* of Theorem 2.4.

Proposition 6.1. *Let* $f, g \in \mathrm{Hol}(\Delta,\Delta)\setminus\mathrm{Aut}(\Delta)$. *If* $f\circ g = g \circ f$ *and* $\lim_{r\to 1^-} f'(r\tau) = \lim_{r\to 1^-} g'(r\tau) < 1$ *at their common Wolff point* $\tau \in \partial\Delta$ *then* $f \equiv g$.

Proof. Let (Ω, σ, Φ) be a fractional linear model for f and let $\widetilde{V}$ be the fundamental set for f given by the very definition of linear fractional model. As explained in *Proposition 3.1* of [Co1], we can choose $\widetilde{V}$ in such a way that it contains small sectors of vertex τ. By Theorem 4.13, g is in the pseudo-iteration semigroup of f, that is there exists a Möbius transformation Ψ such that $\Psi\circ\Phi = \Phi\circ\Psi$ and $\sigma \circ g = \Psi \circ \sigma$.

Since $\text{K-}\lim_{z\to\tau} g'(z) < 1$ then for any compact subset K of Δ the sequence of iterates $\{g^k(K)\}$ converges to τ non-tangentially (see *Lemma 2.2* in [Co1]), and therefore we can repeat the Cowen construction (see [Co1] p. 77 and [Co2] p. 690) in order to get a fundamental set V for g such that V contains small sectors, $V \subset \widetilde{V}$ and g is univalent on V. Moreover, again in *Proposition 3.1* of [Co1], it is shown that $\sigma(V)$ is a fundamental set for Ψ and hence (Ω, σ, Ψ) is actually a fractional linear model for g. Now, by definition, $\Phi(w) = \alpha w + \beta$ for some $\alpha, \beta \in \mathbb{C}$. By applying Theorem 4.7 to both Ψ and Φ, since Ψ commutes with Φ and $\lim_{r\to 1^-} f'(r\tau) = \lim_{r\to 1^-} g'(r\tau) < 1$ we find that $\Psi \equiv \Phi$. This implies that

$$\sigma \circ f = \sigma \circ g. \tag{6.1}$$

Now, since V is fundamental for g then given a compact set $K \subset f(V)$ with non-empty interior, the sequence of iterates $\{g^n(K)\}$ is eventually contained in V. Since

$$g(f(V)) = f(g(V)) \subseteq f(V),$$

then we get $\emptyset \neq g^n(K) \subset V \bigcap f(V)$ for some $n > 0$. Therefore there exists a non-empty open set $U \subset V$ such that $f(U) \subset V$. Hence, since σ is injective on U and equation (6.1) holds in U then $f \equiv g$. □

7. The Parabolic Case: Part I

In this section we prove case *(6)* of Theorem 2.4.

Remark 7.1. Let $f, g \in \mathrm{Hol}(\Delta, \Delta) \setminus \mathrm{Aut}(\Delta)$ be two commuting maps. If f has its Wolff point $\tau \in \partial\Delta$ with $f'(\tau) = 1$ then we know that τ is the Wolff point also for g and $g'(\tau) = 1$. We can assume that $\tau = 1$ up to conjugation in $\mathrm{Aut}(\Delta)$. If $f, g \in C^1(\Delta \cup \{1\})$ then, by Lemma 4.2, there is a neighborhood U of 1 such that both f and g are injective in $U \cap \Delta$. Let $R > 0$ be such that $E(1, R) \subset U$. Since $f(E(1, R)) \subset E(1, R)$, and the same for g, then up to restricting the domain to $E(1, R)$, conjugating f and g by the linear transformation $\varrho(w) = \frac{R}{R+1}w + \frac{1}{R+1}$, which maps Δ onto $E(1, R)$ fixing 1, we can assume f and g to be univalent, $C^1(\overline{\Delta})$ and, by the Schwarz Lemma, to send $\overline{\Delta} \setminus \{1\}$ in Δ. Notice that, if f is regular enough, such a conjugation acts on the n-th derivative of f (and g) at 1 as a multiplication by $\left(\frac{R}{R+1}\right)^{n-1}$, which is strictly positive.

Proposition 7.2. *Let $f, g \in \mathrm{Hol}(\Delta, \Delta) \setminus \mathrm{Aut}(\Delta)$ be such that $f \in C^{3+\epsilon}(1)$, $g \in C^2(1)$. Suppose that $f(1) = g(1) = 1$, $f'(1) = g'(1) = 1$ and $f''(1) = g''(1) = a \neq 0$. If $f \circ g = g \circ f$ then $f \equiv g$.*

Proof. As in Remark 7.1 we can suppose f, g univalent on Δ, f, g continuous up to $\partial\Delta$ and $f, g : \overline{\Delta} \setminus \{1\} \mapsto \Delta$. Moreover up to conjugation by the Cayley transformation C we can transfer our considerations on H. Let (Ω, σ, Φ) be a univalent linear fractional model for f given by Theorem 4.8:

$$\begin{array}{ccc} H & \xrightarrow{f} & H \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

where $\Phi(w) = w + a$ and

$$\lim_{w \to \infty} \frac{\sigma(w)}{w} = 1. \tag{7.1}$$

By Proposition 4.14 $g \in \mathrm{PIS}(f)$. Then there exists a Möbius transformation Ψ such that $\Psi \circ \Phi = \Phi \circ \Psi$ and the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{g} & H \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \Omega & \xrightarrow{\Psi} & \Omega \end{array}$$

Since $\Phi(w) = w + a$ then it is easy to verify that $\Psi(w) = w + b$ with $b \in \mathbb{C}$ and $b \neq 0$ (since $g \neq \mathrm{id}_\Delta$). Then the Proposition will hold whenever we prove that $b = a$, for $g = \sigma^{-1} \circ \Psi \circ \sigma$.

Since, as $n \to \infty$,

$$\frac{g^n(w)}{n} = \frac{w + na + \sum_{j=0}^{n-1} \Gamma(g^j(w))}{n} \to a,$$

then

$$\frac{\sigma(g^n(w))}{g^n(w)} = \frac{\sigma(w) + nb}{g^n(w)} \to \frac{b}{a}.$$

Therefore, by (7.1), we find that $a = b$ and $f \equiv g$. $\square$

Remark 7.3. The previous proposition implies that if $g \in \mathrm{Hol}(\Delta, \Delta)$ is $C^2(1)$, $g(1) = 1$, $g'(1) = 1$ and $g''(1) \neq 0$, then g earns regularity $C^k(1)$ whenever we are able to produce another holomorphic self-map f of Δ, $f \in C^k(1)$ $(k > 3)$, such that f commutes with g and $f''(1) = g''(1)$.

8. The parabolic case: part II

In this section we prove case *(7)* of Theorem 2.4, that is:

Proposition 8.1. *Let $f, g \in \mathrm{Hol}(\Delta, \Delta) \setminus \mathrm{Aut}(\Delta)$ be such that $f \in C^{5+\epsilon}(1)$, $g \in C^4(1)$. Suppose that $f(1) = g(1) = 1$, $f'(1) = g'(1) = 1$, $f''(1) = g''(1) = 0$ and $f'''(1) = g'''(1)$. If $f \circ g = g \circ f$ then $f \equiv g$.*

We start proving that if $f \in C^{5+\epsilon}(1)$ then there exists an invariant subset of the unit disc Δ on which f is conjugated to a suitable translation of the half-plane. For $r \geq 0$, let

$$T_r(z) \stackrel{\text{def}}{=} \frac{1}{(z-1)^2} - r \quad \forall z \in \Delta.$$

Notice that T_r is a biholomorphism from Δ to $T_r(\Delta) \supset H_{\frac{1}{4}-r}$.

Lemma 8.2. *Let $f \in \mathrm{Hol}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$ be such that*

$$f(z) = z + \sum_{k=3}^{5} a_k (z-1)^k + O(|z-1|^{5+\epsilon})$$

for some $0 \leq \epsilon < 1$. Then there exists $r_0 > \frac{1}{4}$ such that, for all $r \geq r_0$ f restricted to the set $T_r^{-1}(H) \subset \Delta$ is conjugated, by T_r to a map F which is continuous on $\overline{H}$, holomorphic in H and such that $F(\overline{H}) \subset H$. Moreover F has an expansion of the form

$$F(w) = w + b_0 + \frac{b_1}{(w+r)^{\frac{1}{2}}} + \frac{b_2}{(w+r)} + O\left(\frac{1}{(w+r)^{1+\frac{\epsilon}{2}}}\right), \tag{8.1}$$

with $b_0 = -2a_3 > 0$, $b_1 = -2a_4$ and $b_2 = 3a_3^2 - 2a_5$.

Proof. If f is not the identity then a_3 is a real negative number (see Theorem 5.1 for $\tau = 1$). The transformation $T_0(z) = (z-1)^{-2}$ is a biholomorphism from Δ to the domain $D = T_0(\Delta)$ with inverse $T_0^{-1}(u) = u^{-\frac{1}{2}} + 1$ for $u \in D$.

Conjugating f by T_0 we have the map $\widetilde{f} \stackrel{\text{def}}{=} T_0 \circ f \circ T_0^{-1} \in \mathrm{Hol}(D, D)$ with the following expansion

$$\begin{aligned}
\widetilde{f}(u) &= u\left(1 + a_3 u^{-1} + a_4 u^{-\frac{3}{2}} + a_5 u^{-2} + O(u^{-(2+\frac{\epsilon}{2})})\right)^{-2} \\
&= u\left(1 + 2a_3 u^{-1} + 2a_4 u^{-\frac{3}{2}} + (a_3^2 + 2a_5) u^{-2} + O(u^{-(2+\frac{\epsilon}{2})})\right)^{-1} \\
&= u\left(1 - 2a_3 u^{-1} - 2a_4 u^{-\frac{3}{2}} + (3a_3^2 - 2a_5) u^{-2} + O(u^{-(2+\frac{\epsilon}{2})})\right) \\
&= u - 2a_3 - 2a_4 u^{-\frac{1}{2}} + (3a_3^2 - 2a_5) u^{-1} + O(u^{-(1+\frac{\epsilon}{2})}).
\end{aligned}$$

Notice that the domain D contains the half-plane $H_{\frac{1}{4}}$. Let $0 < \delta < -2a_3$. Hence there exists $r_0 > \frac{1}{4}$ such that $|\widetilde{f}(u) - u + 2a_3| < \delta$ for all $\mathrm{Re}(u) \geq r_0$. Therefore, if $\mathrm{Re}(u) \geq r$, for all $r \geq r_0$ we have

$$\mathrm{Re}(\widetilde{f}(u)) \geq \mathrm{Re}(u - 2a_3) - |\widetilde{f}(u) - u + 2a_3| > \mathrm{Re}(u) - 2a_3 - \delta > r,$$

that is the half-plane $H_r \subset D$ is invariant for $\widetilde{f}$. Then $F \stackrel{\text{def}}{=} T_r \circ f \circ T_r^{-1}$ has the requested properties. $\square$

We will use the Fundamental Orbital Estimates (see [BoSh] p.70):

Lemma 8.3 (Bourdon-Shapiro). *Let F be a map continuous on $\overline{H}$ and holomorphic in H and such that $F(\overline{H}) \subset H$. If it has the following representation*

$$F(w) = w + b_0 + h(w)$$

where b_0 is a non zero complex number with $\mathrm{Re}(b_0) \geq 0$ and $\lim_{w\to\infty} h(w) = 0$, then there exist c_1, c_2 and R positive numbers such that

$$c_1(|w| + n) \leq |F^n(w)| \leq c_2(|w| + n)$$

for all $n \geq 0$ and for all $w \in H_R$.

Now we build a "fractional linear model" for maps of the form (8.1):

Theorem 8.4. *For $w \in H$ let*

$$F(w) = w + b_0 + \frac{b_1}{(w+r)^{\frac{1}{2}}} + \frac{b_2}{(w+r)} + \Gamma(w+r)$$

be such that $\mathrm{Re}(b_0) > 0$ with $\Gamma(w) = O(\frac{1}{|w|^{1+\epsilon}})$. Then there exists an injective map ν holomorphic in H_R for some $R > 0$ such that for all $w \in H_R$

$$\nu(w) = w + h(w) \quad \text{with} \quad \lim_{w\to\infty} \frac{h(w)}{w} = 0 \quad \text{and} \quad \nu(F(w)) = \nu(w) + b_0.$$

Moreover $\nu(H_R)$ is fundamental for $w \mapsto w + b_0$.

Proof. For each $w \in \overline{H}$ and for all $n \geq 0$, let

$$\begin{aligned}
w(n) &\stackrel{\text{def}}{=} F^n(w) + r,\\
\Delta w(n) &\stackrel{\text{def}}{=} w(n+1) - w(n) = b_0 + \frac{b_1}{w(n)^{\frac{1}{2}}} + \frac{b_2}{w(n)} + \Gamma(w(n)),\\
\nu_n(w) &\stackrel{\text{def}}{=} w(n) - w_0(n),\\
\Delta\nu_n(w) &\stackrel{\text{def}}{=} \nu(n+1) - \nu(n) = \Delta w(n) - \Delta w_0(n).
\end{aligned}$$

Therefore

$$\begin{aligned}
\nu_n(w) = \nu_0(w) + \sum_{j=0}^{n-1} \Delta\nu_j(w) &= w - w_0 + b_1 \sum_{j=0}^{n-1} \left[\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}}\right]\\
&+ b_2 \sum_{j=0}^{n-1} \left[\frac{1}{w(j)} - \frac{1}{w_0(j)}\right] + \sum_{j=0}^{n-1} \left[\Gamma(w(j)) - \Gamma(w_0(j))\right].
\end{aligned} \tag{8.2}$$

By Lemma 8.3 for $w \in H_R$, since $\mathrm{Re}(F^n(w)) \geq 0$

$$\begin{aligned}
|w(n)| &\geq \frac{1}{\sqrt{2}}(|F^n(w)| + r) \geq \frac{1}{\sqrt{2}}(c_1(|w| + n) + r),\\
|w(n)| &\leq |F^n(w)| + r \leq c_2(|w| + n) + r.
\end{aligned} \tag{8.3}$$

Therefore for all $w, w_0 \in H_R$

$$\left|\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}}\right| \leq \frac{2\sqrt[4]{2}}{(c_1 j + r)^{\frac{1}{2}}} \quad \text{and} \quad \left|\frac{1}{w(j)} - \frac{1}{w_0(j)}\right| \leq \frac{2\sqrt{2}}{c_1 j + r}. \tag{8.4}$$

We can assume R to be large enough so that $|\Gamma(w)| \leq \frac{M}{|w|^{1+\epsilon}}$ for all $w \in H_R$ for some $M > 0$. Then

$$|\Gamma(w(j)) - \Gamma(w_0(j))| \leq \frac{2M}{(c_1 j + r)^{1+\epsilon}} \tag{8.5}$$

Moreover, by (8.4) and (8.5), there is $M_1 > 0$ such that

$$|\Delta\nu_j(w)| \leq \frac{M_1}{(c_1 j + r)^{\frac{1}{2}}}. \tag{8.6}$$

The general term of the second sum in (8.2) is

$$\frac{1}{w(j)} - \frac{1}{w_0(j)} = -\frac{\nu_j(w)}{w(j)w_0(j)} = -\frac{\nu_0(w) + \sum_{k=0}^{n-1} \Delta\nu_k(w)}{w(j)w_0(j)},$$

and, using (8.5) and (8.6), there are $M_2, M_3 > 0$ such that

$$\begin{aligned}\left|\frac{1}{w(j)} - \frac{1}{w_0(j)}\right| &\leq \frac{1}{(c_1 j + r)(c_1(j + |w|) + r)}\Big(|w| + |w_0| \\ &\qquad + \sum_{k=0}^{j-1} \frac{M_1}{(c_1 k + r)^{\frac{1}{2}}}\Big) \\ &\leq \frac{|w| + M_2}{(c_1 j + r)(c_1(j + |w|) + r)} + \frac{M_3}{(c_1 j + r)^{\frac{3}{2}}}.\end{aligned} \tag{8.7}$$

The general term of the first sum in (8.2) is

$$\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}} = -\frac{\nu_j(w)}{w(j)^{\frac{1}{2}} w_0(j)^{\frac{1}{2}}} \cdot \frac{1}{w(j)^{\frac{1}{2}} + w_0(j)^{\frac{1}{2}}}.$$

Now if $w = \rho e^{i\theta} \in H$ then $\cos(\theta) > 0$ and

$$\mathrm{Re}(w^{\frac{1}{2}}) = \sqrt{\rho}\cos(\frac{\theta}{2}) = \sqrt{\rho}\sqrt{\frac{1+\cos(\theta)}{2}} \geq \sqrt{\rho\cos(\theta)} = \mathrm{Re}(w)^{\frac{1}{2}}.$$

If $\mathrm{Re}(b_0) > 0$, by Theorem 4.1, there is $c > 0$ such that

$$\mathrm{Re}(F(w) - w) \geq c$$

and therefore iterating we have

$$\mathrm{Re}(F^n(w)) \geq cn + \mathrm{Re}(w).$$

We can assume $c_1 \leq c$. Then

$$|w(j)^{\frac{1}{2}} + w_0(j)^{\frac{1}{2}}| \geq \mathrm{Re}(w(j)^{\frac{1}{2}} + w_0(j)^{\frac{1}{2}}) \geq \mathrm{Re}(w(j))^{\frac{1}{2}} + \mathrm{Re}(w_0(j))^{\frac{1}{2}} \geq 2(c_1 j + r)^{\frac{1}{2}}$$

and, by (8.3) and (8.4), there is $M_4 > 0$ such that

$$\begin{aligned}\left|\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}}\right| &\leq \frac{1}{2(c_1 j + r)(c_1(j + |w|) + r)^{\frac{1}{2}}}\Big(|w| + |w_0| \\ &\qquad + |b_1| \sum_{k=0}^{j-1} \left|\frac{1}{w(k)^{\frac{1}{2}}} - \frac{1}{w_0(k)^{\frac{1}{2}}}\right| + \sum_{k=0}^{j-1} \frac{M_4}{c_1 k + r}\Big).\end{aligned} \tag{8.8}$$

Moreover, for some $M_5, M_6, M_7, M_8 > 0$,

$$\begin{aligned}\left|\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}}\right| &\le \frac{|w| + M_5 + M_6(c_1 j + r)^{\frac{1}{2}} + M_7 \log(c_1 j + r)}{2(c_1 j + r)^{\frac{3}{2}}} \\ &\le \frac{|w| + M_5}{2(c_1 j + r)^{\frac{3}{2}}} + \frac{M_8}{c_1 j + r},\end{aligned}$$

and there are $M_9, M_{10}, M_{11} > 0$ such that

$$\begin{aligned}\left|\frac{1}{w(j)^{\frac{1}{2}}} - \frac{1}{w_0(j)^{\frac{1}{2}}}\right| &\le \frac{1}{2(c_1 j + r)(c_1(j + |w|) + r)^{\frac{1}{2}}}\Big(|w| + |w_0| \\ &\quad + |b_1| \sum_{k=0}^{j-1} \frac{|w| + M_5}{2(c_1 k + r)^{\frac{3}{2}}} + \sum_{k=0}^{j-1} \frac{M_4 + |b_1| M_8}{c_1 k + r}\Big) \\ &\le \frac{M_9 |w| + M_{10}}{(c_1 j + r)(c_1(j + |w|) + r)^{\frac{1}{2}}} + \frac{M_{11} \log(c_1 j + r)}{(c_1 j + r)^{\frac{3}{2}}}.\end{aligned} \tag{8.9}$$

The estimates (8.5), (8.7) and (8.9) together with (8.2) imply the uniform convergence of $\{\nu_n\}$ on compacts subsets of H_R to an injective (by Hurwitz theorem) map ν holomorphic in H_R with the following representation

$$\nu(w) = w + h(w) \quad \text{with} \quad \lim_{w \to \infty} \frac{h(w)}{w} = 0.$$

In fact notice that the bounds in (8.5), (8.7) and (8.9) depend on w in such a way that, dividing them by w, they are infinitesimal when w tends to ∞. Moreover

$$\begin{aligned}\nu_n(F(w)) = F^{n+1}(w) - F^n(w_0) &= \nu_{n+1}(w) + F^{n+1}(w_0) - F^n(w_0) \\ &= \nu_{n+1}(w) + b_0 + O\Big(\frac{1}{|F^n(w_0)|^{\frac{1}{2}}}\Big)\end{aligned}$$

and taking the limit for $n \to \infty$, by Proposition 8.3 and the convergence of ν_n just proved we find

$$\nu(F(w)) = \nu(w) + b_0.$$

As in Remark 4.9, we can show that the set $\nu(H_R)$ is fundamental for the translation $w \to w + b_0$ because

$$\operatorname{Re}(b_0) > 0 \quad \text{and} \quad \lim_{y \to +\infty} \frac{\operatorname{Re}(\nu(R + iy))}{\operatorname{Im}(\nu(R + iy))} = 0.$$

□

Proof of Proposition 8.1. By Remark 7.1 we can assume f, g univalent on Δ, continuous up to $\partial\Delta$ and mapping $\overline{\Delta} \setminus \{1\}$ into Δ. By Lemma 8.2, using T_r, for some $r > 0$, we can conjugate f restricted to $T_r^{-1}(H)$ to a holomorphic map F whose expansion in H is:

$$F(w) = w + b_0 + \frac{b_1}{(w + r)^{\frac{1}{2}}} + \frac{b_2}{(w + r)} + O\Big(\frac{1}{(w + r)^{1 + \frac{\epsilon}{2}}}\Big)$$

with $b_0 > 0$. Moreover, since $g'''(1) = f'''(1) = -3b_0$, we can take r so large that $g(T_r^{-1}(H)) \subset T_r^{-1}(H)$ and g restricted to the set $T_r^{-1}(H)$ is conjugated by the map T_r to a holomorphic map G whose expansion in H is

$$G(w) = w + b_0 + O\Big(\frac{1}{(w + r)^{\frac{1}{2}}}\Big).$$

Now, proceeding as in Proposition 7.2 but using Theorem 8.4 for F and G, instead of Theorem 4.8, we find that F and G coincide. Therefore $f \equiv g$. $\square$

9. Representativeness of models

Let $f \in \mathrm{Hol}(H, H) \cap C^2(\infty)$ have the following expansion:

$$f(w) = w + a + \Gamma(w)$$

where $a = f''(\infty) \neq 0$, and $\Gamma(w) \to 0$ as $w \to \infty$. Let (Ω, σ, Φ) be a fractional linear model for f. From Theorem 4.7 it follows that $\Phi(w) = w + \beta$ for some $\beta \neq 0$. Let us study the ratio

$$\frac{|\sigma(f^n(w))|}{|f^n(w)|}$$

for a fixed $w \in H$. Since $\Phi \circ \sigma = \sigma \circ f$ then $\sigma(f^n(w)) = \Phi^n(\sigma(w)) = \sigma(w) + \beta n$. Therefore $\frac{|\sigma(f^n(w))|}{|f^n(w)|}$ can be compared to $\frac{|\beta n|}{|f^n(w)|}$ for n big enough. But now

$$\begin{aligned} \frac{f^n(w)}{\beta n} &= \frac{1}{\beta n}\left[f^{n-1}(w) + a + \Gamma(f^{n-1}(w))\right] = \\ &= \frac{1}{\beta n}\left[w + an + \sum_{j=0}^{n-1} \Gamma(f^j(w))\right] \end{aligned}$$

and then, since $f^n(w) \to \infty$ as $n \to \infty$, we find that

$$\lim_{n\to\infty} \frac{|\sigma(f^n(w))|}{|f^n(w)|} = \frac{\beta}{a}.$$

Hence the model is really *representative* of the behavior of the distribution of orbits of f in H, *i.e.* f approches to its Wolff point ∞ as fast as Φ does. Using the representativeness of the model we are able to prove:

Proposition 9.1. *Let $f, g \in \mathrm{Hol}(H, H) \setminus \mathrm{Aut}(H)$. Let $f, g \in C^{3+\epsilon}(\infty)$. Suppose that $f \circ g = g \circ f$, ∞ is the Wolff point of f, $f'(\infty) = 1$ and $f''(\infty) = a \neq 0$. Then ∞ is the Wolff of g, $g'(\infty) = 1$ and $g''(\infty) \neq 0$.*

Proof. By Theorem 2.3, g has Wolff point ∞. In [Co1] it is proved that $g'(\infty) = 1$. So we need only to prove that $g''(1) \neq 0$. Suppose not. As in Remark 7.1, we can assume f and g univalent on H. Let $h := f \circ g$. Note that $h \in C^{3+\epsilon}(\infty)$, $h(\infty) = \infty$, $h'(\infty) = 1$ and $h''(\infty) = a \neq 0$. By *Theorem 3.1 of* [Co2] the map $g \in \mathrm{PIS}(h)$. Hence, if (Ω, σ, Φ) is the univalent fractional linear model for h given by Theorem 4.8, there exists a Möbius transformation Ψ such that Ψ commutes with Φ and $\sigma \circ g = \Psi \circ \sigma$. Now, since $\Phi(w) = w + a$ and Ψ commutes with Φ it follows that $\Psi(w) = w + b$ for some $b \in \mathbb{C}$, $b \neq 0$ (since $g \neq \mathrm{id}_H$). As before we find:

$$\lim_{n\to\infty} \frac{|\sigma(g^n(1))|}{|g^n(1)|} = \infty.$$

But this contradicts the fact that, by Theorem 4.8, $\lim\limits_{n\to\infty} \frac{\sigma(g^n(1))}{g^n(1)} = 1$. $\square$

Notice that even if $a = 0$ (and $f \neq \mathrm{id}_H$) then there always exists a fractional linear model (Ω, σ, Φ). But in this case, after repeating the above arguments we find that $\lim\limits_{n\to\infty} \frac{|\sigma(f^n(w))|}{|f^n(w)|} = \infty$, *i.e.* the iterates of Φ tend to ∞ faster than those of f. However, if $f \in C^{5+\epsilon}(\infty)$ we produced a "representative (partial) fractional linear model". Indeed, with the notations of Lemma 8.2 and Theorem 8.4, let

$\sigma \stackrel{\text{def}}{=} \nu \circ T_r \circ C^{-1}$. Then $(H, \sigma, w \mapsto w + b_0)$ is a fractional linear model for $f|_{C(T_r^{-1}(H_R))}$. It is representative since $\sigma(f^n(z))$ tends to ∞ as fast as $f^n(z)$ does.

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